DETERMINATION OF AN ANISOTROPIC PERTURBATION IN ELASTIC INVERSE SCATTERING

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Abstract. We consider a linearized inverse scattering problem for elastic waves. We prove that a fully anisotropic perturbation of the elastic parameters around an isotropic and homogeneous reference can be uniquely determined by (single-)scattered waves. We also give a quantitative stability estimate for an isotropic perturbation, and as a consequence a rigidity result is established.

1. Introduction

The elastic wave equation describes the propagation of mechanical disturbances in solid media, forming a fundamental model in seismology, geophysics, and material science [18]. The equation reads

$$\rho\partial_t^2 u_i - \partial_j(c_{ijk\ell}\partial_k u_\ell) = 0 \quad \text{in } \mathbb{R}^3\times\mathbb{R}, \tag{1.1}$$

where ρ is the density and $\mathbf{C} = (c_{ijk\ell}(x))$ is the elastic tensor satisfying the following symmetry properties

$$c_{ijk\ell} = c_{jik\ell} = c_{k\ell ij}.$$

Assume that there exists a $c_0 > 0$ such that $\rho(x) > c_0$ and

$$\sum_{i,j,k,\ell=1}^{3} c_{ijk\ell}(x)\varepsilon_{ij}\varepsilon_{k\ell} \geq c_0 \sum_{i,j=1}^{3} \varepsilon_{ij}^2 \tag{1.2}$$

for any nonzero 3×3 real-valued symmetric matrix (ε_{ij}).

Further we assume that $\mathbf{C}, \rho \in C^\infty(\mathbb{R}^3)$ and there are three constants λ_0, μ_0, ρ_0 such that quantities $C_{ijk\ell} - \lambda_0\delta_{ij}\delta_{k\ell} - \mu_0(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$ and $\rho - \rho_0$ are compactly supported in $\Omega := \{x \in \mathbb{R}^3 : |x| < 1\}$. Throughout the paper we assume $\rho_0 = 1$ without loss of any generality.

Let u_0 be a solution to (1.1) with $\mathbf{C} = \mathbf{C}_0 := (\lambda_0\delta_{ij}\delta_{k\ell} + \mu_0(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}))$ and $\rho = 1$ (the medium is isotropic and homogeneous), that is,

$$\partial_t^2 u_0 - (\lambda_0 + \mu_0)\nabla\nabla\cdot u_0 - \mu_0\Delta u_0 = 0. \tag{1.3}$$

The ellipticity condition (1.2) reduces to

$$\mu_0 > 0, \quad 3\lambda_0 + 2\mu_0 > 0.$$

In this study we only use special solutions u_0 to (1.3). Denote

$$c_p = \sqrt{\lambda_0 + 2\mu_0}, \quad c_s = \sqrt{\mu_0},$$

to be the P-wave and S-wave speeds respectively. Note that $c_p > c_s$. Notice that we can take a plane P-wave solution

$$u_0 = \alpha H_0(c_p t - \theta\cdot x),$$

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where $\alpha = \theta$, or a plane S-wave solution

$$u_0 = \alpha H_0(c_s t - \theta \cdot x),$$

where $\theta \in \mathbb{S}^2$ and $\alpha \perp \theta$. Here H_0 is the Heaviside function. It is easy to verify that they both satisfy equation (1.3). By the standard hyperbolic theory (cf., for example, [8, Chapter 23]), there exists a unique distributional solution $U(x,t;\theta,\alpha)$, where $\alpha = \theta$ or $\alpha \perp \theta$, to the equation

$$\begin{aligned} \rho \partial_t^2 U_i - \partial_j(c_{ijk\ell}\partial_k U_\ell) = 0, \quad & \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ U = u_0, \quad & \text{for } t \ll 0. \end{aligned} \tag{1.4}$$

This model describes the wave scattering caused by the inhomogeneity encoded in $(\mathbf{C}, \rho)$ with incident plane wave u_0 in the isotropic and homogeneous surrounding background.

We consider the inverse scattering problem associated with the above equation. In this setting, u_0 represents the incident wave, and from the total wave field U measured outside Ω we aim to recover the elastic tensor $\mathbf{C}$ and the density ρ. In general, we are interested in the invertibility of the forward map

$$\mathcal{A}\colon (\mathbf{C}, \rho) \mapsto \big\{ U(\cdot,\cdot;\theta,\alpha)|_{\Omega^c \times (-\infty, T]} \big| \theta \in \mathbb{S}^2, \alpha = \theta \text{ or } \alpha \perp \theta \big\}$$

with $T > 0$ sufficiently large, where U satisfies the equation (1.4). When $\mathbf{C}$ is isotropic, inverse boundary value problems have been studied [24, 21, 22, 27, 4, 29], while the fully anisotropic problem is far from reach. When the elastic tensor is piecewise analytic, certain types of anisotropy can be dealt with [5]. While the Boundary Control method, invented by Belishev [1], can be used to study the anisotropic acoustic wave equation [3], it does not work well for the elastic wave equation even for the isotropic case [2]. We also refer to [11, 12, 14, 15] for further development of the Boundary Control method.

While the fully nonlinear problem, that is, the injectivity of $\mathcal{A}$, is widely open, in this article we consider the invertibility of the Frechét derivative of $\mathcal{A}$ at $(\mathbf{C}_0, 1)$, denoted as $\mathcal{A}'[\mathbf{C}_0, 1]$. This problem is sometimes called the Born approximation in physical literature. Denote $\dot{\mathbf{C}} = (\dot{c}_{ijk\ell})$ and $u_0 = (u_{01}, u_{02}, u_{03})$. One can check that $\mathcal{A}'[\mathbf{C}_0, 1](\dot{\mathbf{C}}, \dot{\rho}) = \dot{u}|_{\Omega^c \times (-\infty, T]}$, where $\dot{u}$ is the solution to

$$\begin{aligned} \partial_t^2 \dot{u}_i - (\lambda_0 + \mu_0)\nabla_i \nabla \cdot \dot{u} - \mu_0 \Delta \dot{u}_i = \partial_j(\dot{c}_{ijk\ell}\partial_k u_{0\ell}) - \dot{\rho}\partial_t^2 u_{0i}, \quad & \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ \dot{u} = 0, \quad & \text{for } t \ll 0. \end{aligned}$$

Here by our assumption, $(\dot{\mathbf{C}}, \dot{\rho})$ vanishes outside Ω. The perturbation of the elastic tensor $\dot{c}_{ijk\ell}$ inherits the symmetry conditions

$$\dot{c}_{ijk\ell} = \dot{c}_{jik\ell} = \dot{c}_{k\ell ij}.$$

We study the invertibility of $\mathcal{A}'[\mathbf{C}_0, 1]$ allowing the perturbation $\dot{\mathbf{C}}$ to be fully anisotropic. The main result of this paper is that $(\dot{\mathbf{C}}, \dot{\rho})$ can be fully determined by $\mathcal{A}'[\mathbf{C}_0, 1](\dot{\mathbf{C}}, \dot{\rho})$. We mention here that a similar linearized problem in frequency domain has been considered in [28], where a transversely isotropic perturbation is proved to be unique. In this paper, we only use incident waves with finitely many incoming angles θ and polarization directions α. Let $\mathbf{e}_i$ denote the unit vector parallel to the positive x_i-axis. The main result can be stated as follows.

Theorem 1.1. *If $\mathcal{A}'[\mathbf{C}_0, 1](\dot{\mathbf{C}}, \dot{\rho})(x,t;\mathbf{e}_i,\mathbf{e}_j) = 0$ for $(x,t) \in \Omega^c \times (-\infty, T]$ and $i,j = 1,2,3$, then $(\dot{\mathbf{C}}, \dot{\rho}) = 0$.*

We mention here that the linearized inverse boundary value problem (at an isotropic reference, not necessarily homogeneous) can be reduced to certain tensor tomography problems of fourth order tensors [6]. To be more precise, using P-waves we end up with the longitudinal ray transform, from which one can determine 5 parameters; using S-waves we end up with the mixed ray transform,

from which 9 parameters can be determined. But the most general anisotropic elastic tensor has 21 independent parameters. So our result determines a total number of 22 parameters in $(\dot{\mathbf{C}}, \dot{\rho})$. One surprising fact about this result is there is no gauge freedom in the determination, while for the anisotropic acoustic wave equation or Maxwell's equations certain gauge freedoms exist. We also refer to [10] for a detailed study on the gauge freedoms of elastic parameters in a Riemannian geometric setting (instead of the Euclidean setting considered in this article). In this study, we will adopt the strategy recently used for fixed angle inverse scattering problems [25, 23, 16, 17, 20, 19], which is intrinsically different from the geometrical approaches.

For the case where $\dot{\mathbf{C}} = \dot{\lambda}\delta_{ij}\delta_{k\ell} + \dot{\mu}(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$ is isotropic, we will also establish a Lipschitz stability estimate for the inverse of $\mathcal{A}'[\mathbf{C}_0, 1]$, which would give a rigidity result of an isotropic perturbation using Stefanov-Uhlmann's argument [26]. This scheme has been established for an acoustic wave equation in [16]. Now we assume $\mathbf{C} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$ where $\lambda - \lambda_0, \mu - \mu_0, \rho - 1 \in C_c^\infty(\Omega)$. Assume that

$$\rho > 0, \quad \mu > 0, \quad 3\lambda + 2\mu > 0.$$

Then the elastic wave equation (1.1) reduces to

$$\rho\partial_t^2 u - \nabla \cdot (\lambda(\nabla \cdot u)I + (\nabla u + \nabla^T u)) = 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R}.$$

The equation (1.4) satisfied by U can be written as

$$\begin{aligned} \rho\partial_t^2 U - \nabla \cdot (\lambda(\nabla \cdot U)I + (\nabla U + \nabla^T U)) = 0, &\qquad \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ U = u_0, &\quad \text{for } t \ll 0. \end{aligned}$$

We write the operator $\mathcal{A}$ as

$$\mathcal{A}\colon (\lambda, \mu, \rho) \mapsto \big\{U(\cdot, \cdot; \theta, \alpha)|_{\Omega^c \times (-\infty, T]} \big| \theta \in \mathbb{S}^2, \alpha = \theta \text{ or } \alpha \perp \theta\big\}.$$

Here we only need to use one incident angle θ with two polarization directions: θ for the P-wave and $\alpha \perp \theta$ for the S-wave. The main result can be summarized in the following theorem.

Theorem 1.2. *Fix $\theta, \alpha \in \mathbb{S}^2$ with $\alpha \perp \theta$. There exist constants $C > 0$, $s_0 > 3/2$, $k \geq 0$ being an integer, $s \in (-s_0, s_0)$, $\gamma \in (0, 1)$ such that for (λ, μ, ρ) sufficiently close to $(\lambda_0, \mu_0, 1)$, there holds*

$$\begin{aligned} C\|(\lambda, \mu, \rho) - (\lambda_0, \mu_0, 1)\|_{H^{s_0}(\Omega)} \leq{}& \|\mathcal{A}(\lambda, \mu, \rho)(\cdot, \cdot; \theta, \theta) - \mathcal{A}(\lambda_0, \mu_0, 1)(\cdot, \cdot; \theta, \theta)\|^\gamma_{H^{-k}((-\infty, T]; H^{s+2}(\Omega^c))} \\ &+ \|\mathcal{A}(\lambda, \mu, \rho)(\cdot, \cdot; \theta, \alpha) - \mathcal{A}(\lambda_0, \mu_0, 1)(\cdot, \cdot; \theta, \alpha)\|^\gamma_{H^{-k}((-\infty, T]; H^{s+2}(\Omega^c))}. \end{aligned}$$

The above result implies that if $\mathcal{A}(\lambda, \mu, \rho) = \mathcal{A}(\lambda_0, \mu_0, 1)$ and (λ, μ, ρ) is sufficiently close to constant $(\lambda_0, \mu_0, 1)$, then $(\lambda, \mu, \rho) = (\lambda_0, \mu_0, 1)$. We mention that a similar rigidity result was recently established in [9] using the Dirichlet-to-Neumann (DtN) map without the closeness assumption. However, DtN map encodes scattering data with all incident angles, while we have only used a single angle.

The rest of the paper is organized as follows. In Section 2.1, we prove Theorem 1.1 for the determination of a fully anisotropic perturbation in the linearized problem. In Section 3, we revisit the linearized problem with an isotropic perturbation and single-angle data. With Stefanov-Uhlmann's scheme, we also prove Theorem 1.2 for the nonlinear map.

2. Analysis of scattered waves and injectivity of the linearized map

In this section we prove Theorem 1.1. We will carefully analyze the single scattered waves $\dot{u}$ generated by incident wave u_0.

