Scattering in QCD : 1 gluon exchange

Now let us move from pre-QCD models of hadronic interactions back to QCD.

We will try to compute the high energy behavior of scattering amplitudes in QCD, with the goal of understanding whether one can find a microscopical explanation for the success of the Regge parametrization.

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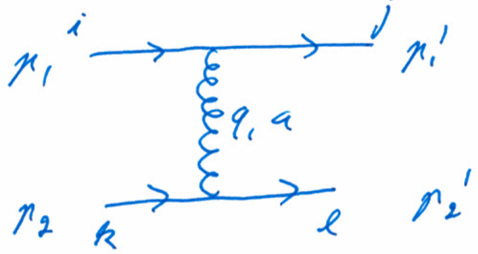
For simplicity let us look at qq-scattering. Remember: at high energies all particles interact similarly via the eikonal vertex, so our results here ~~apply~~ apply more generally.

Recall light cone variables from lecture 1: $A^\pm = \frac{1}{\sqrt{2}}(A^0 \pm A^3)$

$A^{(0)} =$ 

$$p_1 = (p_1^+, 0, \underset{\sim}{0})$$

$$p_2 = (0, p_2^-, \underset{\sim}{0})$$

$$p_1' = (p_1^+ - q^+, -q^-, -\underset{\sim}{q})$$

$$p_2' = (q^+, p_2^- + q^-, \underset{\sim}{q})$$

$$s = 2 p_1^+ p_2^-$$

H.E. limit $\quad p_1^+ \gg q^+ \quad p_2^- \gg q^-$

$$(p_1')^2 = 0 \quad \Rightarrow \quad q^- \approx \frac{-\underset{\sim}{q}^2}{2p_1^+}$$

$$(p_2')^2 = 0 \quad \Rightarrow \quad q^+ \approx \frac{\underset{\sim}{q}^2}{2p_2^-}$$

$$t = q^2 \approx -\underset{\sim}{q}^2 \quad \leftarrow$$ Typical at high s: t is a transverse momentum

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$\overline{|A|^2}$:

• color $\underbrace{\frac{1}{N_c^2}}_{\text{avg}} \underbrace{t^a_{ji} t^a_{lk}}_{A} \underbrace{t^b_{ij} t^b_{kl}}_{A^*}$

Note $t^a_{ij} = (t^a_{ji})^*$, because $t^a = t^{a\dagger}$

$$= \frac{1}{N_c^2} \operatorname{tr} t^a t^b \operatorname{tr} t^a t^b$$

$$= \frac{1}{N_c^2} \frac{1}{2}\delta^{ab} \frac{1}{2}\delta^{ab} = \frac{N_c^2-1}{4N_c^2}$$

• rest: $A = -i(-2ip_1^\mu g) \frac{-ig_{\mu\nu}}{q^2}(-2igp^\nu)$

$$= 2g^2 \frac{s}{t}$$

$$\Rightarrow \overline{|A|^2} = g^4 \frac{N_c^2-1}{4N_c^2} \frac{4s^2}{t^2}$$

• Without the eikonal approximation the answer would be $2\left(\frac{s^2+u^2}{t^2}\right)$ in stead of $\frac{4s^2}{t^2}$

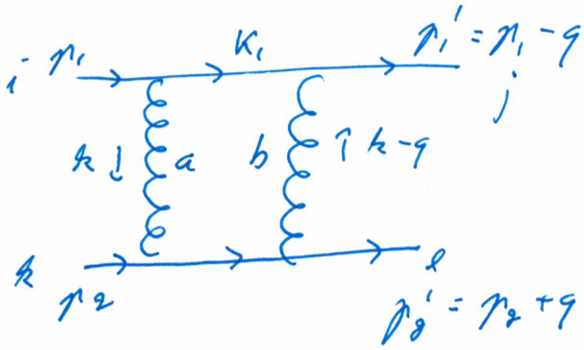
• Note 1: The eikonal vertex conserves spin: no separate sum/average over spins written explicitly above

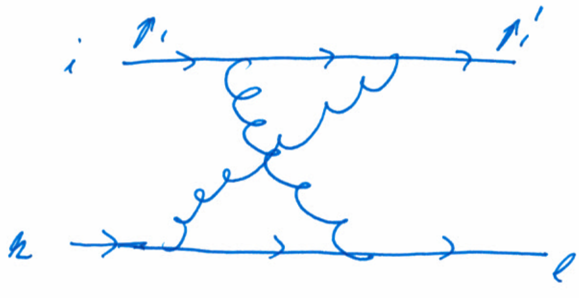
• Note 2: $\delta_{ij} t^a_{ij} = 0$. The final state quarks have a different color than the initial ones. The amplitude is not elastic. ~~has no imag~~

• Note 3: The amplitude is real: at this order no intermediate state with on-shell particles.

2 gluon exchange

In the high energy limit, the leading behavior is given by the 2 diagrams:





$$A^{(1)} = A_A^{(1)} + A_B^{(1)}$$

There are many other diagrams also at the same order in α_s, but these are subleading at large s. For example the vertex correction diagrams  or  are corrections to  and have the same s-dependence as . Likewise for propagator correction diagrams like . Another way to deduce this is that to get the biggest large-s contribution one needs the maximum amount (4) of (big) eikonal couplings to the quark lines, and internal gluon propagators that can have only a transverse momentum so that they do not suppress the result at high s.

