FYSS9000, High energy scattering in QCD, spring 2023

Exercise 2, tutorial session Mon April 24th at 10-12 (FYS5 [!, room changed 17.4.2023]), return by Wed April 26th at 18.00.

1. Let us work out the color structure of the 2-gluon exchange. We have an amplitude with 4 external colored legs $A_{ij,lk}$, with real and imaginary parts

$$\mathrm{Im}\mathcal{A}^{ij}_{k\ell} \sim (t^b t^a)_{ji} (t^b t^a)_{\ell k}$$

and

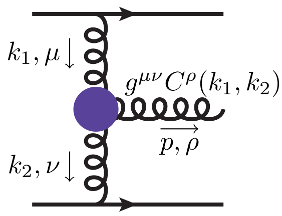
$$\mathrm{Re}\mathcal{A}^{ij}_{k\ell} \sim (t^b t^a)_{ji} [t^b, t^a]_{\ell k}.$$

Decompose these into parts that correspond to color singlet (the colors of the $+$-going and $-$-going quarks are unchanged) and octet (color structure as in one-gluon exchange) parts, i.e determine the coefficients $\mathrm{I}_{\underline{1}}$, $\mathrm{R}_{\underline{1}}$, $\mathrm{I}_{\underline{8}}$ and $\mathrm{R}_{\underline{8}}$ in

$$(t^b t^a)_{ji} (t^b t^a)_{\ell k} = \mathrm{I}_{\underline{1}} \delta_{ij} \delta_{k\ell} + \mathrm{I}_{\underline{8}} t^c_{ji} t^c_{\ell k} \tag{1}$$

$$(t^b t^a)_{ji} [t^b, t^a]_{\ell k} = \mathrm{R}_{\underline{1}} \delta_{ij} \delta_{k\ell} + \mathrm{R}_{\underline{8}} t^c_{ji} t^c_{\ell k} \tag{2}$$

You can do this using the projectors introduced in Barone & Predazzi (but then in particular $\mathrm{I}_{\underline{8}}$ becomes cumbersome). But perhaps a more straightforward way is to just Fierz away all the generators using (1.15) in the lecture note and compare the coefficients.



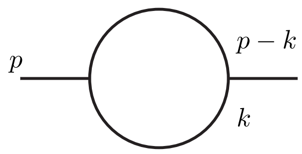
2. The Lipatov vertex is $C^\rho = (C^+, C^-, \mathbf{C}_T) = (k_1^+ + \mathbf{k}_{T1}^2/k_2^-, k_2^- + \mathbf{k}_{T2}^2/k_1^+, -\mathbf{k}_{T1} - \mathbf{k}_{T2})$. The outgoing gluon has momentum $p = k_1 - k_2$
   (a) Using the approximations in multi-Regge kinematics, express C^ρ in terms of p^+, p^- and $\mathbf{k}_{T1}, \mathbf{k}_{T2}$.
   (b) Remembering that $p^2 = 0$ show that $p^\rho C_\rho(k_1, k_2) = 0$
   (c) Calculate $C^\rho(k_1, k_2) C_\rho(k_1, k_2)$ for on-shell p: this would appear in the $q + q \to q + q + g$ cross section.

3. Show how a Laplace transform deconvolutes the nested rapidity integrals in the BFKL ladder. I.e. if

$$f(y) = \int_0^y \mathrm{d}y_1 \int_0^{y_1} \mathrm{d}y_2 \cdots \int_0^{y_{n-1}} \mathrm{d}y_n e^{(y-y_1)(\varepsilon(k_1)+\varepsilon(q-k_1))} e^{(y_1-y_2)(\varepsilon(k_2)+\varepsilon(q-k_2))} \times \cdots e^{(y_{n-1}-y_n)(\varepsilon(k_n)+\varepsilon(q-k_n))} e^{y_n(\varepsilon(k_{n+1})+\varepsilon(q-k_{n+1}))}$$

calculate the Laplace transform $f(\omega) = \int_0^\infty \mathrm{d}y e^{-\omega y} f(y)$ and show that it factorizes into a product. Hint: take the rapidity differences $y_n - y_{n+1}$ as integration variables.

4. In these 3 questions we will demonstrate dispersion relations and cutting rules with the simplest possible example: a loop correction to a propagator in $\lambda\phi^3$-theory. Consider the following loop diagram

$i\mathcal{M} =$ 

Let the particles have mass m. Then, according to the cutting rule, the imaginary part of the amplitude is

$$\operatorname{Im}\mathcal{M}(s) = \lambda^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4}(2\pi)\delta(k^2 - m^2)\theta(k^0)(2\pi)\delta((k-p)^2 - m^2)\theta(p^0 - k^0) \tag{3}$$

Calculate this integral. For convenience we denote $s \equiv p^2$ and $v = \sqrt{1 - \frac{4m^2}{s}}$ (why is this a velocity?). Hint: for example you first integrate over k^0 using the first delta function, which sets $k^0 = E_{\mathbf{k}}$. You then go the rest frame of p where $p^\mu = (\sqrt{s}, \mathbf{0})$ and change integration variables from $|\mathbf{k}|$ to $E_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$. In the end the result should be proportional to v, so that $\operatorname{Im}\mathcal{M}$ is only nonzero for $s > 4m^2$ (why?).

5. Then reconstruct the real part of the amplitude using a once-subtracted dispersion relation

$$\mathcal{M}(s) = \mathcal{M}(0) + \frac{s}{\pi}\int_{4m^2}^{\infty} \mathrm{d}s' \frac{\operatorname{Im}\mathcal{M}(s')}{s'(s'-s)} \tag{4}$$

We won't be able to determine the constant $\mathcal{M}(0)$. At least I need mathematica to calculate the integral. For simplicity let's just calculate the amplitude for $s < 4m^2$ where it is real (!).

6. *[This problem is more work, will give you points if you do it but will not count towards the maximum for the course.]* Now let's calculate the whole diagram. It is (simplifying some i's [sign corrected 17.4.2023])

$$\mathcal{M}(s) = -i\lambda^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\delta} \frac{1}{(p-k)^2 - m^2 + i\delta} \tag{5}$$

You do this integral by first changing the variable $k^0 = ik^4$ (note this brings out an additional i from the Jacobian) and then making a Wick rotation where k^4 is again integrated over the real axis and you have a 4-dimensional Euclidean k-integral. In fact you need to make it $4 - 2\varepsilon$-dimensional and calculate it using the Feynman trick

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{[xA + (1-x)B]^2} \tag{6}$$

with $A = k^2 + m^2$ and $B = (p-k)^2 + m^2$ (you can forget about the δ's at this point, unless you want to be careful), and then integrating over $d^{4-2\varepsilon}k$. See e.g. https://en.wikipedia.org/wiki/Dimensional_regularization; this website has precisely the integral you need. At this point we just want to calculate $\mathcal{M}(s) - \mathcal{M}(0)$ where the $1/\varepsilon$ and the other funny stuff cancels, you should be left with a logarithm $\sim \ln\big(m^2 - x(1-x)s\big)$ that must be integrated over x. At this point it is enough to do two things: a) take the imaginary part of the logarithm and reproduce the result of problem 4 (when does the logarithm have an imaginary part? I think we lost the sign of the imaginary part when we dropped the $i\delta$'s, so let's not worry about the sign . . .) and b) look at $s < 4m^2$ (why is the integral simpler here?) and check that you reproduce the answer of problem 5. *All in all the lesson here is that doing the whole integral is a lot more work than just calculating the imaginary part and using the dispersion relation.*