2.1. **Send P-wave, observe P-wave.** First, take a P-wave solution

$$u_0 = \theta H_0(c_p t - \theta \cdot x),$$

as the incidence wave. We now have

$$\partial_t^2 \dot{u}_i - (\lambda_0+\mu_0)\nabla_i \nabla \cdot \dot{u} - \mu_0 \Delta \dot{u}_i = -\partial_j \dot{c}_{ijk\ell}\theta_k\theta_\ell \delta(c_p t - \theta\cdot x) + \dot{c}_{ijk\ell}\theta_j\theta_k\theta_\ell \delta' - c_p^2 \dot{\rho}\theta_i \delta'.$$

Here δ is the delta distribution. The solution $\dot{u}$ satisfies the above equation in the distributional sense. We first use the scattered P-wave information contained in $\dot{u}|_{\Omega^c\times(-\infty,T]}$. For this, we take divergence of both sides of the above equation,

$$\Box_p(\nabla\cdot\dot{u}) = -\partial_i\partial_j \dot{c}_{ijk\ell}\theta_k\theta_\ell\delta + 2\partial_i \dot{c}_{ijk\ell}\theta_j\theta_k\theta_\ell\delta' - \dot{c}_{ijk\ell}\theta_i\theta_j\theta_k\theta_\ell\delta'' - c_p^2\theta\cdot\nabla\dot{\rho}\delta' + c_p^2\dot{\rho}\delta'', \tag{2.1}$$

where $\Box_p := \partial_t^2 - \Delta_p$ and $\Delta_p := (\lambda_0+2\mu_0)\Delta$. Denote $H_j(s) = s^j H_0(s)$ for $j = 1, 2, \cdots$. We will show that $\nabla\cdot\dot{u}$ can be written as the progressive wave expansion

$$\nabla\cdot\dot{u} = w_2(x)\delta' + w_1(x)\delta + w_0(x)H_0 + w_{-1}(x)H_1 + w'(x,t), \tag{2.2}$$

where $w_2, w_1, w_0, w_{-1} \in C^\infty(\mathbb{R}^3)$. Inserting the ansatz on the right hand side of (2.2) into the equation (2.1), we end up with

$$\begin{aligned}
&- c_p^2[\Delta w_2\delta' - 2\theta\cdot\nabla w_2\delta''] \\
&- c_p^2[\Delta w_1\delta - 2\theta\cdot\nabla w_1\delta'] \\
&- c_p^2[\Delta w_0 H_0 - 2\theta\cdot\nabla w_0\delta] \\
&- c_p^2[\Delta w_{-1}H_1 - 2\theta\cdot\nabla w_{-1}H_0] \\
&+ \Box_p w' \\
=& - \partial_i\partial_j\dot{c}_{ijk\ell}\theta_k\theta_\ell\delta + 2\partial_i\dot{c}_{ijk\ell}\theta_j\theta_k\theta_\ell\delta' - \dot{c}_{ijk\ell}\theta_i\theta_j\theta_k\theta_\ell\delta'' - c_p^2\theta\cdot\nabla\dot{\rho}\delta' + c_p^2\dot{\rho}\delta''.
\end{aligned}$$

In this subsection the argument for δ', δ, H_j is always $c_p t - \theta\cdot x$ and we occasionally omit it. We then choose w_2, w_1, w_0, w_{-1} and w' such that

$$\partial_t^2 w' - c_p^2\Delta w' - c_p^2\Delta w_{-1}H_1 = 0, \qquad \text{in } \mathbb{R}^3\times\mathbb{R}, \tag{2.3}$$

$$2c_p^2\theta\cdot\nabla w_{-1} - c_p^2\Delta w_0 = 0, \qquad \text{in } \mathbb{R}^3, \tag{2.4}$$

$$2c_p^2\theta\cdot\nabla w_0 - c_p^2\Delta w_1 = -\partial_i\partial_j\dot{c}_{ijk\ell}\theta_k\theta_\ell, \qquad \text{in } \mathbb{R}^3, \tag{2.5}$$

$$2c_p^2\theta\cdot\nabla w_1 - c_p^2\Delta w_2 = 2\partial_i\dot{c}_{ijk\ell}\theta_j\theta_k\theta_\ell - c_p^2\theta\cdot\nabla\dot{\rho}, \quad \text{in } \mathbb{R}^3, \tag{2.6}$$

$$2c_p^2\theta\cdot\nabla w_2 = -\dot{c}_{ijk\ell}\theta_i\theta_j\theta_k\theta_\ell + c_p^2\dot{\rho}, \qquad \text{in } \mathbb{R}^3. \tag{2.7}$$

For the construction, we can take

$$w_2(x) = \frac{1}{2c_p^2}\int_0^\infty -\dot{c}_{ijk\ell}(x-s\theta)\theta_j\theta_k\theta_\ell + c_p^2\dot{\rho}(x-s\theta)\mathrm{d}s,$$

$$w_1(x) = \frac{1}{2c_p^2}\int_0^\infty c_p^2\Delta w_2(x-s\theta) + 2\partial_i\dot{c}_{ijk\ell}(x-s\theta)\theta_j\theta_k\theta_\ell - c_p^2\theta\cdot\nabla\dot{\rho}(x-s\theta)\mathrm{d}s,$$

$$w_0(x) = \frac{1}{2c_p^2}\int_0^\infty c_p^2\Delta w_1(x-s\theta) - \partial_i\partial_j\dot{c}_{ijk\ell}(x-s\theta)\theta_k\theta_\ell\mathrm{d}s,$$

and

$$w_{-1}(x) = \frac{1}{2}\int_0^\infty \Delta w_0(x-s\theta)\mathrm{d}s.$$

By above construction w_2, w_1, w_0, w_{-1} all vanish in $\{x\cdot\theta < -1\}$, then w' has to satisfy (2.3) such that $w' = 0$ for $t \ll 0$. By standard hyperbolic theory (cf., for example, [7, Chapter 7]), there exists

a unique solution $w' \in H^2_{\mathrm{loc}}$, and it vanishes in $\{c_p t < \theta \cdot x\}$. This justifies our ansatz (2.2). We also refer the proof of [16, Proposition 3.1] for more details.

We can now write

$$\nabla \cdot \dot{u} = w_2(x)\delta' + w_1(x)\delta + v'(x,t)H_0, \tag{2.8}$$

by taking

$$v'(x,t) = w_0(x) + w_{-1}(x)(c_p t - \theta \cdot x) + w'(x,t) \in H^2_{\mathrm{loc}}.$$

Now let us assume that

$$\mathcal{A}'[\mathbf{C}_0, 1](\dot{\mathbf{C}}, \dot{\rho}) = 0,$$

or equivalently $\dot{u} = 0$ in $\Omega^c \times (-\infty, T]$. Then $\nabla \cdot \dot{u} = 0$ in $\Omega^c \times (-\infty, T]$. By the ansatz (2.8) we conclude that

$$v' = 0 \quad \text{in } \{c_p t > \theta \cdot x\} \cap (\Omega^c \times (-\infty, T]).$$

Also, since $\nabla \cdot \dot{u} = v'$ in $\{c_p t > \theta \cdot x\}$, we have

$$\Box_p v' = 0, \quad \text{in } \{c_p t > \theta \cdot x\}.$$

We recall the unique continuation principle, see e.g. [13, Theorem 3.16].

Proposition 2.1. *Let $\omega \subset \mathbb{R}^n$ be open, and $T, c \in (0, +\infty)$, $t_0 \in \mathbb{R}$ all be constants. Define*

$$K = K(\omega, T, t_0) = \{(x,t) \in \mathbb{R}^{n+1} \mid d(x,\omega)/c < T - |t - t_0|\},$$

where d is the distance function on $\mathbb{R}^n$. Let $u \in H^1(K)$ satisfy $\Box_c u = 0$ in K and $u = 0$ in $\omega \times (t_0 - T, t_0 + T)$. Then $u = 0$ in K.

Lemma 2.2. *Let $s, a \in \mathbb{R}$. Suppose that $u \in H^1(\mathbb{R}^{n+1})$ satisfies $\Box_c u = 0$ in the set*

$$R = \{(x,t) \in \mathbb{R}^{n+1} \mid \theta \cdot x < \min(c(t-s), a)\}$$

and that $u = 0$ in $\Omega^c \times (-\infty, T)$ for T sufficiently large. Then $u = 0$ in R.

Proof. After a rotation in space and translation in time we may assume that $\theta = (1,0,0)$ and $s = 0$. Let $x_0 = (x_0^1, x_0^2, x_0^3) \in \Omega$ satisfy $x_0^1 < a$ and let $t_0 > x_0^1/c$. Choose large enough $r > 0$ so that $y := x_0 - r\theta$ has a neighborhood ω that is contained in Ω^c. Then the closed diamond $\overline{K(\{y\}, r/c, t_0)}$ contains (x_0, t_0) and is contained in R. For a small enough neighborhood $\omega \subset \Omega^c$ of y, the open diamond $K(\omega, r/c, t_0)$ also contains (x_0, t_0) and is contained in R. If T is taken sufficiently large, we have $u = 0$ in $\omega \times (t_0 - r/c, t_0 + r/c)$. It follows from Proposition 2.1 that $v' = 0$ near (x_0, t_0).

To see $\overline{K(\{y\}, r/c, t_0)}$ is indeed contained in R, we argue as follows. Let $(x,t) \in \overline{K(\{y\}, r/c, t_0)}$. It follows from

$$|x^1 - x_0^1 + r|/c \le r/c$$

that $x^1 \le x_0^1$. In particular $x^1 < a$. To get a contradiction, suppose that $ct \le x^1$. We have

$$t \le x^1/c \le x_0^1/c < t_0, \quad |t_0 - t| = t_0 - t > |x_0^1 - x^1|/c$$

and

$$|x^1 - x_0^1 + r|/c + |x_0^1 - x^1|/c < |x^1 - x_0^1 + r|/c + |t_0 - t| \le r/c.$$

Finally, we arrive to the contradiction

$$r = r - |x^1 - x_0^1| + |x_0^1 - x^1| \le |x^1 - x_0^1 + r| + |x_0^1 - x^1| < r.$$

The proof is done. □

Taking $s = 0$ and large enough $a > 0$, it follows from Lemma 2.2 that that $v' = 0$ in $\{c_p t > \theta \cdot x\}$ and then also on $\{c_p t = \theta \cdot x\}$ by continuity.

Notice that $v'(x,t) = w_0(x)$ on $\{c_p t = \theta \cdot x\}$, thus now we have $w_0 = 0$ in $\mathbb{R}^3$. Then the identity (2.5) reduces to

$$-c_p^2 \Delta w_1 = -\partial_i \partial_j \dot{c}_{ijk\ell} \theta_k \theta_\ell \quad \text{in } \mathbb{R}^3.$$

Together with (2.6) and (2.7), we end up with the equation

$$4(\theta \cdot \nabla)^2 \partial_i \partial_j \dot{c}_{ijk\ell} \theta_k \theta_\ell + \Delta^2 \dot{c}_{ijk\ell} \theta_i \theta_j \theta_k \theta_\ell - c_p^2 \Delta^2 \dot{\rho} = 4(\theta \cdot \nabla) \Delta \partial_i \dot{c}_{ijk\ell} \theta_j \theta_k \theta_\ell - 2c_p^2 (\theta \cdot \nabla)^2 \Delta \dot{\rho}.$$

In particular, if we take $\theta = \mathbf{e}_1$, the above equation reads

$$4\partial_1^2 \partial_i \partial_j \dot{c}_{ij11} + \Delta^2 \dot{c}_{1111} - c_p^2 \Delta^2 \dot{\rho} = 4\partial_1 \Delta \partial_i \dot{c}_{i111} - 2c_p^2 \partial_1^2 \Delta \dot{\rho}. \tag{2.9}$$

2.2. **Send S-wave, observe P-wave.** Now take an S-wave solution

$$u_0 = \alpha H_0(c_s t - \theta \cdot x), \quad \alpha \perp \theta,$$

we have

$$\partial_t^2 \dot{u}_i - (\lambda_0 + \mu_0) \nabla_i \nabla \cdot \dot{u} - \mu_0 \Delta \dot{u}_i = -\partial_j \dot{c}_{ijk\ell} \theta_k \alpha_\ell \delta(c_s t - \theta \cdot x) + \dot{c}_{ijk\ell} \theta_j \theta_k \alpha_\ell \delta' - c_s^2 \dot{\rho} \alpha_i \delta'.$$

In this subsection the argument for $\delta'', \delta', \delta, H_j$ is always $c_s t - \theta \cdot x$ and we occasionally omit it. Taking divergence of both sides, we have

$$[\partial_t^2 - c_p^2 \Delta](\nabla \cdot \dot{u}) = -\partial_i \partial_j \dot{c}_{ijk\ell} \theta_k \alpha_\ell \delta(c_s t - \theta \cdot x) + 2\partial_i \dot{c}_{ijk\ell} \theta_j \theta_k \alpha_\ell \delta' - \dot{c}_{ijk\ell} \theta_i \theta_j \theta_k \alpha_\ell \delta'' - c_s^2 \alpha_i \partial_i \dot{\rho} \delta'. \tag{2.10}$$

We use the ansatz

$$\nabla \cdot \dot{u} = w_1(x) \delta(c_s t - \theta \cdot x) + w_0(x) H_0 + w_{-1}(x) H_1 + w_{-2}(x) H_2 + w'(x,t), \tag{2.11}$$

where $w_1, w_0, w_{-1}, w_{-2} \in C^\infty(\mathbb{R}^3)$ are all smooth functions. Inserting the above ansatz into (2.10), one obtains

$$\begin{aligned}
&c_s^2 w_1 \delta'' - c_p^2 \left[\Delta w_1 \delta - 2\theta \cdot \nabla w_1 \delta' + w_1(x) \delta''\right] \\
&+ c_s^2 w_0 \delta' - c_p^2 \left[\Delta w_0 H_0 - 2\theta \cdot \nabla w_0 \delta + w_0 \delta'\right] \\
&+ c_s^2 w_{-1} \delta - c_p^2 \left[\Delta w_{-1} H_1 - 2\theta \cdot \nabla w_{-1} H_0 + w_{-1} \delta\right] \\
&+ c_s^2 w_{-2} H_0 - c_p^2 \left[\Delta w_{-2} H_2 - 2\theta \cdot \nabla w_{-2} H_1 + w_{-2} H_0\right] \\
&+ \Box_p w' \\
=& - \partial_i \partial_j \dot{c}_{ijk\ell} \theta_k \alpha_\ell \delta(c_s t - \theta \cdot x) + 2\partial_i \dot{c}_{ijk\ell} \theta_j \theta_k \alpha_\ell \delta' - \dot{c}_{ijk\ell} \theta_i \theta_j \theta_k \alpha_\ell \delta'' - c_s^2 \alpha_i \partial_i \dot{\rho} \delta'.
\end{aligned}$$