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These are <u>loop diagrams</u>, so we have to integrate over an internal momentum.
We ~~will~~ do this <u>dispersively</u>, calculating first the imaginary part. The optical theorem gives

$$\operatorname{Im} A_A^{(1)} = \frac{1}{2} \underbrace{\int d\Omega_2(k_1, k_2)}_{\text{calculate first this phase space}} A^{(0)}(s,k) \left(A^{(0)}(s, k-q)\right)^+$$

$$\int d\Omega_2 = \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} (2\pi)^2 \delta(k_1^2)\delta(k_2^2)(2\pi)^4 \delta^4(p_1+p_2-k_1-k_2)$$

$$= \int \frac{d^4k}{(2\pi)^2} \delta((p_1-k)^2)\delta((p_2+k)^2)$$

<u>Sudakov-variables</u> α, β:

$$k = (\alpha p_1^+, -\beta p_2^-, \underset{\sim}{k})$$

$$\int d^4k = \underbrace{p_1^+ p_2^-}_{s/2} \int d\alpha\, d\beta\, d^2\underset{\sim}{k} = \frac{s}{2}\int d\alpha\, d\beta\, d^2\underset{\sim}{k}$$

$$\left.\begin{aligned} (p_1-k)^2 &= (1-\alpha)\beta s - \underset{\sim}{k}^2 = 0 \\ (p_2+k)^2 &= \alpha(1-\beta)s - \underset{\sim}{k}^2 = 0 \end{aligned}\right\} \quad \beta \approx \alpha \approx \frac{\underset{\sim}{k}^2}{s} \quad \text{small!}$$

$$\delta((p_1-k)^2) \approx \frac{1}{s}\delta\left(\beta - \frac{\underset{\sim}{k}^2}{s}\right) \quad \left(= \delta(\beta s - \underset{\sim}{k}^2)\right)$$

$$\delta((p_2+k)^2) \approx \frac{1}{s}\delta\left(\alpha - \frac{\underset{\sim}{k}^2}{s}\right)$$

$$\Rightarrow \int d^4\Omega_2 = \frac{1}{2s}\int \frac{d^2\underset{\sim}{k}}{(2\pi)^2}$$

This is very typical for the high energy limit: everything longitudinal ($\pm$) is scaled out into the appropriate power of s, and what remains are only transverse integrals

Now we put the pieces together:

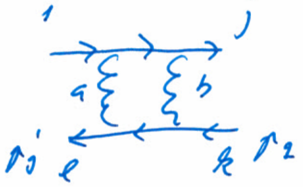
$$\operatorname{Im} A_A^{(1)} = \frac{1}{4s}\int \frac{d^2 k}{(2\pi)^2} (t^b t^a)_{ji} (t^b t^a)_{lk} \underbrace{\frac{+2g^2 s}{-k^2}}_{A^{(0)}(k)} \underbrace{\frac{2g^2 s}{-(k-q)^2}}_{(A^{(0)}(k-q))^\dagger}$$

$$= 16\pi^2 \alpha_s^2 (t^b t^a)_{ji} (t^b t^a)_{lk} \frac{1}{N_c \alpha_s} \frac{s}{t} \underbrace{N_c \alpha_s \int \frac{d^2 k}{(2\pi)^2} \frac{-q^2}{k^2 (k-q)^2}}_{\equiv\, \varepsilon(t) \,\sim\, \ln q^2}$$

$$\frac{s}{t} = \operatorname{Im}\left(-\frac{1}{\pi}\frac{s}{t}\ln\frac{s}{t}\right)$$

Note: $t<0$, so this has an imaginary part. This can be derived using dispersion relations (exercise)

Note 2: Now the imaginary part is $\sim s$, and we deduce that the real part is $\sim s\ln s$. This $\ln s$ cannot come from any other diagram.

$A_B^{(1)} =$ 

same, except $p_2 \leftrightarrow p_2'$ i.e. $s \leftrightarrow u \simeq -s$, $l \leftrightarrow k$, $t^b t^a \leftrightarrow t^a t^b$

$$\Rightarrow A_B^{(1)} = \frac{16\pi\alpha_s}{N_c} (t^a t^b)_{ji} (t^a t^b)_{lk} \frac{s}{t} \ln\frac{s}{|t|}\, \varepsilon(t)$$

Collect the real and imaginary parts separately from A+B:

$$\mathrm{Im}\, A^{(1)} = \frac{16\pi^2\alpha_s}{N_c}(t^b t^a)_{ji}(t^b t^a)_{ek}\,\frac{s}{t}\,\varepsilon(t)$$

$$\mathrm{Re}\, A^{(1)} = -\frac{16\pi^2\alpha_s}{N_c}(t^b t^a)_{ji}[t^b, t^a]_{ek}\,\frac{s}{t}\ln\frac{s}{|t|}\quad \varepsilon(t)$$

Color projections   details as exercise

- singlet $\sim \delta_{ij}\delta_{ke}$ incoming and outgoing q have same color
- octet $\sim t^a_{ji}t^a_{ek}$ color structure as 1-gluon exchange

$$\boxed{\begin{aligned}\mathrm{Re}A_s^{(1)} &= 0\\ \mathrm{Im}A_s^{(1)} &\sim s\,\varepsilon(t)\end{aligned}}$$

Interpretation as QCD pomeron $(1+\varepsilon(t)=\alpha(t))$ see page 3-9

$$A \approx \left(1+e^{-i\pi(1+\varepsilon(t))}\right)s^{1+\varepsilon(t)}$$

$\eta_P = 1 \leftarrow$ just like pomeron in data

- Pomeron amplitude at leading order is imaginary; i.e. scattering (elastic) is purely absorptive
-

$$\mathrm{Re}\, A^{(1)}_{octet} \sim s\ln s \gg \mathrm{Im} \sim s$$

$$A^{(0)}_{octet} + A^{(1)}_{octet} = t^a_{ij}t^a_{ke}\,8\pi\alpha_s\,\underbrace{\frac{s}{t}\left(1+\varepsilon(t)\ln\frac{s}{|t|}\right)}$$

$$\approx s^{1+\varepsilon(t)}(-1)\Big(\underbrace{-1+e^{-i\pi(1+\varepsilon(t))}}_{-2}\Big)$$

This is interpreted as an effective particle/mediator called the reggeized gluon, $\eta = -1$ (see p. 3-9)