We can take w_1, w_0, w_{-1}, w_{-2} and w' such that

$$\Box_p w' - c_p^2 \Delta w_{-1} H_1 - c_p^2 \Delta w_{-2} H_2 + 2c_p^2 \theta \cdot \nabla w_{-2} H_1 = 0, \qquad \text{in } \mathbb{R}^3 \times \mathbb{R}, \tag{2.12}$$

$$(c_s^2 - c_p^2) w_{-2} + 2c_p^2 \theta \cdot \nabla w_{-1} - c_p^2 \Delta w_0 = 0, \qquad \text{in } \mathbb{R}^3, \tag{2.13}$$

$$(c_s^2 - c_p^2) w_{-1} + 2c_p^2 \theta \cdot \nabla w_0 - c_p^2 \Delta w_1 = -\partial_i \partial_j \dot{c}_{ijk\ell} \theta_k \alpha_\ell, \qquad \text{in } \mathbb{R}^3, \tag{2.14}$$

$$(c_s^2 - c_p^2) w_0 + 2c_p^2 \theta \cdot \nabla w_1 = 2\partial_i \dot{c}_{ijk\ell} \theta_j \theta_k \alpha_\ell - c_s^2 \alpha_i \partial_i \dot{\rho}, \quad \text{in } \mathbb{R}^3, \tag{2.15}$$

$$(c_s^2 - c_p^2) w_1 = -\dot{c}_{ijk\ell} \theta_i \theta_j \theta_k \alpha_\ell, \qquad \text{in } \mathbb{R}^3. \tag{2.16}$$

Since $c_p \neq c_s$, we just successively take

$$w_1(x) = -\frac{1}{(c_s^2 - c_p^2)} \dot{c}_{ijk\ell}(x) \theta_i \theta_j \theta_k \alpha_\ell,$$

$$w_0(x) = \frac{1}{c_s^2 - c_p^2} (-2c_p^2 \theta \cdot \nabla w_1(x) + 2\partial_i \dot{c}_{ijk\ell}(x) \theta_j \theta_k \alpha_\ell - c_s^2 \alpha_i \partial_i \dot{\rho}),$$

$$w_{-1}(x) = \frac{1}{c_s^2 - c_p^2}\left(-2c_p^2\theta\cdot\nabla w_0 + c_p^2\Delta w_1 - \partial_i\partial_j\dot{c}_{ijk\ell}\theta_k\alpha_\ell\right),$$

and

$$w_{-2}(x) = \frac{1}{c_s^2 - c_p^2}\left(-2c_p^2\theta\cdot\nabla w_{-1} + c_p^2\Delta w_0\right).$$

Note that by the construction above, w_j for $j \in \{1, 0, -1, -2\}$ are all smooth functions and compactly supported in Ω. Then we let w' satisfy the equation (2.12) with $w' = 0, t \ll 0$, which has a unique solution in H^2_{loc}. This justifies our ansatz (2.11).

Now we can write

$$\nabla\cdot\dot{u} = w_1(x)\delta(c_st - \theta\cdot x) + v'(x,t)H_0(c_st - \theta\cdot x) + w'(x,t) \tag{2.17}$$

by taking

$$v'(x,t) = w_0(x) + (c_st - \theta\cdot x)w_{-1}(x) + (c_st - \theta\cdot x)^2 w_{-2}(x). \tag{2.18}$$

Then v' and w' solve

$$\Box_p(v' + w') = 0, \quad \text{in } \{c_st > \theta\cdot x\},$$

and

$$\Box_p w' = 0, \quad \text{in } \{c_st < \theta\cdot x\}.$$

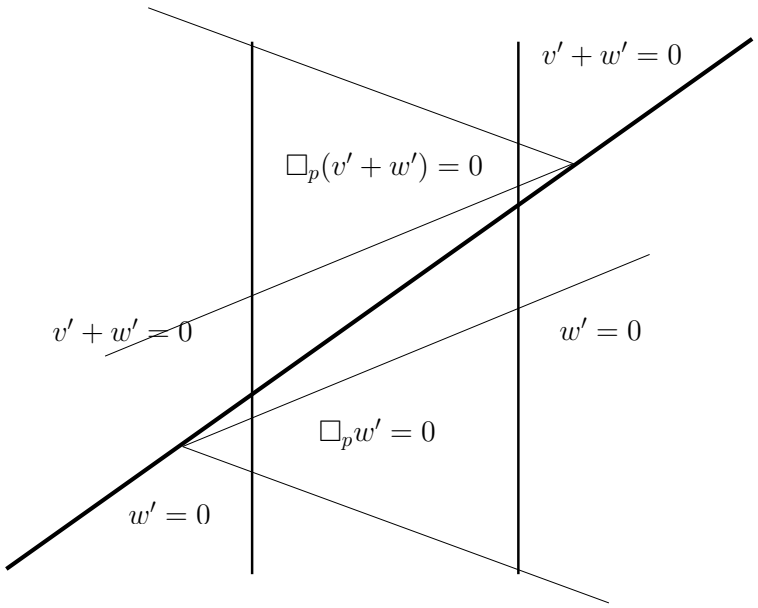


FIGURE 1. Unique continuation for S-P scattering.

Notice that $\nabla\cdot\dot{u} = v' + w'$ in $\{c_st > \theta\cdot x\}$ and $\nabla\cdot\dot{u} = w'$ in $\{c_st < \theta\cdot x\}$. For the inverse problem, we start with the fact $\nabla\cdot\dot{u} =$ in $\Omega^c\times(-\infty, T]$. By unique continuation for the Cauchy problem

$$\begin{aligned}\Box_p w' &= 0, \quad \text{in } \{c_st < \theta\cdot x\},\\ w' &= 0, \quad \text{in } \{c_st < \theta\cdot x\}\cap(\Omega^c\times(-\infty, T]),\end{aligned}$$

we can conclude $w' \equiv 0$ in $\{c_s t < \theta \cdot x\}$. This follows from a time-reversed version of the argument that we will give next. By unique continuation for the Cauchy problem

$$\begin{aligned} \Box_p(v' + w') &= 0, \quad \text{in } \{c_s t > \theta \cdot x\}, \\ v' + w' &= 0, \quad \text{in } \{c_s t > \theta \cdot x\} \cap (\Omega^c \times (-\infty, T]), \end{aligned}$$

we have $v' + w' \equiv 0$ in $\{c_s t > \theta \cdot x\}$. See Fig. 1 for an illustration. Indeed, suppose that (x_0, t_0) satisfies $c_s t_0 > \theta \cdot x_0$. We apply Lemma 2.2 with

$$a = \theta \cdot x_0, \quad s = \frac{(c_p - c_s)a}{c_p c_s}.$$

With this choice of (a, s), $\min(c_p(t-s), a) \le c_s t$ for all t since either $a \le c_s t$ or $c_s t < a$, and in the latter case

$$c_p(t-s) = c_p t - \frac{(c_p - c_s)a}{c_s} < c_p t - (c_p - c_s)t = c_s t.$$

In particular, the set R in Lemma 2.2 is contained in the region $c_s t > \theta \cdot x$, and we obtain $v' + w' = 0$ in R. Moreover, $(x_0, t_0) \in \overline{R}$ since

$$c_p(t_0 - s) = c_p t_0 - \frac{(c_p - c_s)a}{c_s} > c_p t_0 - (c_p - c_s)t_0 = c_s t_0 > \theta \cdot x_0.$$

By perturbing (x_0, t_0) slightly, we conclude that $v' + w' = 0$ near (x_0, t_0).

Notice that v' is smooth and w' is in H^2_{loc}. Taking the traces on $\{c_s t = \theta \cdot x\}$, we have $v' + w' = w' = 0$ and $(\partial_t + c_p \theta \cdot \nabla)(v' + w') = (\partial_t + c_p \theta \cdot \nabla) w' = 0$ on $\{c_s t = \theta \cdot x\}$. This means that $v' = (\partial_t + c_p \theta \cdot \nabla) v' = 0$ on $\{c_s t = \theta \cdot x\}$ and thus $w_0(x) = w_{-1}(x) = 0$ by (2.18).

Then the identities (2.14) and (2.15) become

$$\begin{aligned} -c_p^2 \Delta w_1 &= -\partial_i \partial_j \dot{c}_{ijk\ell} \theta_k \alpha_\ell, && \text{in } \mathbb{R}^3, \\ 2c_p^2 \theta \cdot \nabla w_1 &= 2\partial_i \dot{c}_{ijk\ell} \theta_j \theta_k \alpha_\ell - c_s^2 \alpha_i \partial_i \dot{\rho}, && \text{in } \mathbb{R}^3. \end{aligned}$$

Together with the equation (2.16), we obtain

$$\begin{aligned} 2\frac{c_p^2}{c_p^2 - c_s^2} \theta \cdot \nabla \dot{c}_{ijk\ell} \theta_i \theta_j \theta_k \alpha_\ell &= 2\partial_i \dot{c}_{ijk\ell} \theta_j \theta_k \alpha_\ell - c_s^2 \alpha_i \partial_i \dot{\rho}, && \text{in } \mathbb{R}^3, \\ \frac{c_p^2}{c_p^2 - c_s^2} \Delta \dot{c}_{ijk\ell} \theta_i \theta_j \theta_k \alpha_\ell &= \partial_i \partial_j \dot{c}_{ijk\ell} \theta_k \alpha_\ell, && \text{in } \mathbb{R}^3. \end{aligned}$$

In particular, if we take $\theta = \mathbf{e}_1, \alpha = \mathbf{e}_2$ in above equations, we have

$$2\frac{c_p^2}{c_p^2 - c_s^2} \partial_1 \dot{c}_{1112} = 2\partial_i \dot{c}_{i112} - c_s^2 \partial_2 \dot{\rho}, \quad \text{in } \mathbb{R}^3, \tag{2.19}$$

$$\frac{c_p^2}{c_p^2 - c_s^2} \Delta \dot{c}_{1112} = \partial_i \partial_j \dot{c}_{ij12}, \quad \text{in } \mathbb{R}^3. \tag{2.20}$$

Taking $\theta = \mathbf{e}_2, \alpha = \mathbf{e}_1$, we have

$$2\frac{c_p^2}{c_p^2 - c_s^2} \partial_2 \dot{c}_{2221} = 2\partial_i \dot{c}_{i221} - c_s^2 \partial_1 \dot{\rho}, \quad \text{in } \mathbb{R}^3, \tag{2.21}$$

$$\frac{c_p^2}{c_p^2 - c_s^2} \Delta \dot{c}_{2221} = \partial_i \partial_j \dot{c}_{ij21}, \quad \text{in } \mathbb{R}^3. \tag{2.22}$$

The equations (2.20) and (2.22) immediately imply that $\Delta(\dot{c}_{1112} - \dot{c}_{2221}) = 0$ in $\mathbb{R}^3$, and since $\dot{\mathbf{C}}$ is compactly supported in Ω,

$$\dot{c}_{1112} = \dot{c}_{2221}, \quad \text{in } \mathbb{R}^3.$$

Now we write (2.19) and (2.21) as

$$2\frac{c_s^2}{c_p^2 - c_s^2}\partial_1 \dot{c}_{1112} = 2\partial_2 \dot{c}_{2112} + 2\partial_3 \dot{c}_{3112} - c_s^2 \partial_2 \dot{\rho}, \tag{2.23}$$

$$2\frac{c_s^2}{c_p^2 - c_s^2}\partial_2 \dot{c}_{2221} = 2\partial_1 \dot{c}_{1221} + 2\partial_3 \dot{c}_{3221} - c_s^2 \partial_1 \dot{\rho}. \tag{2.24}$$

2.3. **Send P-wave, observe S-wave.** In this step, we take a P-wave solution

$$u_0 = \theta H_0(c_p t - \theta \cdot x),$$

and observed the scattered S-waves. Now we have

$$\partial_t^2 \dot{u}_i - (\lambda_0 + \mu_0)\nabla_i \nabla \cdot \dot{u} - \mu_0 \Delta \dot{u}_i = -\partial_j \dot{c}_{ijk\ell}\theta_k\theta_\ell\delta + \dot{c}_{ijk\ell}\theta_j\theta_k\theta_\ell\delta' - c_p^2\dot{\rho}\theta_i\delta'.$$

We introduce the notation e_{ijk} that is skew-symmetric and $e_{123} = 1$. Taking curl of both sides

$$\begin{aligned}\Box_s(\nabla \times u)_i =& e_{ipq}\partial_q(-\partial_j \dot{c}_{pjk\ell}\theta_k\theta_\ell\delta(c_p t - \theta\cdot x) + \dot{c}_{pjk\ell}\theta_j\theta_k\theta_\ell\delta') - c_p^2 e_{ipq}\partial_q(\dot{\rho}\theta_p\delta')\\ =& - e_{ipq}\dot{c}_{pjk\ell}\theta_j\theta_k\theta_\ell\theta_q\delta'' + e_{ipq}\partial_q\dot{c}_{pjk\ell}\theta_j\theta_k\theta_\ell\delta' + e_{ipq}\partial_j\dot{c}_{pjk\ell}\theta_k\theta_\ell\theta_q\delta' - e_{ipq}\partial_q\partial_j\dot{c}_{pjk\ell}\theta_k\theta_\ell\delta.\\ & - c_p^2 e_{ipq}\partial_q\dot{\rho}\theta_p\delta' + c_p^2 e_{ipq}\dot{\rho}\theta_p\theta_q\delta''.\end{aligned}$$

In this subsection the argument for $\delta'', \delta', \delta, H_j$ is always $c_p t - \theta\cdot x$ and we occasionally omit it. We use the ansatz

$$\nabla \times \dot{u} = w_1(x)\delta(c_p t - \theta\cdot x) + w_0(x)H_0 + w_{-1}(x)H_1 + w_{-2}(x)H_2 + w'(x,t), \tag{2.25}$$

where v, w are all smooth functions. Insert into the equation above,

$$\begin{aligned}&c_p^2 w_{1i}\delta'' - c_s^2\left[\Delta w_{1i}\delta - 2\theta\cdot\nabla w_{1i}\delta' + w_{1i}\delta''\right]\\ &+ c_p^2 w_{0i}\delta' - c_s^2\left[\Delta w_{0i}H_0 - 2\theta\cdot\nabla w_{0i}\delta + w_{0i}\delta'\right]\\ &+ c_p^2 w_{-1i}\delta - c_s^2\left[\Delta w_{-1i}H_1 - 2\theta\cdot\nabla w_{-1i}H_0 + w_{-1i}\delta\right]\\ &+ c_p^2 w_{-2i}H_0 - c_s^2\left[\Delta w_{-2i}H_2 - 2\theta\cdot\nabla w_{-2i}H_1 + w_{-2i}H_0\right]\\ &+ \Box_s w_i'\\ =& e_{ipq}\dot{c}_{pjk\ell}\theta_j\theta_k\theta_\ell\theta_q\delta'' + e_{ipq}\partial_q\dot{c}_{pjk\ell}\theta_j\theta_k\theta_\ell\delta' + e_{ipq}\partial_j\dot{c}_{pjk\ell}\theta_k\theta_\ell\theta_q\delta' - e_{ipq}\partial_q\partial_j\dot{c}_{pjk\ell}\theta_k\theta_\ell\delta\\ &- c_p^2 e_{ipq}\partial_q\dot{\rho}\theta_p\delta.\end{aligned}$$

Now we take w_1, w_0, w_{-1}, w_{-2} such that

$$(c_p^2 - c_s^2)w_{1i} = -e_{ipq}\dot{c}_{pjk\ell}\theta_j\theta_k\theta_\ell\theta_q, \qquad \text{in } \mathbb{R}^3, \tag{2.26}$$

$$\begin{aligned}(c_p^2 - c_s^2)w_{0i} =& -2c_s^2\theta\cdot\nabla w_{1i} + e_{ipq}\partial_q\dot{c}_{pjk\ell}\theta_j\theta_k\theta_\ell + e_{ipq}\partial_j\dot{c}_{pjk\ell}\theta_k\theta_\ell\theta_q\\ &- c_p^2 e_{ipq}\partial_q\dot{\rho}\theta_p, \qquad \text{in } \mathbb{R}^3,\end{aligned} \tag{2.27}$$

$$(c_p^2 - c_s^2)w_{-1i} = c_s^2\Delta w_{1i} - 2c_s^2\theta\cdot\nabla w_{0i} - e_{ipq}\partial_q\partial_j\dot{c}_{pjk\ell}\theta_k\theta_\ell, \qquad \text{in } \mathbb{R}^3, \tag{2.28}$$

$$(c_p^2 - c_s^2)w_{-2i} = c_s^2\Delta w_{0i} - 2c_s^2\theta\cdot\nabla w_{-1i}, \qquad \text{in } \mathbb{R}^3, \tag{2.29}$$

The function w' satisfies the equation

$$\begin{aligned} \Box_s w' - c_s^2 \Delta w_{-1} H_1 - c_s^2 \Delta w_{-2} H_2 + 2c_s^2 \theta \cdot \nabla w_{-2} H_1 = 0, \quad &\text{in } \mathbb{R}^3 \times \mathbb{R}, \\ w' = 0, \quad &t \ll 0. \end{aligned}$$

Then above equation has a unique solution $w' \in H^2_{\mathrm{loc}}$ such that $w' = 0$ in $\{c_p t < \theta \cdot x\}$. This justifies our ansatz (2.25).

Therefore we can write

$$\nabla \times \dot{u} = w_1(x)\delta(c_p t - \theta \cdot x) + v'(x,t) H_0,$$

by taking

$$v'(x,t) = w_0(x) + w_{-1}(x)(c_p t - \theta \cdot x) + w_{-2}(x)(c_p t - \theta \cdot x)^2 + w'(x,t),$$

which is in H^2_{loc}. Then v' solves

$$\Box_s v' = 0, \quad \text{in } \{c_p t > \theta \cdot x\}.$$

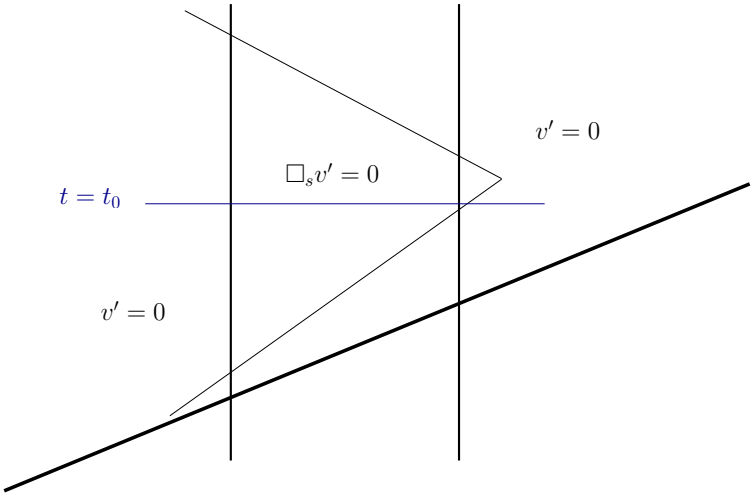


FIGURE 2. Unique continuation for P-S scattering.

For the inverse problem, if $\nabla \times \dot{u} = 0$ in $\Omega^c \times (-\infty, T]$, we have

$$\begin{aligned} \Box_s v' = 0, \quad &\text{in } \{c_p t > \theta \cdot x\}, \\ v' = 0, \quad &\text{in } \{c_p t > \theta \cdot x\} \cap (\Omega^c \times (-\infty, T]). \end{aligned}$$

We apply Lemma 2.2 with large enough $a, s > 0$. Then $R \cap (\Omega \times \mathbb{R})$ is in the region $c_p t > \theta \cdot x$ and contains a neighborhood of $\Omega \times \{t_0\}$ for some $t_0 \in \mathbb{R}$. So $v' = 0$ in a neighborhood of $\Omega \times \{t_0\}$. We conclude that $v' = 0$ in $\{c_p t > \theta \cdot x\}$ by noticing that $v' = 0$ is the unique solution to

$$\Box_s v' = 0, \quad \text{in } \{c_p t > \theta \cdot x\} \cap (\Omega \times (-\infty, t_0)), \tag{2.30}$$

$$v' = 0, \quad \text{in } \{c_p t > \theta \cdot x\} \cap (\partial\Omega \times (-\infty, t_0)), \tag{2.31}$$

$$v' = \partial_t v' = 0, \quad \text{in } \Omega \times \{t_0\}. \tag{2.32}$$

By continuity, we obtain $v' = (\partial_t + c_s \theta \cdot \nabla) v' = 0$ on $\{c_p t = \theta \cdot x\}$. This implies that

$$w_0(x) = w_{-1}(x) = 0 \quad \text{in } \mathbb{R}^3.$$

See Fig. 2 for an illustration. Then the equations (2.27) and (2.28) become

$$\begin{aligned} 2c_s^2\theta\cdot\nabla w_{1i} &= e_{ipq}\partial_q\dot{c}_{pjk\ell}\theta_j\theta_k\theta_\ell + e_{ipq}\partial_j\dot{c}_{pjk\ell}\theta_k\theta_\ell\theta_q - c_p^2 e_{ipq}\partial_q\dot{\rho}\theta_p, && \text{in } \mathbb{R}^3\\ c_s^2\Delta w_{1i} &= e_{ipq}\partial_q\partial_j\dot{c}_{pjk\ell}\theta_k\theta_\ell, && \text{in } \mathbb{R}^3. \end{aligned}$$

Together with the expression $w_{1i} = -\frac{1}{c_p^2-c_s^2}e_{ipq}\dot{c}_{pjk\ell}\theta_j\theta_k\theta_\ell\theta_q$, we obtain

$$\begin{aligned} -2\frac{c_s^2}{c_p^2-c_s^2}e_{ipq}\theta\cdot\nabla\dot{c}_{pjk\ell}\theta_j\theta_k\theta_\ell\theta_q &= e_{ipq}\partial_q\dot{c}_{pjk\ell}\theta_j\theta_k\theta_\ell + e_{ipq}\partial_j\dot{c}_{pjk\ell}\theta_k\theta_\ell\theta_q - c_p^2 e_{ipq}\partial_q\dot{\rho}\theta_p,\\ \frac{c_s^2}{c_p^2-c_s^2}e_{ipq}\Delta\dot{c}_{pjk\ell}\theta_j\theta_k\theta_\ell\theta_q &= -e_{ipq}\partial_q\partial_j\dot{c}_{pjk\ell}\theta_k\theta_\ell. \end{aligned}$$

Take $\theta = \mathbf{e}_1$ in above identities, we obtain

$$i=1: \qquad e_{123}\partial_3\dot{c}_{2111} + e_{132}\partial_2\dot{c}_{3111} = 0, \tag{2.33}$$

$$-e_{123}\partial_3\partial_j\dot{c}_{2j11} - e_{132}\partial_2\partial_j\dot{c}_{3j11} = 0, \tag{2.34}$$

$$i=2: \quad -2\frac{c_s^2}{c_p^2-c_s^2}e_{231}\partial_1\dot{c}_{3111} = e_{213}\partial_3\dot{c}_{1111} + e_{231}\partial_1\dot{c}_{3111} + e_{231}\partial_j\dot{c}_{3j11} - c_p^2 e_{213}\partial_3\dot{\rho}, \tag{2.35}$$

$$\frac{c_s^2}{c_p^2-c_s^2}e_{231}\Delta\dot{c}_{3111} = -e_{213}\partial_3\partial_j\dot{c}_{1j11} - e_{231}\partial_1\partial_j\dot{c}_{3j11}, \tag{2.36}$$

$$i=3: \quad -2\frac{c_s^2}{c_p^2-c_s^2}e_{321}\partial_1\dot{c}_{2111} = e_{312}\partial_2\dot{c}_{1111} + e_{321}\partial_1\dot{c}_{2111} + e_{321}\partial_j\dot{c}_{2j11} - c_p^2 e_{312}\partial_2\dot{\rho}, \tag{2.37}$$

$$\frac{c_s^2}{c_p^2-c_s^2}e_{321}\Delta\dot{c}_{2111} = -e_{312}\partial_2\partial_j\dot{c}_{1j11} - e_{321}\partial_1\partial_j\dot{c}_{2j11}. \tag{2.38}$$

Notice that the above six equations are linearly dependent, and two of them are redundant.