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Lipatov vertex

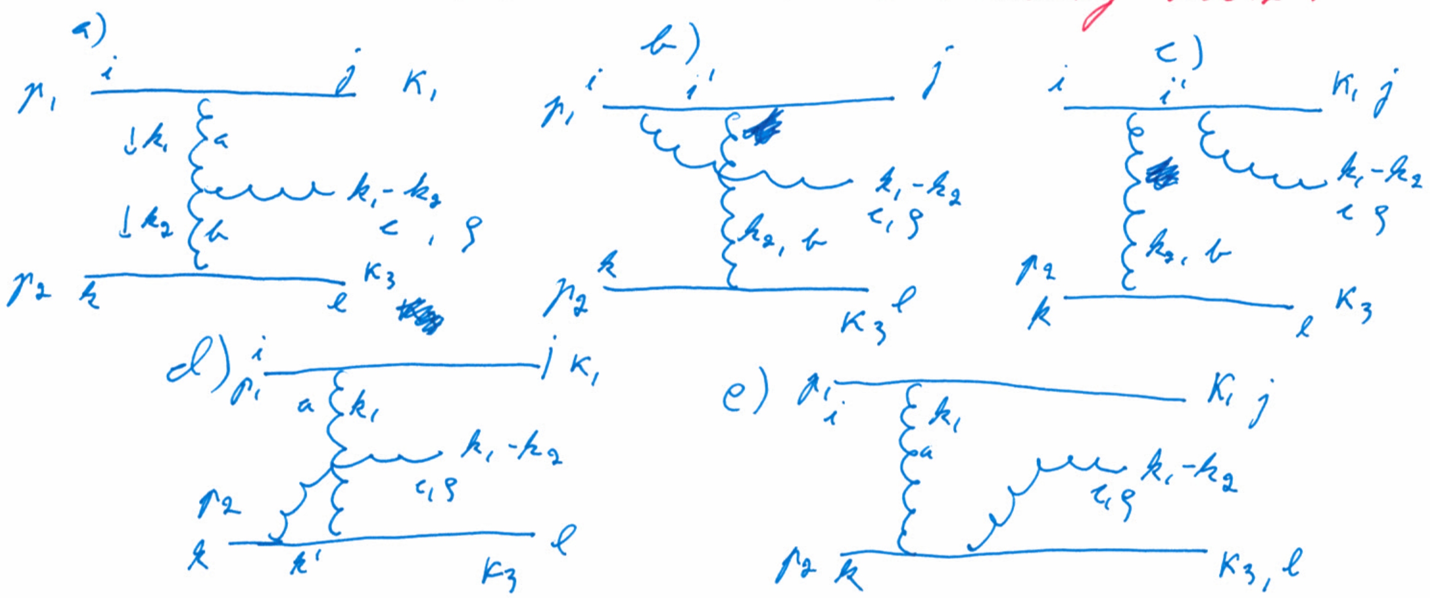
What about the next order in α_s? This should give us more insight into whether a pomeron / reggeized gluon structure is true at a higher order.

We will still want to calculate the next order corrections to leading order in s and using dispersive techniques. I.e. we will cut diagrams to obtain the imaginary part of the diagram that corresponds to physical on-shell particles in an intermediate state.

These cut diagrams are divided into 2 kinds of contributions:

- "real": cut $2q + g$ ↗ N.B. Not 'real' as in Re / Im, but 'real' as in real vs. virtual = unreal
- "virtual": cut $2q$ →

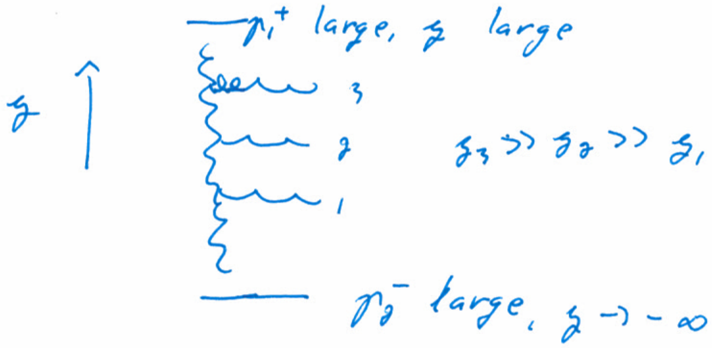
We concentrate first on the real contribution. We need the diagrams that produce $qq \to qqg$. There are 5 that contribute at leading $\ln s$:



Sudakov:

$$k_{1,2} = \left(\alpha_{1,2}\, p_1^+ ,\ \beta_{1,2}\, p_2^- ,\ \underset{\sim}{k}_{1,2}\right)$$

The leading contribution in the large s limit comes from the <u>multi-Regge</u> kinematical regime, where the rapidities of the gluons are strongly ordered.



= multi-Regge kinematics

$$\Rightarrow \quad p_1^+ \gg k_1^+ \gg k_2^+ \gg p_2^+ \quad (=0)$$

$$p_2^- \gg k_2^- \gg k_1^- \gg p_1^- \quad (=0)$$

$$\Rightarrow \quad \alpha_2 \ll \alpha_1 \ll 1, \qquad \beta_1 \ll \beta_2 \ll 1$$

Why is multi-Regge the dominant regime? Because all the "t-channel" propagators only have transverse momenta and are thus not suppressed by large momentum components $\sim \sqrt{s}$. In MRK all transverse momenta are assumed to be of the same order.

A practical result of this:

Produced gluon:

$$0 = (k_1-k_2)^2 = 2(\alpha_1-\alpha_2)(\beta_1-\beta_2)\,p_1^+p_2^- - (\underset{\sim}{k}_1-\underset{\sim}{k}_2)^2$$

$$\alpha_1\beta_2\, s \approx (\underset{\sim}{k}_1-\underset{\sim}{k}_2)^2$$

$$k_1^2 = \alpha_1\beta_1 s - \underset{\sim}{k}_1^2 = \underbrace{\frac{\beta_1}{\beta_2}}_{\ll 1}\underbrace{(\underset{\sim}{k}_1-\underset{\sim}{k}_2)^2}_{\text{same order of magnitude as } \underset{\sim}{k}_1} - \underset{\sim}{k}_1^2 \approx -\underset{\sim}{k}_1^2$$

$k_2^2 \approx -\underset{\sim}{k}_2^2 \to$ Regions of large α, β will give much larger $k_{1,2}^2$, which will suppress amplitude.