2.4. **Send S-wave, observe S-wave.** Take an S-wave solution

$$u_0 = \alpha H_0(c_s t - \theta\cdot x), \quad \alpha\perp\theta,$$

as the incident wave, we have

$$\partial_t^2\dot{u}_i - (\lambda_0+\mu_0)\nabla_i\nabla\cdot\dot{u} - \mu_0\Delta\dot{u}_i = -\partial_j c_{ijk\ell}\theta_k\alpha_\ell\delta + c_{ijk\ell}\theta_j\theta_k\alpha_\ell\delta' - c_s^2\rho\alpha_i\delta'.$$

Taking curl of both sides

$$\begin{aligned} \Box_s(\nabla\times u)_i = &- e_{ipq}\dot{c}_{pjk\ell}\theta_j\theta_k\alpha_\ell\theta_q\delta'' + e_{ipq}\partial_q\dot{c}_{pjk\ell}\theta_j\theta_k\alpha_\ell\delta' + e_{ipq}\partial_j\dot{c}_{pjk\ell}\theta_k\alpha_\ell\theta_q\delta' - e_{ipq}\partial_q\partial_j\dot{c}_{pjk\ell}\theta_k\alpha_\ell\delta\\ &- c_s^2 e_{ipq}\partial_q\dot{\rho}\alpha_p\delta' + c_s^2 e_{ipq}\dot{\rho}\alpha_p\theta_q\delta''. \end{aligned}$$

By the same consideration as for PP-scattering, we can use the ansatz

$$\nabla\times\dot{u} = w_2(x)\delta' + w_1(x)\delta + v'(x,t)H_0.$$

with $v'\in H^2_{\mathrm{loc}}$. We end up with

$$\partial_t^2 v' - c_s^2\Delta v' = 0, \qquad \text{in } \{c_s t > \theta\cdot x\}, \tag{2.39}$$

$$2c_s(\partial_t v' + c_s\theta\cdot\nabla v') - c_s^2\Delta w_1 = -e_{ipq}\partial_q\partial_j\dot{c}_{pjk\ell}\theta_k\alpha_\ell, \qquad \text{on } \{c_s t = \theta\cdot x\}, \tag{2.40}$$

$$\begin{aligned} 2c_s^2\theta\cdot\nabla w_1 - c_s^2\Delta w_2 = &\, e_{ipq}\partial_q\dot{c}_{pjk\ell}\theta_j\theta_k\alpha_\ell + e_{ipq}\partial_j\dot{c}_{pjk\ell}\theta_k\alpha_\ell\theta_q\\ &- c_s^2 e_{ipq}\partial_q\dot{\rho}\alpha_p, \qquad \text{in } \mathbb{R}^3, \end{aligned} \tag{2.41}$$

$$2c_s^2\theta\cdot\nabla w_2 = -e_{ipq}\dot{c}_{pjk\ell}\theta_j\theta_k\alpha_\ell\theta_q + c_s^2 e_{ipq}\dot{\rho}\alpha_p\theta_q, \qquad \text{in } \mathbb{R}^3. \tag{2.42}$$

Similar to the consideration for the case of PP-scattering. applying the unique continuation principle to the Cauchy problem

$$\begin{aligned}\Box_s v' &= 0, \quad \text{in } \{c_s t > \theta\cdot x\},\\ v' &= 0, \quad \text{in } \{c_s t > \theta\cdot x\}\cap(\Omega^c\times(-\infty,T]),\end{aligned}$$

we conclude that $v'=0$ on $\{c_s t = \theta\cdot x\}$. Then

$$c_s^2\Delta w_1 = e_{ipq}\partial_q\partial_j\dot{c}_{pjk\ell}\theta_k\alpha_\ell$$

by equation (2.40). Together with (2.41) and (2.42) we end up with the identity

$$\begin{aligned}&4e_{ipq}(\theta\cdot\nabla)^2\partial_q\partial_j\dot{c}_{pjk\ell}\theta_k\alpha_\ell + e_{ipq}\Delta^2\dot{c}_{pjk\ell}\theta_j\theta_k\alpha_\ell\theta_q - c_s^2 e_{ipq}\Delta^2\dot{\rho}\alpha_p\theta_q\\ =&2e_{ipq}(\theta\cdot\nabla)\Delta\partial_q\dot{c}_{pjk\ell}\theta_j\theta_k\alpha_\ell + 2e_{ipq}(\theta\cdot\nabla)\Delta\partial_j\dot{c}_{pjk\ell}\theta_k\alpha_\ell\theta_q - 2c_s^2 e_{ipq}(\theta\cdot\nabla)\Delta\partial_q\dot{\rho}\alpha_p.\end{aligned}$$

Take $\theta=\mathbf{e}_1$ and $\alpha=\mathbf{e}_2$ in the above equation.

$$\begin{aligned}&4e_{123}\partial_1^2\partial_3\partial_j\dot{c}_{2j12}+4e_{132}\partial_1^2\partial_2\partial_j\dot{c}_{3j12}\\ &\qquad = 2e_{123}\partial_1\partial_3\Delta\dot{c}_{2112} + 2e_{132}\partial_1\Delta\partial_2\dot{c}_{3112} - 2c_s^2 e_{123}\partial_1\partial_3\Delta\dot{\rho},\end{aligned} \tag{2.43}$$

$$\begin{aligned}&4e_{213}\partial_1^2\partial_3\partial_j\dot{c}_{1j12}+4e_{231}\partial_1^3\partial_j\dot{c}_{3j12} + e_{231}\Delta^2\dot{c}_{3112}\\ &\qquad = 2e_{213}\partial_1\Delta\partial_3\dot{c}_{1112} + 2e_{231}\partial_1\Delta\partial_1\dot{c}_{3112} + 2e_{231}\partial_1\Delta\partial_j\dot{c}_{3j12},\end{aligned} \tag{2.44}$$

$$\begin{aligned}&4e_{312}\partial_1^2\partial_2\partial_j\dot{c}_{1j12}+4e_{321}\partial_1^3\partial_j\dot{c}_{2j12} + e_{321}\Delta^2\dot{c}_{2112} - c_s^2 e_{321}\Delta^2\dot{\rho}\\ &\qquad = 2e_{312}\partial_1\Delta\partial_2\dot{c}_{1112} + 2e_{321}\partial_1\Delta\partial_1\dot{c}_{2112}\\ &\qquad\quad + 2e_{321}\partial_1\Delta\partial_j\dot{c}_{2j12} - 2c_s^2 e_{321}\partial_1^2\Delta\dot{\rho}.\end{aligned} \tag{2.45}$$

We note that the above three equations are linearly dependent, and one of them is redundant.

Taking $\theta=\mathbf{e}_2$ and $\alpha=\mathbf{e}_1$, we have

$$\begin{aligned}&4e_{123}\partial_2^2\partial_3\partial_j\dot{c}_{2j21}+4e_{132}\partial_2^3\partial_j\dot{c}_{3j21} + e_{132}\Delta^2\dot{c}_{3221}\\ &\qquad = 2e_{123}\partial_2\Delta\partial_3\dot{c}_{2221} + 2e_{132}\partial_2\Delta\partial_2\dot{c}_{3221} + 2e_{132}\partial_2\Delta\partial_j\dot{c}_{3j21},\end{aligned} \tag{2.46}$$

$$4e_{213}\partial_2^2\partial_3\partial_j\dot{c}_{1j21} + 4e_{231}\partial_2^2\partial_1\partial_j\dot{c}_{3j21} = 2e_{213}\partial_2\Delta\partial_3\dot{c}_{1221} + 2e_{231}\partial_2\Delta\partial_1\dot{c}_{3221} - 2c_s^2 e_{213}\partial_2\partial_3\Delta\dot{\rho}, \tag{2.47}$$

$$\begin{aligned}&4e_{321}\partial_2^2\partial_1\partial_j\dot{c}_{2j21}+4e_{312}\partial_2^3\partial_j\dot{c}_{1j21} + e_{312}\Delta^2\dot{c}_{1221} - c_s^2 e_{312}\Delta^2\dot{\rho}\\ &\qquad = 2e_{321}\partial_2\Delta\partial_1\dot{c}_{2221} + 2e_{312}\partial_2\Delta\partial_2\dot{c}_{1221}\\ &\qquad\quad + 2e_{312}\partial_2\Delta\partial_j\dot{c}_{1j21} - 2c_s^2 e_{312}\partial_2^2\Delta\dot{\rho}.\end{aligned} \tag{2.48}$$

2.5. **Uniqueness of the parameters.** Keep in mind that all the parameters $\dot{c}_{ijk\ell}$ are compactly supported. Applying ∂_1^{-1} to (2.43) and ∂_2^{-1} to (2.47), we get

$$4\partial_1\partial_3\partial_j\dot{c}_{2j12} - 4\partial_1\partial_2\partial_j\dot{c}_{3j12} = 2\partial_3\Delta(\dot{c}_{2112} - c_s^2\dot{\rho}) - 2\Delta\partial_2\dot{c}_{3112}$$

and

$$4\partial_2\partial_3\partial_j\dot{c}_{1j21} - 4\partial_2\partial_1\partial_j\dot{c}_{3j21} = 2\partial_3\Delta(\dot{c}_{2112} - c_s^2\dot{\rho}) - 2\Delta\partial_1\dot{c}_{3221},$$

since $\dot{\mathbf{C}}$ is compactly supported. The above two equations lead to

$$2\partial_1\partial_3\partial_j\dot{c}_{2j12} - 2\partial_2\partial_3\partial_j\dot{c}_{1j21} = \Delta(\partial_1\dot{c}_{3221} - \partial_2\dot{c}_{3112}).$$

Inserting (2.19) and (2.21) into the above equation, we have

$$\frac{2c_p^2}{c_p^2 - c_s^2}(\partial_1\partial_3\partial_2\dot{c}_{2221} - \partial_2\partial_3\partial_1\dot{c}_{1112}) = \Delta(\partial_1\dot{c}_{3221} - \partial_2\dot{c}_{3112}).$$

Since we already have $\dot{c}_{2221} = \dot{c}_{1112}$, we now have $\Delta(\partial_1 \dot{c}_{3221} - \partial_2 \dot{c}_{3112}) = 0$ in $\mathbb{R}^3$ and consequently

$$\partial_1 \dot{c}_{3221} - \partial_2 \dot{c}_{3112} = 0. \tag{2.49}$$

Using again (2.23) and (2.24), which imply that

$$\partial_2^2(2\dot{c}_{1212} - c_s^2\dot{\rho}) + 2\partial_2\partial_3\dot{c}_{1213} = \partial_1^2(2\dot{c}_{1212} - c_s^2\dot{\rho}) + 2\partial_1\partial_3\dot{c}_{1223},$$

we have

$$(\partial_1^2 - \partial_2^2)(2\dot{c}_{1212} - c_s^2\dot{\rho}) = 0.$$

So

$$2\dot{c}_{1212} - c_s^2\dot{\rho} = 0.$$

Now (2.45) reduces to

$$\begin{aligned}&4\partial_1^3\partial_2\dot{c}_{1112} + 4\partial_1^2\partial_2\partial_3\dot{c}_{1312} - 4\partial_1^3\partial_2\dot{c}_{2212} - 4\partial_1^3\partial_3\dot{c}_{2312} + 4\partial_1^2\partial_2^2\dot{c}_{1212} - 4\partial_1^4\dot{c}_{1212} + \Delta^2\dot{c}_{1212}\\ =&2\Delta\partial_1\partial_2\dot{c}_{1112} - 2\Delta\partial_1\partial_2\dot{c}_{2212} - 2\Delta\partial_1\partial_3\dot{c}_{2312}.\end{aligned}$$

Recalling that $\dot{c}_{1112} - \dot{c}_{2212} = 0$ and $\partial_2\dot{c}_{1312} - \partial_1\dot{c}_{2312} = 0$, the above identity leads to

$$2\Delta\partial_1\partial_3\dot{c}_{2312} + 4\partial_1^2\partial_2^2\dot{c}_{1212} - 4\partial_1^4\dot{c}_{1212} + \Delta^2\dot{c}_{1212} = 0. \tag{2.50}$$

Similar consideration applied to (2.48) gives

$$2\Delta\partial_2\partial_3\dot{c}_{1312} + 4\partial_1^2\partial_2^2\dot{c}_{1212} - 4\partial_2^4\dot{c}_{1212} + \Delta^2\dot{c}_{1212} = 0. \tag{2.51}$$

Subtracting (2.51) from (2.50), we obtain

$$(\partial_1^4 - \partial_2^4)\dot{c}_{1212} = 0,$$

from which we get

$$\dot{c}_{1212} = 0.$$

Then $\dot{\rho} = 0$ and also, by (2.50) and (2.51),

$$\Delta\partial_1\partial_3\dot{c}_{2312} = \Delta\partial_2\partial_3\dot{c}_{1312} = 0$$

Consequently

$$\dot{c}_{2312} = \dot{c}_{3112} = 0.$$

Using again (2.23) and (2.24), we get $\partial_1\dot{c}_{1112} = \partial_2\dot{c}_{2221} = 0$ and then

$$\dot{c}_{1112} = \dot{c}_{2221} = 0.$$

Now we already have $\dot{c}_{1112} = \dot{c}_{1212} = \dot{c}_{2212} = \dot{c}_{1312} = \dot{c}_{2312} = 0$. Using (2.20), we have

$$\dot{c}_{3312} = 0.$$

Summarizing above results, we have

$$\dot{c}_{ij12} = 0, \quad i,j = 1,2,3.$$

By the same consideration with different choices of $\theta = \mathbf{e}_i$ and $\alpha = \mathbf{e}_j$, with $i,j = 1,2,3$, $i \neq j$, we can also have

$$\dot{c}_{ij13} = \dot{c}_{ij23} = 0, \quad i,j = 1,2,3.$$

The identities (2.36) and (2.38) then reduce to

$$\begin{aligned}\partial_3\partial_1\dot{c}_{1111} - \partial_1\partial_3\dot{c}_{3311} = 0,\\ \partial_2\partial_1\dot{c}_{1111} - \partial_1\partial_2\dot{c}_{2211} = 0.\end{aligned}$$

This yields

$$\dot{c}_{1111} = \dot{c}_{2211} = \dot{c}_{3311}.$$

Now the equation (2.9) reduces to

$$4\partial_1^2\Delta\dot{c}_{1111} + \Delta^2\dot{c}_{1111} = 4\partial_1^2\Delta\dot{c}_{1111}.$$

So

$$\Delta^2\dot{c}_{1111} = 0,$$

and consequently $\dot{c}_{2211} = \dot{c}_{3311} = \dot{c}_{1111} = 0$. Similarly, we have

$$\dot{c}_{2222} = \dot{c}_{3333} = \dot{c}_{2233} = 0.$$

Now we have shown that all components of $\dot{\mathbf{C}}$ and $\dot{\rho}$ vanish. This proves the injectivity of $\mathcal{A}'[\mathbf{C}_0, 1]$, while we have only used θ, α to be $\mathbf{e}_j$'s.

3. Rigidity of an isotropic perturbation

In this section we consider the isotropic case, that is, when

$$\mathbf{C} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}),$$

and prove Theorem 1.2. Let us first go back to the linearized problem, but now we need to give it a quantitative study. For this isotropic case, we do not need to use P-S or S-P scattering. Also, we will only use incident waves in a single direction.

3.1. **Send P-wave, obverse P-wave.** By using a P-wave $u_0 = \theta H_0(c_p t - \theta\cdot x)$ as the incident wave, the total wave field u satisfies the equation

$$\begin{cases} (\rho\partial_t^2 - \mathcal{L}_{\lambda,\mu})u = 0, & \text{in} \quad \mathbb{R}^3\times\mathbb{R}, \\ u = \theta H_0(c_p t - \theta\cdot x), & \text{for} \quad t \ll 0, \end{cases} \tag{3.1}$$

where $\mathcal{L}_{\lambda,\mu}u = \mu\Delta u + (\lambda+\mu)\nabla\nabla\cdot u$, and $|\theta| = 1$. By linearizing (ρ, λ, μ) at $(\rho_0 = 1, \lambda_0, \mu_0)$, we obtain

$$\begin{cases} (\partial_t^2 - \mathcal{L}_{\lambda_0,\mu_0})\dot{u} = (\mathcal{L}_{\dot{\lambda},\dot{\mu}} - \dot{\rho}\partial_t^2)u_0, & \text{in} \quad \mathbb{R}^3\times\mathbb{R}, \\ \dot{u} = 0, & \text{for} \quad t \ll 0. \end{cases}$$

We compute

$$\begin{aligned} \mathcal{L}_{\dot{\lambda},\dot{\mu}}u_0 &= -\nabla\cdot(\dot{\lambda}\delta I + 2\dot{\mu}(\theta\times\theta)\delta) = -\nabla(\dot{\lambda}\delta) - 2\theta(\theta\cdot\nabla(\dot{\mu}\delta)) \\ &= -[(\nabla\dot{\lambda})\delta - \dot{\lambda}\theta\delta'] - [2(\theta\cdot\nabla\dot{\mu})\theta\delta - 2\dot{\mu}\theta\delta'] \\ &= (\dot{\lambda} + 2\dot{\mu})\theta\delta' - [(\nabla\dot{\lambda}) + 2\theta(\theta\cdot\nabla\dot{\mu})]\delta, \end{aligned} \tag{3.2}$$

and

$$\dot{\rho}\partial_t^2 u_0 = \dot{\rho}c_p^2\theta\delta'.$$

So the single scattered wave $\dot{u}$ solves the following equation.

$$\begin{cases} (\partial_t^2 - \mathcal{L}_{\lambda_0,\mu_0})\dot{u} = (\dot{\lambda} + 2\dot{\mu} - \dot{\rho}c_p^2)\theta\delta' - [\nabla\dot{\lambda} + 2\theta(\theta\cdot\nabla\dot{\mu})]\delta, & \text{in} \quad \mathbb{R}^3\times\mathbb{R}, \\ \dot{u} = 0, & \text{for} \quad t \ll 0, \end{cases} \tag{3.3}$$

We consider $\nabla\cdot\dot{u}$ which encodes the compressional wave. By taking the divergence of both sides, we have

$$\begin{cases} \Box_p(\nabla\cdot\dot{u}) = -[\Delta\dot{\lambda} + 2(\theta\cdot\nabla)^2\dot{\mu}]\delta \\ \qquad\qquad + (\theta\cdot\nabla)(2\dot{\lambda} + 4\dot{\mu} - \dot{\rho}c_p^2)\delta' \\ \qquad\qquad - (\dot{\lambda} + 2\dot{\mu} - \dot{\rho}c_p^2)\delta'', & \text{in} \quad \mathbb{R}^3\times\mathbb{R}, \\ \nabla\cdot\dot{u} = 0, & \text{for} \quad t \ll 0, \end{cases} \tag{3.4}$$

By the discussion in Section 2.1, we can use the following ansatz

$$\nabla\cdot\dot{u}=w_2(x)\delta'(c_pt-\theta\cdot x)+w_1(x)\delta(c_pt-\theta\cdot x)+w_0(x,t)H_0(c_pt-\theta\cdot x),$$

with $w_2(x)=w_1(x)=0$ when $\theta\cdot x\ll 0$ and $w_0(x,t)=0$ when $t\ll 0$. By computation, we have

$$\Box_p(\nabla\cdot\dot{u})=(\Box_pw_0)H_0+(L_pw_0-c_p^2\Delta w_1)\delta+(2c_p^2\theta\cdot\nabla w_1-c_p^2\Delta w_2)\delta'+(2c_p^2\theta\cdot\nabla w_2)\delta'', \tag{3.5}$$

where $L_p:=2c_p(\partial_t+c_p\theta\cdot\nabla)$. Note that when acting on functions independent of t such as w_1 and w_2, L_p will be the same as $2c_p^2\theta\cdot\nabla$, so we also use the notation

$$L_\theta:=\theta\cdot\nabla$$

when it is appropriate. Comparing different singularities in (3.4) and (3.5) gives

$$\begin{cases}\Box_pw_0=0, & \text{in } \{c_pt-\theta\cdot x>0\},\\ L_pw_0-c_p^2\Delta w_1=-[\Delta\dot{\lambda}+2(\theta\cdot\nabla)^2\dot{\mu}]=:F_0, & \text{on } \{c_pt-\theta\cdot x=0\},\\ 2c_p^2\theta\cdot\nabla w_1-c_p^2\Delta w_2=(\theta\cdot\nabla)(2\dot{\lambda}+4\dot{\mu}-\dot{\rho}c_p^2)=:F_1, & \text{in } \mathbb{R}^3,\\ 2c_p^2\theta\cdot\nabla w_2=-(\dot{\lambda}+2\dot{\mu}-\dot{\rho}c_p^2)=:F_2, & \text{in } \mathbb{R}^3.\end{cases}$$

The last two equations on $\mathbb{R}^3$ are transport equations for $w_1(x), w_2(x)$. Note that F_1 and F_2 are compactly supported, so we can successively solve the transport equations to determine w_2, w_1. That means that we can define the inverse of L_θ, e.g., L_θ^{-1} and write

$$w_2=\frac{1}{2c_p^2}L_\theta^{-1}F_2,\quad w_1=\frac{1}{2c_p^2}L_\theta^{-1}(c_p^2\Delta w_2+F_1).$$

So we can simplify the system as follows,

$$\begin{cases}\Box_pw_0=0, & \text{in } \{c_pt-\theta\cdot x>0\},\\ L_pw_0=\displaystyle\sum_{r=0}^{2}\left(\frac{\Delta}{2}L_\theta^{-1}\right)^rF_r, & \text{on } \{c_pt-\theta\cdot x=0\}.\end{cases} \tag{3.6}$$

Recall $\Omega:=\{x\in\mathbb{R}^3:|x|<1\}$. Denote

$$\Sigma_{T,p}:=\Sigma_{T,p,\theta}:=\big(\partial\Omega\times(-\infty,T]\big)\cap\{c_pt-\theta\cdot x>0\}$$

and

$$\mathcal{D}_p(\theta):=\mathcal{D}_p(\dot{\lambda},\dot{\mu},\dot{\rho};\theta):=\sum_{r=0}^{2}\left(\frac{\Delta}{2}L_\theta^{-1}\right)^rF_r.$$

Applying [16, Proposition 1.4] to (3.6), we can have

$$\|\mathcal{D}_p(\theta)\|_{L^2(\Omega)}\lesssim\|w_0(\cdot,\cdot;\theta,\theta)\|_{H^2(\Sigma_{T,p})}=\|\nabla\cdot\dot{u}(\cdot,\cdot;\theta,\theta)\|_{H^2(\Sigma_{T,p})}. \tag{3.7}$$

Here $w_0(\cdot,\cdot;\theta,\theta)$ signifies w_0 with an emphasis on the dependence of w_0 on θ. The last equal sign in (3.7) is due to the fact that $\nabla\cdot\dot{u}=w_0$ on $\Sigma_{T,p}$.

We directly compute

$$\begin{aligned}L_\theta^2\mathcal{D}_p(\theta)&=-(L_\theta^2\Delta\dot{\lambda}+2L_\theta^4\dot{\mu})+\frac{\Delta}{2}L_\theta^2(2\dot{\lambda}+4\dot{\mu}-\dot{\rho}c_p^2)-\frac{\Delta^2}{4}(\dot{\lambda}+2\dot{\mu}-\dot{\rho}c_p^2)\\&=-\frac{\Delta^2}{4}\dot{\lambda}+2\left(L_\theta^2\Delta-L_\theta^4-\frac{\Delta^2}{4}\right)\dot{\mu}+c_p^2\left(\frac{\Delta^2}{4}-\frac{L_\theta^2\Delta}{2}\right)\dot{\rho}.\end{aligned} \tag{3.8}$$

3.2. **Send S-wave, observe S-wave.** By using a shear wave $u_0 = \alpha H_0(c_s t - \theta \cdot x)$ as the incident wave, we have

$$\begin{cases} (\rho \partial_t^2 - \mathcal{L}_{\lambda,\mu}) u = 0, & \text{in} \quad \mathbb{R}^3 \times \mathbb{R}, \\ u = \alpha H_0(c_s t - \theta \cdot x), & \text{for} \quad t \ll 0, \end{cases} \tag{3.9}$$

with $|\alpha| = 1$ and $\alpha \perp \theta$. Similar to Section 3.1, we now have

$$\begin{cases} (\partial_t^2 - \mathcal{L}_{\lambda_0,\mu_0}) \dot{u} = (\dot{\mu} - \dot{\rho} c_s^2) \alpha \delta' - (\theta(\alpha \cdot \nabla \dot{\mu}) + \alpha(\theta \cdot \nabla \dot{\mu}))\delta, & \text{in} \quad \mathbb{R}^3 \times \mathbb{R}, \\ \dot{u} = 0, & \text{for} \quad t \ll 0. \end{cases} \tag{3.10}$$

Consider $\nabla \times \dot{u}$ which encodes the S wave. By taking the curl of both sides, we obtain

$$\begin{cases} \Box_s(\nabla \times \dot{u}) = -\big[(\theta \times \nabla)(\alpha \cdot \nabla) + (\alpha \times \nabla)(\theta \cdot \nabla)\big] \dot{\mu} \delta \\ \qquad\qquad + \big[(\theta \times \alpha)(\theta \cdot \nabla)\dot{\mu} - (\alpha \times \nabla)(\dot{\mu} - \dot{\rho} c_s^2)\big] \delta' \\ \qquad\qquad + (\alpha \times \theta)(\dot{\mu} - \dot{\rho} c_s^2) \delta'', & \text{in} \quad \mathbb{R}^3 \times \mathbb{R}, \\ \quad \nabla \times \dot{u} = 0, & \text{for} \quad t \ll 0, \end{cases} \tag{3.11}$$

By the discussions in Section 2.4, we use the ansatz

$$\nabla \times \dot{u} = w_2(x) \delta'(c_s t - \theta \cdot x) + w_1(x) \delta(c_s t - \theta \cdot x) + w_0(x,t) H(c_s t - \theta \cdot x),$$

where w_2, w_1 and w_0 are vector-valued functions, and $w_2(x) = w_1(x) = 0$ when $\theta \cdot x \ll 0$ and $w_0(x,t) = 0$ when $t \ll 0$. We compute

$$\Box_s(\nabla \times \dot{u}) = (\Box_s w_0) H_0 + (L_s w_0 - c_s^2 \Delta w_1)\delta + (2c_s^2 \theta \cdot \nabla w_1 - c_s^2 \Delta w_2)\delta' + (2c_s^2 \theta \cdot \nabla w_2)\delta'', \tag{3.12}$$

where $L_s = 2c_s(\partial_t + c_s \theta \cdot \nabla)$. Note that when acting on functions independent of t such as w_1 and w_2, L_s will be the same as $2c_s^2 \theta \cdot \nabla = 2c_s^2 L_\theta$. Comparing different singularities in (3.11) and (3.12) gives

$$\begin{cases} \Box_s w_0 = 0, & \text{in} \ \ \{c_s t - \theta \cdot x > 0\}, \\ L_s w_0 - c_s^2 \Delta w_1 = -\big[(\theta \times \nabla)(\alpha \cdot \nabla) + (\alpha \times \nabla)(\theta \cdot \nabla)\big] \dot{\mu} =: G_0, & \text{on} \ \ \{c_s t - \theta \cdot x = 0\}, \\ 2c_s^2 \theta \cdot \nabla w_1 - c_s^2 \Delta w_2 = \big[(\theta \times \alpha)(\theta \cdot \nabla)\dot{\mu} - (\alpha \times \nabla)(\dot{\mu} - \dot{\rho} c_s^2)\big] =: G_1, & \text{on} \ \ \mathbb{R}^3, \\ 2c_s^2 \theta \cdot \nabla w_2 = (\alpha \times \theta)(\dot{\mu} - \dot{\rho} c_s^2) =: G_2, & \text{on} \ \ \mathbb{R}^3. \end{cases}$$

Similar to the P-wave case, we can simplify the system as follows,

$$\begin{cases} \Box_s w_0 = 0, & \text{in} \ \ \{c_s t - \theta \cdot x > 0\}, \\ L_s w_0 = \displaystyle\sum_{r=0}^{2} \left(\frac{\Delta}{2} L_\theta^{-1} \right)^r G_r, & \text{on} \ \ \{c_s t - \theta \cdot x = 0\}. \end{cases} \tag{3.13}$$

Denote

$$\Sigma_{T,s} := \Sigma_{T,s,\theta} := \big(\partial\Omega \times (-\infty, T]\big) \cap \{c_s t - \theta \cdot x > 0\}$$

and

$$\mathcal{D}_s(\theta, \alpha) := \mathcal{D}_s(\dot{\lambda}, \dot{\mu}, \dot{\rho}; \theta, \alpha) := \sum_{r=0}^{2} \left(\frac{\Delta}{2} L_\theta^{-1} \right)^r G_r.$$