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$$A_a = (-i)(-2igp_{1\mu})\, t^a_{ji} \left(\frac{-i}{k_1^2}\right)(-f_{abc}) \times$$

$$\left[(k_1+k_2)^\rho g^{\mu\nu} + \underbrace{(k_1 - 2k_2)^\mu}_{} g^{\nu\rho} + \underbrace{(k_2 - 2k_1)^\nu}_{} g^{\mu\rho} \right] \times$$

$$p_1^+ (k_1^- - 2k_2^-) \approx -2k_2^- p_1^+ \qquad p_2^-(k_2^+ - 2k_1^+) \simeq -2p_2^- k_1^+$$

$$\left(\frac{-i}{k_2^2}\right)(-2igp_{2\nu})\, t^b_{lk}$$

$$\approx \frac{+4ig^2 f_{abc}}{\underset{\sim}{k}_1^2\, \underset{\sim}{k}_2^2}\, p_{1\mu} p_{2\nu} \Big[(k_1+k_2)^\rho g^{\mu\nu} - \underbrace{2k_2^\mu g^{\nu\rho}}_{(1)} - \underbrace{2k_1^\nu g^{\mu\rho}}_{(2)} \Big]\, t^a_{ji} t^b_{lk}$$

(1) : $p_{1\mu} k_2^\mu = p_1^+ k_2^- = \beta_2 p_1^+ p_2^- = \beta_2 p_{1\mu} p_{2\nu} g^{\mu\nu}$

$p_{2\nu} g^{\nu\rho} = p_2^\rho$

(2) : $p_{2\nu} k_1^\nu = p_2^- k_1^+ = \alpha_1 p_2^- p_1^+ = \alpha_1 p_{1\mu} p_{2\nu} g^{\mu\nu}$

$p_{1\mu} g^{\mu\rho} = p_1^\rho$

$$= \frac{4ig^2 f_{abc}\, t^a_{ji} t^b_{lk}}{\underset{\sim}{k}_1^2\, \underset{\sim}{k}_2^2}\, p_{1\mu} p_{2\nu} g^{\mu\nu} \left[(k_1+k_2)^\rho - 2\beta_2 p_2^\rho - 2\alpha_1 p_1^\rho \right]$$

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$\rightarrow$ |

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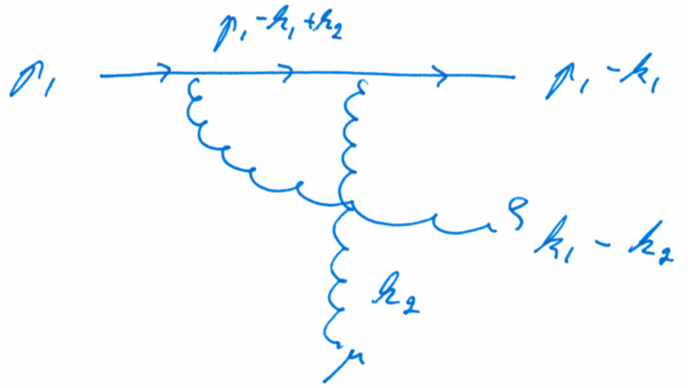
For diagram b, c, d and e we need a kind of a "double eikonal vertex" for the radiation of an additional gluon off the high energy quark line:

Suppressing colour indices:

$V_2 =$



$\not{p}\not{p} = p^2$, so

$\not{p}\,\frac{\not{p}}{p^2} = 1$ and

$\frac{1}{\not{p}} = \frac{\not{p}}{p^2}$

$$\underbrace{\bar{u}_\sigma(p_1 - k_1)}_{\approx \bar{u}_\sigma(p_1)} \gamma^\mu \underbrace{\frac{i}{\not{p}_1 - \not{k}_1 + \not{k}_2}}\, \gamma^\rho u_{\sigma'}(p_1)$$

$$= \frac{i(\not{p}_1 - \not{k}_1 + \not{k}_2)}{(p_1 - k_1 + k_2)^2} \approx \frac{i\not{p}_1}{\underbrace{(p_1 - k_1 + k_2)^2}}$$

skalaaripropagaattori

$$\not{p}_1 = \sum_\lambda u_\lambda(p_1)\bar{u}_\lambda(p_1)$$

$$(p_1 - k_1 + k_2)^2 = \underbrace{p_1^2}_{=0} + \underbrace{(k_1 - k_2)^2}_{=0} - 2p_1\cdot(k_2 - k_1)$$

$$= -2p_1^+(\underbrace{k_2^-}_{\beta_2 p_2^-} - \underbrace{k_1^-}_{\text{small}}) \approx -s\beta_2$$

$$V_2 = \frac{-ig^2}{\beta_2 s}\sum_\lambda \underbrace{\bar{u}_\sigma(p_1)\gamma^\mu u_\lambda(p_1)}^{2p_1^\mu \delta_{\sigma\lambda}}\ \underbrace{\bar{u}_\lambda(p_1)\gamma^\rho u_\sigma(p_1)}^{2p_1^\rho \delta_{\lambda\sigma'}}$$

$$= \frac{ig^2}{\beta_2 s}(-2p_1^\mu i)(-2p_1^\rho i)\,\delta_{\sigma\sigma'}$$

kuin 2 eik. verteksiä ja skalaaripropagaattori välissä

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$$A_b = (-i)(-2p_1^{\mu} ig)\frac{i}{\beta_2 s}(-2p_1^{\rho} ig)\, t^b_{ji'} t^c_{i'i}\, (-2p_2^{\nu} ig)\frac{-ig_{\mu\nu}}{k_2^2} t^b_{ek}$$

$$A_c = (-i)(-2p_1^{\rho} ig)\frac{i}{\beta_2 s}(-2p_1^{\mu} ig)\, t^c_{ji'} t^b_{i'i}\, (-2p_2^{\nu} ig)\frac{-ig_{\mu\nu}}{k_2^2} t^b_{ek}$$

$$A_b + A_c = \frac{8p_1^{\rho} p_1^{\mu} p_2^{\nu}}{\beta_2 s\,(-k_2^2)}\underbrace{[t^b, t^c]_{ji}}_{= if^{abc} t^a_{ji}}\, t^b_{ek}\, g_{\mu\nu}$$