Using [16, Proposition 1.4] to (3.13) we can have

$$\|\mathcal{D}_s(\theta, \alpha)\|_{L^2(\Omega)} \lesssim \|w_0(\cdot, \cdot; \theta, \alpha)\|_{H^2(\Sigma_{T,s})} = \|\nabla \times \dot{u}(\cdot, \cdot; \theta, \alpha)\|_{H^2(\Sigma_{T,s})}. \tag{3.14}$$

The last equal sign in (3.14) is due to the fact that $\nabla \times \dot{u} = w_0$ on $\Sigma_{T,s}$. We can directly compute

$$\begin{aligned} L_\theta^2 \mathcal{D}_s(\theta,\alpha) &= -L_\theta^2\big[(\theta\times\nabla)(\alpha\cdot\nabla)+(\alpha\times\nabla)(\theta\cdot\nabla)\big]\dot{\mu} \\ &\quad + \frac{\Delta}{2}L_\theta\big[(\theta\times\alpha)(\theta\cdot\nabla)\dot{\mu} - (\alpha\times\nabla)(\dot{\mu}-\dot{\rho}c_s^2)\big] - \frac{\Delta^2}{4}\big[(\alpha\times\theta)(\dot{\mu}-\dot{\rho}c_s^2)\big]. \end{aligned} \tag{3.15}$$

Now we choose a coordinate frame $\{\boldsymbol{e_1}=\theta, \boldsymbol{e_2}=\alpha, \boldsymbol{e_3}=\theta\times\alpha\}$. Rewriting (3.8) and (3.15) under this coordinates system gives

$$\partial_1^2\mathcal{D}_p(\boldsymbol{e_1}) = -\frac{\Delta^2}{4}\dot{\lambda} + 2\left(\partial_1^2\Delta - \partial_1^4 - \frac{\Delta^2}{4}\right)\dot{\mu} + \left(\frac{\Delta^2}{4} - \frac{\Delta}{2}\partial_1^2\right)\dot{\rho}c_p^2, \tag{3.16}$$

and

$$\begin{aligned} \partial_1^2\mathcal{D}_s(\boldsymbol{e_1},\boldsymbol{e_2}) &= \boldsymbol{e_1}\left[\partial_{13}^2\frac{\Delta}{2}\dot{\rho}c_s^2 - \left(\partial_{1113}^4 + \frac{\Delta}{2}\partial_{13}^2\right)\dot{\mu}\right] + \boldsymbol{e_2}\partial_{1123}^4\dot{\mu} \\ &\quad + \boldsymbol{e_3}\left[\left(\partial_1^4 + \Delta\partial_1^2 + \frac{\Delta^2}{4} - \partial_{12}^2\right)\dot{\mu} - \left(\partial_1^2+\frac{\Delta}{2}\right)\frac{\Delta}{2}\dot{\rho}c_s^2\right]. \end{aligned} \tag{3.17}$$

We first recover $\dot{\mu}$. From (3.17), we see the second component of the vector $\partial_1^2\mathcal{D}_s(\theta,\alpha)$ equals to $\partial_{1123}^4\dot{\mu}$, and actually the second component of the vector $\mathcal{D}_s(\theta,\alpha)$ equals to $\partial_{23}^2\dot{\mu}$. Hence, by (3.14) we can have

$$\|\dot{\mu}\|_{L^2(\Omega)} \lesssim \|\partial_{23}^2\dot{\mu}\|_{L^2(\Omega)} \lesssim \|\mathcal{D}_s(\boldsymbol{e_1},\boldsymbol{e_2})\|_{L^2(\Omega)} \lesssim \|\nabla\times\dot{u}(\cdot,\cdot;\boldsymbol{e_1},\boldsymbol{e_2})\|_{H^2(\Sigma_{T,s})}.$$

Therefore, when either acting differential operators $\partial_t + c_s\theta\cdot\nabla$ or $\alpha\cdot\nabla$ multiple times to (3.13) and using similar arguments as above, we can also obtain for any multi-index β,

$$\|\partial^\beta\dot{\mu}\|_{L^2(\Omega)} \lesssim \|\nabla_{x,t}^{|\beta|}\nabla\times\dot{u}(\cdot,\cdot;\boldsymbol{e_1},\boldsymbol{e_2})\|_{H^2(\Sigma_{T,s})}. \tag{3.18}$$

Next we recover $\dot{\rho}$. From (3.17), we see the third component of $\partial_1^2\mathcal{D}_s(\boldsymbol{e_1},\boldsymbol{e_2})$ equals to $(\partial_1^4 + \Delta\partial_1^2 + \frac{\Delta^2}{4} - \partial_{12}^2)\dot{\mu} - (\partial_1^2+\frac{\Delta}{2})\frac{\Delta}{2}\dot{\rho}c_s^2$, so by (3.14) we have

$$\begin{aligned} \|\nabla_{x,t}^2\nabla\times\dot{u}(\cdot,\cdot;\boldsymbol{e_1},\boldsymbol{e_2})\|_{H^2(\Sigma_{T,s})} &\gtrsim \|\partial_1^2\mathcal{D}_s(\boldsymbol{e_1},\boldsymbol{e_2})\|_{L^2(\Omega)} \\ &\gtrsim \left\|\left(\partial_1^4 + \Delta\partial_1^2 + \frac{\Delta^2}{4} - \partial_{12}^2\right)\dot{\mu} - \left(\partial_1^2+\frac{\Delta}{2}\right)\frac{\Delta}{2}\dot{\rho}c_s^2\right\|_{L^2(\Omega)} \\ &\gtrsim \|(2\partial_1^2+\Delta)\Delta\dot{\rho}\|_{L^2(\Omega)} - \|\dot{\mu}\|_{H^4(\Omega)}. \end{aligned}$$

Note that the operator $-(2\partial_1^2+\Delta)$ is elliptic and $\dot{\rho}$ has zero trace on $\partial\Omega$, so by the interior regularity of elliptic operator, together with the Ponicaré inequality, we have

$$\|(2\partial_1^2+\Delta)\Delta\dot{\rho}\|_{L^2(\Omega)} \lesssim \|\Delta\dot{\rho}\|_{H^2(\Omega)} \lesssim \|\dot{\rho}\|_{H^4(\Omega)}.$$

Moreover, by (3.18) we have $\|\dot{\mu}\|_{H^4(\Omega)} \lesssim \|\nabla_{x,t}^4\nabla\times\dot{u}(\cdot,\cdot;\boldsymbol{e_1},\boldsymbol{e_2})\|_{H^2(\Sigma_{T,s})}$, thus we can continue the computation,

$$\|\nabla_{x,t}^2\nabla\times\dot{u}(\cdot,\cdot;\boldsymbol{e_1},\boldsymbol{e_2})\|_{H^2(\Sigma_{T,s})} \gtrsim \|\dot{\rho}\|_{H^4(\Omega)} - \|\nabla_{x,t}^4\nabla\times\dot{u}(\cdot,\cdot;\boldsymbol{e_1},\boldsymbol{e_2})\|_{H^2(\Sigma_{T,s})}. \tag{3.19}$$

Finally we recover $\dot{\lambda}$. Similarly, applying $(\partial_t + c_p\theta\cdot\nabla)^2$ to (3.6) and using (3.16), we have

$$\begin{aligned} &\|\nabla_{x,t}^2\nabla\cdot\dot{u}(\cdot,\cdot;\boldsymbol{e_1},\boldsymbol{e_1})\|_{H^2(\Sigma_{T,p})} \\ \gtrsim &\left\|-\left(\frac{\Delta}{2}\right)^2\dot{\lambda} + 2\left[\Delta\partial_1^2 - \partial_1^4 - \left(\frac{\Delta}{2}\right)^2\right]\dot{\mu} + \left[\left(\frac{\Delta}{2}\right)^2 - \frac{\Delta}{2}\partial_1^2\right]\dot{\rho}c_p^2\right\|_{L^2(\Omega)} \end{aligned}$$

$$\gtrsim \|\Delta^2 \dot\lambda\|_{L^2(\Omega)} - \sum_{|\beta|=4} \|\partial^\beta \dot\mu\|_{L^2(\Omega)} - \sum_{|\beta|=4} \|\partial^\beta \dot\rho\|_{L^2(\Omega)}$$
$$\gtrsim \|\dot\lambda\|_{H^4(\Omega)} - \|\dot\mu\|_{H^4(\Omega)} - \|\dot\rho\|_{H^4(\Omega)}.$$

Combining this with (3.18) and (3.19), we finally have

$$\begin{aligned} &\|\dot\lambda\|_{H^4(\Omega)} + \|\dot\mu\|_{H^4(\Omega)} + \|\dot\rho\|_{H^4(\Omega)} \\ &\lesssim \|\nabla^2_{x,t} \nabla \cdot \dot u(\cdot,\cdot;\boldsymbol{e_1},\boldsymbol{e_1})\|_{H^2(\Sigma_{T,p})} + \sum_{j\le 4} \|\nabla^j_{x,t} \nabla \times \dot u(\cdot,\cdot;\boldsymbol{e_1},\boldsymbol{e_2})\|_{H^2(\Sigma_{T,s})} \\ &\lesssim \|\nabla^3_{x,t} \dot u(\cdot,\cdot;\boldsymbol{e_1},\boldsymbol{e_1})\|_{H^2(\Sigma_{T,p})} + \|\nabla^5_{x,t} \dot u(\cdot,\cdot;\boldsymbol{e_1},\boldsymbol{e_2})\|_{H^2(\Sigma_{T,s})}. \end{aligned} \tag{3.20}$$

Remember that $\dot u(\cdot,\cdot;\boldsymbol{e_1},\boldsymbol{e_1})$ is the linearization of the solution of $(\rho\partial_t^2 - \mathcal{L}_{\lambda,\mu})u = 0$ the with incident P wave $u_0 = \boldsymbol{e_1} H_0(c_p t - \boldsymbol{e_1}\cdot x)$, and $\dot u(\cdot,\cdot;\boldsymbol{e_1},\boldsymbol{e_2})$ is the linearization of the solution with the incident S wave $u_0 = \boldsymbol{e_2} H_0(c_s t - \boldsymbol{e_1}\cdot x)$. We obtained the Lipschitz stability.

3.3. **The nonlinear map.** To derive uniqueness and stability for the local rigidity problem, we shall use the following abstract result from [26, Theorem 2].

Proposition 3.1. *Let $\mathcal{X}_j$, $\mathcal{Y}_j$ with $j = 1,2,3$ be Banach spaces with $\mathcal{X}_3 \subset \mathcal{X}_1 \subset \mathcal{X}_2$ and $\mathcal{Y}_3 \subset \mathcal{Y}_2 \subset \mathcal{Y}_1$, such that the following interpolation estimates hold:*

$$\|f\|_{\mathcal{X}_1} \lesssim \|f\|_{\mathcal{X}_2}^{\mu_1} \|f\|_{\mathcal{X}_3}^{1-\mu_1}, \quad \|g\|_{\mathcal{Y}_2} \lesssim \|g\|_{\mathcal{Y}_1}^{\mu_2} \|g\|_{\mathcal{Y}_3}^{1-\mu_2}, \quad \mu_1,\mu_2 \in (0,1],\ \ \mu_1\mu_2 > 1/2. \tag{3.21}$$

Let $\mathcal{A} : \mathcal{V}_1 \to \mathcal{Y}_1$ be a nonlinear map where $\mathcal{V}_1 \subset \mathcal{X}_1$ is an open subset of $\mathcal{X}_1$. Consider $f_0 \in \mathcal{V}_1$ and assume that

$$\mathcal{A}(f) = \mathcal{A}(f_0) + A_{f_0}(f - f_0) + R_{f_0}(f), \quad \|R_{f_0}(f)\|_{\mathcal{Y}_1} \le C(f_0)\|f - f_0\|^2_{\mathcal{X}_1} \tag{3.22}$$

holds for all f in some neighborhood of f_0 in $\mathcal{V}_1$. Here A_{f_0} stands for the Fréchet derivative of $\mathcal{A}$ at f_0. In addition, suppose that

$$\|h\|_{\mathcal{X}_2} \le C\|A_{f_0} h\|_{\mathcal{Y}_2}, \quad h \in \mathcal{X}_1 \tag{3.23}$$

and

$$\|A_{f_0} h\|_{\mathcal{Y}_3} \le C\|h\|_{\mathcal{X}_3}, \quad h \in \mathcal{X}_3. \tag{3.24}$$

Then for any $L > 0$ there exists $\epsilon > 0$, so that for any f with

$$\|f - f_0\|_{\mathcal{X}_1} \le \epsilon, \quad \|f\|_{\mathcal{X}_3} \le L, \tag{3.25}$$

one has the conditional stability estimate

$$\|f - f_0\|_{\mathcal{X}_1} \le C L^{2-\mu_1-\mu_2} \|\mathcal{A}(f) - \mathcal{A}(f_0)\|_{\mathcal{Y}_1}^{\mu_1\mu_2}.$$

In particular, if $\mathcal{A}(f) = \mathcal{A}(f_0)$ for some f satisfying (3.25), then $f = f_0$.