(In the same way)

$$A_d + A_e = -i\frac{8p_2^{\rho} p_1^{\mu} p_2^{\nu}}{\alpha_1 s\, k_1^2} g_{\mu\nu} f^{abc} t^a_{ji} t^b_{ek}$$

Putting all together:

$$A_a + A_b + A_c + A_d + A_e$$

$$= -i\frac{\overbrace{(-2ig\,p_{1\mu})}^{\text{eik.}}\overbrace{(-2ig\,p_{2\nu})}^{\text{eik. vertex}}}{\underbrace{k_1^2\; k_2^2}_{\text{propagators}}} g^{\mu\nu}(-1)\Big[(k_1+k_2)^{\rho} - 2\beta_2 p_2^{\rho} \overbrace{-2\alpha_1 p_1^{\rho}}^{-2k_1^+}$$

$$- \frac{2k_1^2}{\beta_2 s} p_1^{\rho} - \frac{2k_2^2}{\alpha_1 s} p_2^{\rho}\Big]\underbrace{(-f^{abc})\, t^a_{ji}\, t^b_{ek}}_{\text{color factors}}$$

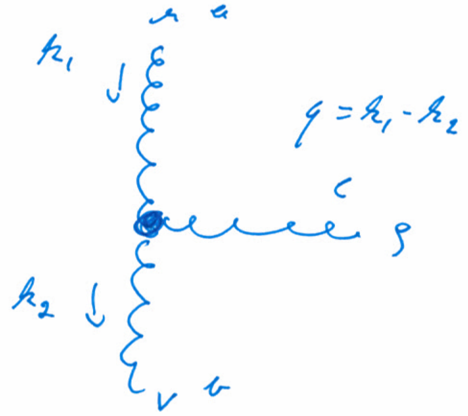
Just like , except 3-gluon vertex replaced by a more complicated structure; the Lipatov vertex.

Finally we arrive at an *effective vertex* that sums up the contribution of gluon radiation off the external (quark) legs of the diagram:

$$C^+ = \underbrace{-k_1^+ - k_2^+}_{\ll k_1^+} + \underbrace{2\alpha_1 p_1^+}_{2k_1^+} + \frac{\cancel{2}\,\mathbf{k}_1^2}{\cancel{2}\cancel{p_1^+}\underbrace{p_2^-\beta_2}_{=k_2^-}}\cancel{p_1^+} = k_1^+ + \frac{\mathbf{k}_1^2}{k_2^-}$$

$$C^- = k_2^- + \frac{\mathbf{k}_2^2}{k_1^+}$$

$$\mathbf{C} = -(\mathbf{k}_1 + \mathbf{k}_2)$$



$$-f_{abc}\, g^{\mu\nu}\, C^\rho(k_1, k_2)$$

This is the <u>Lipatov vertex</u>. It is nonlocal due to the $\frac{1}{k_1^+}$, $\frac{1}{k_2^-}$ in the expression, which in position space correspond to inverse derivatives $\frac{1}{\partial_+}$, $\frac{1}{\partial_-}$.

Some properties:

- $q \cdot C = 0 \quad \to$ "gauge invariant"
- $C^2 = -4\,\frac{\mathbf{k}_1^2 \mathbf{k}_2^2}{q^2}$

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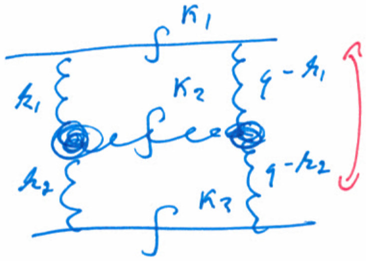
3-particle phase space



$\Delta y, \quad s \sim e^{\Delta y}$

$$\int d\Omega_3 = \frac{(2\pi)^3}{(2\pi)^8}\int d^4k_1\, d^4k_2\, \delta((p_1-k_1)^2)\,\delta((p_2+k_2)^2)\,\delta((k_1-k_2)^2)$$

$$= \frac{s^2}{4(2\pi)^5}\int d^2\underset{\sim}{k}_1\, d^2\underset{\sim}{k}_2\, d\alpha_1\, d\alpha_2\, d\beta_1\, d\beta_2 \;\times$$

$$\delta(-\underline{\underline{\beta_1}}\,\underbrace{(1-\alpha_1)}_{\approx 1}s - \underset{\sim}{k}_1^2)\,\delta(\underline{\underline{\alpha_2}}\,\underbrace{(1+\beta_2)}_{\approx 1}s - \underset{\sim}{k}_2^2)\;\times$$

$$\delta(\underbrace{(\alpha_1-\alpha_2)(\beta_1-\beta_2)}_{-\alpha_1\beta_2}s - (\underset{\sim}{k}_1-\underset{\sim}{k}_2)^2)$$

$\rightarrow \int d\beta_2 : \beta_2 = -\frac{(\underset{\sim}{k}_1-\underset{\sim}{k}_2)^2}{\alpha_1 s} > -1$

$\Rightarrow \alpha_1 > \frac{(\underset{\sim}{k}_1-\underset{\sim}{k}_2)^2}{s}$

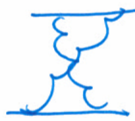
$$= \frac{1}{8\pi s}\int \frac{d^2\underset{\sim}{k}_1}{(2\pi)^2}\frac{d^2\underset{\sim}{k}_2}{(2\pi)^2}\underbrace{\int_{\frac{(\underset{\sim}{k}_1-\underset{\sim}{k}_2)^2}{s}}^{1}\frac{d\alpha_1}{\alpha_1}}_{\boxed{\sim \ln s}}$$

$\rightarrow$ this integral gives a "large logarithm" of s already in the Im part of the amplitude. Remember that Im(II) $\sim s$, no log. Also recall that $C^\mu(k_1,k_2)\,C^\mu(q-k_1,q-k_2)$ is transverse $\rightarrow$ does not affect this. Thus we expect $\ln^2 s$ in the real part.

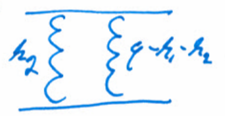
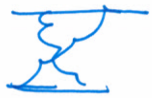
This large logarithm comes from the fact that there is a large rapidity interval $\Delta y \sim \ln s$ and the extra gluon K_2 can be emitted (with a constant probability per unit rapidity) anywhere in this interval.

Virtual correction

We have now calculated (although we did not put together all the pieces) the real $(qq \to qqg)$ contribution, expressed in terms of the Lipatov effective vertex. We will now discuss the other part; the virtual $(qq \to qq)$ correction.

$2\,\mathrm{Im}\,A^{(2)}_{virt} = \Big($  $+$  $\Big) \times \Big($  $\Big)^{\dagger}$

$\qquad\qquad A^{(1)}(k_2) \qquad\qquad\qquad (A^{(0)}(q-k_2))^{\dagger}$

$+ \Big($  $\Big) \times \Big($  $+$  $\Big)^{\dagger}$

$\qquad\qquad A^{(0)}(k_1) \qquad\qquad\qquad (A^{(1)}(q-k_1))^{\dagger}$

$$\mathrm{Im}\,A^{(2)}_{virt} = \frac{1}{2}\int d\Omega_2 \left[A^{(1)}(k_2)\left(A^{(0)}(q-k_2)\right)^{\dagger} + A^{(0)}(k_1)\left(A^{(1)}(q-k_1)\right)^{\dagger} \right]$$

$$A^{(0)}(q-k_2) = 8\pi\alpha_s \frac{s}{(q-k_2)^2}\, t^a_{ji}\, t^a_{lk}$$

y, rapidity

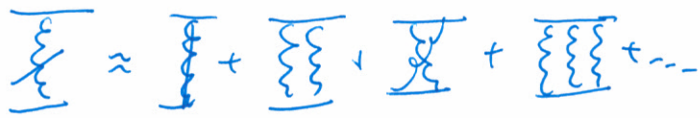
$$A^{(1)}(k_2) = 8\pi\alpha_s \frac{s}{k_2^2} \ln\frac{s}{k_2^2}\, \underbrace{\varepsilon(k_2)}$$

$$= N_c \alpha_s \int \frac{d^2 \underset{\sim}{k}_1}{(2\pi)^2} \frac{-\underset{\sim}{k}_2^2}{\underset{\sim}{k}_1^2(\underset{\sim}{k}_2-\underset{\sim}{k}_1)^2}$$

These are equivalent (to this order in α_s) to the exchange of a reggeized gluon

$$\frac{-i g_{\mu\nu}}{k^2} e^{\Delta y\, \varepsilon(k^2)}$$

reggeized gluon propagator



Total cross section to α_s^3

We now have all the ingredients to compute the total cross section, both for the singlet and octet channels. We will not go through the considerable amount of algebra needed to put together all the pieces here; see Barone & Predazzi chap. 8 for details.

We just quote the result:

octet:

$$\mathrm{Im}\,A_{\underline{8}}^{(2)} = \frac{N_c^2 \alpha_s^3}{2\pi^3} t^a_{ji} t^a_{lk}\, s \ln\frac{s}{|t|} \times$$

$$\int d^2\underset{\sim}{k}_1\, d^2\underset{\sim}{k}_2 \frac{\underset{\sim}{q}^2}{\underset{\sim}{k}_1^2 \underset{\sim}{k}_2^2 (\underset{\sim}{k}_1-\underset{\sim}{q})^2(\underset{\sim}{k}_2-\underset{\sim}{q})^2}$$

(0) + (1) + (2)

$$\mathrm{Im}\,A_{\underset{\sim}{8}} = 8\pi\alpha_s t^a_{ji} t^a_{lk}\left(\frac{s}{t}\right)\left[1 + \varepsilon(t)\ln\frac{s}{|t|} + \frac{1}{2}\varepsilon^2(t)\ln^2\frac{s}{|t|}\right]$$

$$\varepsilon(t) = N_c \alpha_s \int \frac{d^2k}{(2\pi)^2}\frac{-\underset{\sim}{q}^2}{\underset{\sim}{k}^2(\underset{\sim}{q}-\underset{\sim}{k})^2}$$

singlet:

$$\mathrm{Im}\,A_{\underline{1}}^{(2)} = -i\frac{2\alpha_s^3}{\pi^2}\frac{N_c^2-1}{4}\delta_{ij}\delta_{kl}\, s\ln\frac{s}{|t|}\int d^2\underset{\sim}{k}_1\, d^2\underset{\sim}{k}_2 \times$$

$$\left[\frac{\underset{\sim}{q}^2}{\underset{\sim}{k}_1^2\underset{\sim}{k}_2^2(\underset{\sim}{k}_1-\underset{\sim}{q})(\underset{\sim}{k}_2-\underset{\sim}{q})^2} - \frac{1}{2}\frac{1}{\underset{\sim}{k}_2^2(\underset{\sim}{k}_1-\underset{\sim}{q})^2(\underset{\sim}{k}_1-\underset{\sim}{k}_2)^2} - \frac{1}{2}\frac{1}{\underset{\sim}{k}_1^2(\underset{\sim}{k}_2-\underset{\sim}{q})^2(\underset{\sim}{k}_1-\underset{\sim}{k}_2)^2}\right]$$

These similar terms are there also in the octet calculation, but they cancel between the real and virtual terms. In the singlet they do not, because of the different color factors.