We are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Denote $\kappa := (\lambda,\mu,\rho)$ and $\kappa_0 := (\lambda_0,\mu_0,1)$. Abbreviate $P_\kappa := \rho\partial_t^2 - \mathcal{L}_{\lambda,\mu}$ and $P_{\kappa_0} := \partial_t^2 - \mathcal{L}_{\lambda_0,\mu_0}$. Let $P_\kappa u = 0$, $P_{\kappa_0} u_0 = 0$ with initial conditions as in either (3.1) or (3.9). We use u^p to indicate the wave corresponding to the initial condition $u^p = \theta H_0(c_p t - \theta\cdot x)$ when $t \ll 0$, and use u^s to indicate the wave corresponding to the initial condition $u^s = \alpha H_0(c_s t - \theta\cdot x)$ when $t \ll 0$. The superscripts p and s represent compressional and shear waves.

Denote the following nonlinear map

$$\mathcal{A}\colon \kappa \ \mapsto\ (u^p, u^s)|_{\Omega^c\times(-\infty,T]}.$$

We want to apply Proposition 3.1 to the operator $\mathcal{A}$. To that end, we let $\dot{u}$ satisfy $P_{\kappa_0}\dot{u} = -P_{\dot{\kappa}}u_0$ with the initial condition $\dot{u} = 0$ when $t \ll 0$. Let $r = u - u_0 - \dot{u}$ with the initial condition $r = 0$ when $t \ll 0$. To apply Proposition 3.1, we need to check the following

$$\|(r^p, r^s)\|_{H^{-k}((-\infty,T);H^{s+2}(\Omega^c))} \leq C\|\kappa - \kappa_0\|^2_{H^{s_0}(\Omega)}, \tag{3.26}$$

$$\|\dot{\kappa}\|_{L^2(\Omega)} \leq C\|(\dot{u}^p, \dot{u}^s)\|_{H^{10}((-\infty,T);H^{10}(\Omega^c))}, \tag{3.27}$$

$$\|(\dot{u}^p, \dot{u}^s)\|_{H^{\alpha}((-\infty,T);H^{\beta}(\Omega^c))} \leq C\|\dot{\kappa}\|_{H^N(\Omega)}. \tag{3.28}$$

The corresponding function spaces are specified as follows

$$\begin{aligned} &\mathcal{X}_1 = H^{s_0}(\Omega), && \mathcal{X}_2 = L^2(\Omega), && \mathcal{X}_3 = H^{\alpha}((-\infty,T);H^{\beta}(\Omega^c)),\\ &\mathcal{Y}_1 = H^{-k}H^{s+2}(\Omega^c), && \mathcal{Y}_2 = H^{10}((-\infty,T);H^{10}(\Omega^c)), && \mathcal{Y}_3 = H^N(\Omega), \end{aligned}$$

where α, β and N are chosen large enough such that the inclusion relation in Proposition 3.1 is satisfied, and $k \geq 0$ be an integer, and let $-s_0 \leq s \leq s_0$. The conditions (3.26)–(3.28) correspond to (3.22)–(3.24), respectively. We verify conditions (3.26)-(3.28) one by one.

Condition (3.26)**:** r satisfies

$$P_{\kappa_0}r = P_{\kappa_0}(u - u_0 - \dot{u}) = P_{\kappa_0-\kappa}u + P_{\kappa-\kappa_0}u_0 = P_{\kappa-\kappa_0}(u_0 - u).$$

By Helmholtz decomposition we could divide R as $r = \nabla\phi + \nabla\times\vec{\psi}$, then

$$\Box_p(\Delta\phi) = \nabla\cdot\big(P_{\kappa-\kappa_0}(u_0-u)\big), \quad \text{and} \quad \Box_s(\nabla\times\nabla\times\vec{\psi}) = \nabla\times\big(P_{\kappa-\kappa_0}(u_0-u)\big).$$

In the rest of the proof, we abbreviate the $H^b(\mathbb{R}^3)$-valued Sobolev spaces $H^a((-\infty,T);H^b(\mathbb{R}^3))$ as H^aH^b for short. By [16, Lemma A.4], we have

$$\begin{aligned} \|(\nabla\cdot r, \nabla\times r)\|_{H^{-k}H^{s+1}} &= \|(\Delta\phi, \nabla\times\nabla\times\vec{\psi})\|_{H^{-k}H^{s+1}}\\ &\lesssim \|P_{\kappa-\kappa_0}(u_0-u)\|_{H^{-k}H^{s+1}}\\ &\lesssim \|\kappa-\kappa_0\|_{L^\infty(\Omega)}(\|u_0-u\|_{H^{-k+2}H^{s+1}} + \|u_0-u\|_{H^{-k}H^{s+3}}). \end{aligned}$$

Because

$$\|(\nabla\cdot r, \nabla\times r)\|_{L^2(\mathbb{R}^3)} = \|(\xi\cdot\hat{r}, \xi\times\hat{r})\|_{L^2(\mathbb{R}^3)} = \big\||\xi||\hat{r}|\big\|_{L^2(\mathbb{R}^3)} \simeq \|\nabla r\|_{L^2(\mathbb{R}^3)},$$

so

$$\|(\nabla\cdot r, \nabla\times r)\|_{H^{-k}H^{s+1}} \simeq \|\nabla r\|_{H^{-k}H^{s+1}} \simeq \|r\|_{H^{-k}H^{s+2}},$$

thus

$$\|r\|_{H^{-k}H^{s+2}} \lesssim \|\kappa-\kappa_0\|_{L^\infty(\Omega)}(\|u_0-u\|_{H^{-k+2}H^{s+1}} + \|u_0-u\|_{H^{-k}H^{s+3}}). \tag{3.29}$$

Once again, because $P_{\kappa_0}(u_0-u) = -P_{\kappa_0}u = P_{\kappa-\kappa_0}u$, so similar to the derivation of (3.29), we can have

$$\|u_0-u\|_{H^{-k}H^{s+1}} \lesssim \|\kappa-\kappa_0\|_{L^\infty(\Omega)}(\|u\|_{H^{-k+2}H^{s+1}} + \|u\|_{H^{-k}H^{s+3}}). \tag{3.30}$$

Combining (3.29) and (3.30), we obtain

$$\|r\|_{H^{-k}H^{s+2}} \lesssim \|\kappa-\kappa_0\|^2_{H^{s_0}(\Omega)}(\|u\|_{H^{-k}H^{s+5}} + \|u\|_{H^{-k+2}H^{s+3}} + \|u\|_{H^{-k+4}H^{s+1}}), \tag{3.31}$$

where $s_0 > \lceil 3/2\rceil$ so that $H^{s_0}(\Omega) \hookrightarrow L^\infty(\Omega)$. This implies (3.26).

Condition (3.27)**:** This condition is the Lipschitz stability of the linearized problem, and it is implied by (3.20). Indeed, from (3.20) we have

$$\|\dot{\kappa}\|_{L^2(\Omega)} \lesssim \|\dot{\lambda}\|_{L^2(\Omega)} + \|\dot{\mu}\|_{L^2(\Omega)} + \|\dot{\rho}\|_{L^2(\Omega)}$$

$$\lesssim \|\nabla^3_{x,t}\dot{u}(\cdot,\cdot;\boldsymbol{e_1},\boldsymbol{e_1})\|_{H^2(\Sigma_{T,p})} + \|\nabla^5_{x,t}\dot{u}(\cdot,\cdot;\boldsymbol{e_1},\boldsymbol{e_2})\|_{H^2(\Sigma_{T,s})}$$
$$\lesssim \|(\dot{u}^p,\dot{u}^s)\|_{H^{10}((-\infty,T);H^{10}(\Omega^c))}.$$

Condition (3.28)**:** This condition is an *a priori* estimate for the linearized elastic system. We know that $\dot{u}$ satisfies $P_{\kappa_0}\dot{u} = -P_{\dot{\kappa}}u_0$, thus similar to the derivation of (3.29) we have

$$\|\dot{u}\|_{H^{-k}H^{s+2}} \lesssim \|\dot{\kappa}\|_{L^\infty(\Omega)}(\|u_0\|_{H^{-k+2}H^{s+1}} + \|u_0\|_{H^{-k}H^{s+3}}) \lesssim \|\dot{\kappa}\|_{H^{s_0}(\Omega)}. \tag{3.32}$$

which further indicates (3.28).

Therefore, by Proposition 3.1, we obtain the following local Hölder inverse stability:

$$\|(\rho-\rho_0,\lambda-\lambda_0,\mu-\mu_0)\|_{H^{s_0}(\Omega)} \le C\|(u^p-u_0^p,u^s-u_0^s)\|^{\mu}_{H^{-k}((-\infty,T];H^{s+2}(\Omega^c))}.$$

The proof is complete. □

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References

[1] M. I. Belishev. An approach to multidimensional inverse problems for the wave equation. *Dokl. Akad. Nauk*, 297(3):524–527, 1987.
[2] M. I. Belishev. Recent progress in the boundary control method. *Inverse Probl.*, 23(5):R1, 2007.
[3] M. I. Belishev and Y. V. Kurylev. To the reconstruction of a Riemannian manifold via its spectral data (BC-method). *Commun. Partial Differ. Equations*, 17(5-6):767–804, 1992.
[4] S. Bhattacharyya. Local uniqueness of the density from partial boundary data for isotropic elastodynamics. *Inverse Probl.*, 34(12):125001, 2018.
[5] M. V. de Hoop, G. Nakamura, and J. Zhai. Unique recovery of piecewise analytic density and stiffness tensor from the elastic-wave Dirichlet-to-Neumann map. *SIAM J. Appl. Math.*, 79(6):2359–2384, 2019.
[6] M. V. de Hoop, T. Saksala, G. Uhlmann, and J. Zhai. Generic uniqueness and stability for the mixed ray transform. *Trans. Am. Math. Soc.*, 374(9):6085–6144, 2021.
[7] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[8] L. Hörmander. *The analysis of linear partial differential operators III: Pseudo-differential operators*. Springer Science & Business Media, 2007.
[9] J. Ilmavirta, T. Saksala, and L. Yan. Rigidity of homogeneous Lamé systems. *arXiv preprint arXiv:2602.08860*, 2026.
[10] J. Ilmavirta and H. Schlüter. Gauge freedoms in the anisotropic elastic Dirichlet-to-Neumann map. *Inverse Probl. Imaging*, 19(4):816–828, 2025.
[11] A. Katchalov and Y. Kurylev. Multidimensional inverse problem with incomplete boundary spectral data. *Commun. Partial Differ. Equations*, 23(1-2):55–95, 1998.
[12] A. Katchalov, Y. Kurylev, and M. Lassas. *Inverse boundary spectral problems*. Chapman and Hall/CRC, 2001.
[13] A. Katchalov, Y. Kurylev, and M. Lassas. *Inverse boundary spectral problems*, volume 123 of *Chapman Hall/CRC Monogr. Surv. Pure Appl. Math.* Boca Raton, FL: CRC Press, 2001.
[14] Y. Kurylev, M. Lassas, and E. Somersalo. Maxwell's equations with a polarization independent wave velocity: Direct and inverse problems. *J. Math. Pures Appl.*, 86(3):237–270, 2006.
[15] M. Lassas and L. Oksanen. Inverse problem for the Riemannian wave equation with Dirichlet data and Neumann data on disjoint sets. *Duke Math. J.*, 163(6):1071–1103, 2014.
[16] S. Ma, L. Potenciano-Machado, and M. Salo. Fixed angle inverse scattering for sound speeds close to constant. *SIAM J. Math. Anal.*, 55(4):3420–3456, 2023.
[17] S. Ma and M. Salo. Fixed angle inverse scattering in the presence of a Riemannian metric. *J. Inverse Ill-Posed Probl.*, 30(4):495–520, 2022.
[18] J. E. Marsden and T. J. R. Hughes. *Mathematical foundations of elasticity*. Dover Civil and Mechanical Engineering. Dover Publications, Inc., Mineola, NY, 1994.
[19] L. Oksanen, Rakesh, and M. Salo. Rigidity in the Lorentzian Calderón problem with formally determined data. *arXiv preprint arXiv:2409.18604*, 2024.
[20] L. Oksanen, Rakesh, and M. Salo. Rigidity for fixed angle inverse scattering for Riemannian metrics. *SIAM J. Math. Anal.*, 58(1):308–334, 2026.
[21] L. Rachele. An inverse problem in elastodynamics: uniqueness of the wave speeds in the interior. *J. Differential Equations*, 162(2):300–325, 2000.
[22] L. Rachele. Uniqueness of the density in an inverse problem for isotropic elastodynamics. *Trans. Amer. Math. Soc.*, 355(12):4781–4806, 2003.
[23] Rakesh and M. Salo. Fixed angle inverse scattering for almost symmetric or controlled perturbations. *SIAM J. Math. Anal.*, 52(6):5467–5499, 2020.
[24] V. Romanov. An inverse problem of electrodynamics. *Doklady Mathematics*, 66(2):200–205, 2002.
[25] M. Salo and Rakesh. The fixed angle scattering problem and wave equation inverse problems with two measurements. *Inverse Probl.*, 36(3):035005, 2020.
[26] P. Stefanov and G. Uhlmann. Linearizing non-linear inverse problems and an application to inverse backscattering. *J. Funct. Anal.*, 256(9):2842–2866, 2009.
[27] P. Stefanov, G. Uhlmann, and A. Vasy. Local recovery of the compressional and shear speeds from the hyperbolic DN map. *Inverse Probl.*, 34(1):014003, 2017.
[28] Y. Yang and J. Zhai. Unique determination of a transversely isotropic perturbation in a linearized inverse boundary value problem for elasticity. *Inverse Probl. Imaging*, 13(6):1309, 2019.
[29] J. Zhai. Determination of the density in the linear elastic wave equation. *J. Differential Equations*, 459:114064, 2026.

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