4.2.-11

BFKL ladder

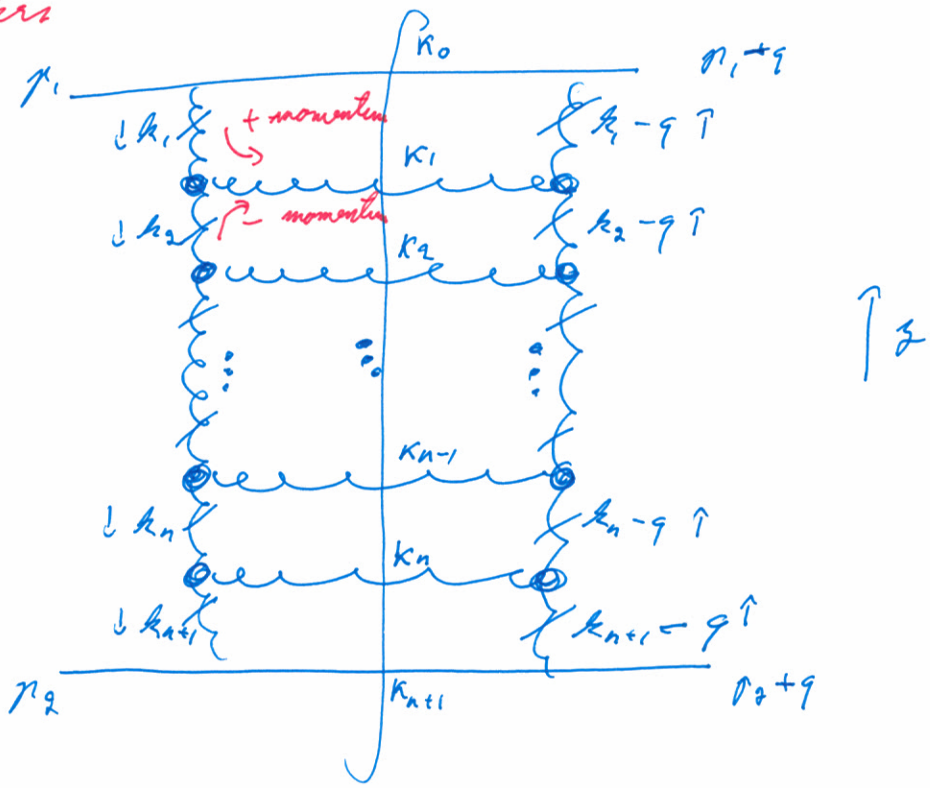
Balitsky, Fadin, Kuraev, Lipatov

It is possible to resum all the dominant diagrams at leading $\ln s$.

Power counting: $\alpha_s << 1$, $\alpha_s \ln s \sim 1$

At each new order in α_s only include terms that come with an additional $\ln s$ to compensate it. Result: the dominant contribution is from ladders

$\mathrm{Im}\, A_{qq \to qq} = \sum_{n=0}^{\infty}$




Lipator vertex


reggeized propagator

MRK:

$\alpha_1 >> \alpha_2 >> \dots >> \alpha_{n+1}$

$\beta_1 << \beta_2 << \dots << \beta_{n+1}$

Sudakov: $k_i = (\alpha_i p_1^+, \beta_i p_2^-, \underset{\sim}{k}_i)$

Momentum flow:

$k_i^+ \approx k_i^+ \sim e^{+y_i}$

$k_i^- \approx k_{i+1}^- \sim e^{-y_i}$

$\Rightarrow y > y_1 > y_2 > \dots > y_n > 0$

rapidity ordered cascade

compare: DGLAP is transverse momentum ordered, dominant when $-q^2 = -t$ is large.

The phase space of the BFKL ladder is that of n+2 particles. This sounds complicated, but we can simplify it a lot in MRK.

$$d\Omega_{n+2} = \prod_{i=0}^{n+1}\left[\frac{d^4 K_i}{(2\pi)^4} 2\pi\delta(K_i^2)\right](2\pi)^4\delta^4(p_1+p_2-\sum_i K_i)$$

rapidity

- $\int dK_i^- dK_i^+ \delta(2K_i^- K_i^+ - \underset{\sim}{K}^2) = \int \frac{dK_i^+}{2K_i^+} = \frac{1}{2}\int dy_i$
- $\delta(p_1^+ - \sum_i K_i^+)\,\delta(p_2^- - \sum_i K_i^-) \approx \delta(p_1^+ - K_0^+)\,\delta(p_2^- - K_{n+1}^-)$

(remember MRK!)

- $\underbrace{\prod_{i=0}^{n+1} \frac{d^2 \underset{\sim}{k}_i}{(2\pi)^2}}_{n+2}\underbrace{(2\pi)^2\delta^2(\sum_{i=0}^{n+1}\underset{\sim}{k}_i)}_{-1} = \underbrace{\prod_{i=1}^{n+1}\frac{d^2\underset{\sim}{k}_i}{(2\pi)^2}}_{n+1}$

$$d\Omega_{n+2} = (2\pi)^2\prod_{i=1}^{n+1}\frac{d^2\underset{\sim}{k}_i}{(2\pi)^2}\;\underbrace{\prod_{i=1}^{n}\frac{1}{4\pi}\int dy_i}_{\sim(\ln s)^n}$$

$\sim (\ln s)^n$ → every rung brings $\alpha_s \ln s$

MRK limits

$$A(y) \sim \int_0^y dy_1 \int_0^{y_1} dy_2 \cdots \int_0^{y_{n-1}} dy_n\, e^{(y-y_1)(\varepsilon(k_1)+\varepsilon(q-k_1))} \times$$

$$\cdots e^{(y_n-0)(\varepsilon(k_{n+1})+\varepsilon(q-k_{n+1}))}$$

Deconvolute with Laplace $\int_0^\infty dy\, e^{-\omega y} A(y)$

Laplace in y = Mellin transform in s

4.2.2011

Recall that the expression for the amplitude (Im part) was a sum over $n = 0 \dots \infty$ of n-rung ladders, each with a $2(n+1)$-dimensional transverse momentum integral.

After the Laplace-transform this can be expressed by a <u>recursive</u> formula:

$R =$ singlet / octet

We define the intermediate quantity

$$F_R(\omega;\underline{k}_1;\underline{q}) = \frac{1 - C_R\int d^2\underline{k}_2 \,\frac{K(\underline{k}_1;\underline{k}_2)}{\underline{k}_2^2(\underline{q}-\underline{k}_2)^2}\, F_R(\omega;\underline{k}_2;\underline{q})}{\omega - \underbrace{\varepsilon(-\underline{k}_1^2) - \varepsilon(-(\underline{k}_1-\underline{q})^2)}_{-\varepsilon_1}}$$

<u>BFKL</u> This is the general form with different coefficients for different channels R.

$$\mathrm{Im}\, A_R(\omega,\underline{q}) = G_R \int d^2\underline{k}_1 \,\frac{F_R(\omega;\underline{k}_1;\underline{q})}{\underline{k}_1^2(\underline{q}-\underline{k}_1)^2}$$

$K_{12} =$ 

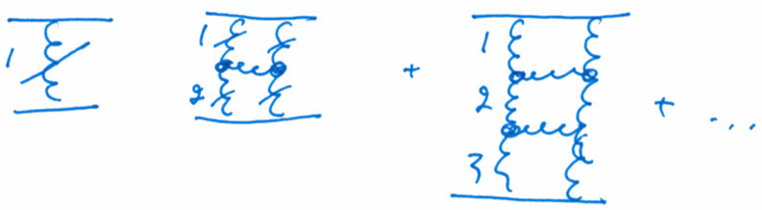
$C_R,\ G_R \sim \alpha_s$

Def. $$\int_1 \equiv \int \frac{d^2\underline{k}_1}{\underline{k}_1^2(\underline{q}-\underline{k}_1)^2} \quad \rightarrow \quad F_1 = \frac{1 - \int_2 K_{12} F_2}{\omega - \varepsilon_1}$$

$$\mathrm{Im}\, A \sim \int_1 F_1 = \int_1 \frac{1 - \int_2 K_{12} F_2}{\omega - \varepsilon_1} = \int_1 \frac{1}{\omega-\varepsilon_1} - \int_{12} \frac{K_{12}}{\omega-\varepsilon_1} F_2$$

$$= \int_1 \frac{1}{\omega-\varepsilon_1} - \int_{12} \frac{K_{12}}{(\omega-\varepsilon_1)(\omega-\varepsilon_2)} \left(1 - \int_3 K_{23} F_3\right) + \dots$$

$F_3 = \frac{1}{\omega - \varepsilon_3} - \dots(\dots)$



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Let us try to give some physical interpretation to this equation.

$$\omega \mathcal{F}_1 = 1 - \int_2 K_{12} \mathcal{F}_2 + \Sigma_1 \mathcal{F}_1 \qquad \omega \leftarrow \text{Laplace}$$

$\omega \uparrow \frac{\partial}{\partial y}$; $1 \uparrow$ init. cond @ $y=0$; $\int_2 K_{12}\mathcal{F}_2 \nwarrow$ real; $\Sigma_1 \mathcal{F}_1 \nwarrow$ virtual

The BFKL equation is often encountered as a differential - integral evolution equation in rapidity

$$\frac{d}{dy} \mathcal{F}_y(\underset{\sim}{k}) = \int_{\underset{\sim}{k}} \left[K_{\text{real}}(\underset{\sim}{k},\underset{\sim}{k}') \mathcal{F}_y(\underset{\sim}{k}') + K_{\text{virt}}(\underset{\sim}{k},\underset{\sim}{k}') \mathcal{F}_y(\underset{\sim}{k}) \right]$$

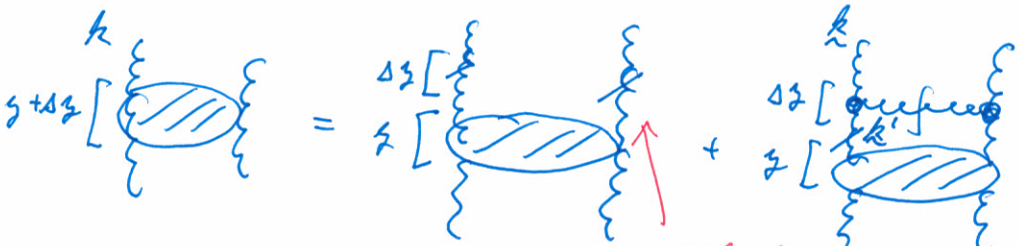
not $\underset{\sim}{k}'$

(Suppressing q, color indices)

This corresponds to the change in the scattering amplitude as a gluon distribution inside a hadron when the rapidity / energy is increased by Δy.

$\mathcal{F}_y(\underset{\sim}{k}) =$ 

$$\mathcal{F}_{y+\Delta y}(\underset{\sim}{k}) = \mathcal{F}_y(\underset{\sim}{k}) e^{\Delta y\, \mathcal{E}(\underset{\sim}{k})} + \Delta y \int_{\underset{\sim}{k}'} K(\underset{\sim}{k},\underset{\sim}{k}') \mathcal{F}_y(\underset{\sim}{k}')$$



$K \sim C^2$, square of Lipatov vertex

virtual correction / reggeized gluon is a Sudakov factor, makes sure probability is conserved.

 4.2.2011

Solutions

- octet: easy → confirm reggeized gluon

$$A_8 = -2\pi\alpha_s\left[1 - e^{-i\pi(1+\varepsilon(t))}\right]\left(\frac{s}{|t|}\right)^{1+\varepsilon(t)}$$

- singlet ("QCD pomeron"). Equation more complicated.
  - eigenfunctions of kernel needed in order to solve it. These can be found, but there is a considerable calculation to do it; takes a few pages in textbooks.
  - result: $A_1 \sim s^{1+\frac{N_c\alpha_s}{\pi}4\ln 2} \approx s^{1+0.5}$

    - analytical structure not quite Regge (Instead of a pole in the l-plane there is a branch cut.) Thus the exact form is not quite as simple as the Regge parametrisation.

$0.5 > 0.08$

Phenomenologically this exercise is a partial success.

+ made connection to successful Regge parametrization
- energy dependence is much faster than $s^{1.08}$ seen in data.
  → weak coupling QCD doesn't quite describe total cross section ($t=0$!)

L9, 14.2.-11 →|

+ will work better at higher $|t|$ and be basis for further development.