SCATTERING IN CORNER DOMAINS WITH ANISOTROPIC INHOMOGENEITIES: THE DEGENERATE BERNOULLI CASE

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ABSTRACT. We study the scattering behavior of an anisotropic inhomogeneous Lipschitz medium at a fixed wave number, continuing our previous work [KSS24] and using free boundary techniques from [SS25]. Our main results can be categorized into two distinct cases. In the first case, we show that in two dimensions, piecewise C^1 or convex penetrable obstacles with corners, and in higher dimensions, obstacles with edge points, always induce nontrivial scattering for any incoming wave. In the second case, we prove that piecewise C^1 obstacles with corners in two dimensions (and with edge points in higher dimensions) with angles $\notin \pi\mathbb{Q}$ always produce nontrivial scattering for any incoming wave.

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1. INTRODUCTION

1.1. **Mathematical model.** Let D be a bounded Lipschitz domain in $\mathbb{R}^n$ (where $n \geq 2$) such that $\mathbb{R}^n \setminus \overline{D}$ is connected. Within this domain, let $\rho \in L^\infty(D)$ be a positive real-valued function. Additionally, let $A \in (C^{0,1}(\overline{D}))^{n \times n}$ be a real matrix-valued symmetric function, satisfying the condition of uniform ellipticity

$$(1.1) \qquad c_{\text{ellip}}^{-1}|\xi|^2 \leq \xi \cdot A(x)\xi \leq c_{\text{ellip}}|\xi|^2 \quad \text{for a.e. } x \in D \text{ and all } \xi \in \mathbb{R}^n$$

for some constant $c_{\text{ellip}} > 0$. Under the assumption that the medium outside D is homogeneous, if we illuminate the anisotropic medium (D, A, ρ) with an incident field u^{inc}

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having a fixed wave number $\kappa > 0$ that satisfies

$$(1.2) \qquad\qquad (\Delta + \kappa^2)u^{\mathrm{inc}} = 0 \quad \text{in } \mathbb{R}^n,$$

classical scattering theory (see, e.g. [CCH23, CK19, KG08]) guarantees the existence of unique scattered field $u^{\mathrm{sc}} \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ that is outgoing (i.e., satisfies the outgoing Sommerfeld radiation condition [KSS24, Definition 1.1]). The total field $u^{\mathrm{to}} = u^{\mathrm{sc}} + u^{\mathrm{inc}}$ satisfies

$$\left(\nabla \cdot \tilde{A}(x)\nabla + \kappa^2 \tilde{\rho}(x) \right) u^{\mathrm{to}} = 0 \quad \text{in } \mathbb{R}^n,$$

where

$$\tilde{A} = A\chi_{\overline{D}} + \mathrm{Id}\chi_{\mathbb{R}^n \setminus \overline{D}} \quad \text{and} \quad \tilde{\rho} = \rho\chi_{\overline{D}} + \chi_{\mathbb{R}^n \setminus \overline{D}}.$$

Remark 1.1. For notational convenience, throughout the paper we use A to denote $\tilde{A}$.

If $A(x_0) \neq \mathrm{Id}$ or $\rho(x_0) \neq 1$ at some point $x_0 \in \partial D$, then one might expect that the obstacle (D, A, ρ) is strong enough to produce scattering effects for many incident waves. Moreover, if the boundary of D has singularity at x_0, it might be true that every incident wave scatters nontrivially (i.e. for any u^{inc} solving (1.2), $u^{\mathrm{sc}}|_{\mathbb{R}^n \setminus \overline{D}}$ is not identically zero).

Conversely, if the anisotropic medium (D, A, ρ) is nonscattering with respect to an incident field u^{inc} in the sense that $u^{\mathrm{sc}} = 0$ in $\mathbb{R}^n \setminus \overline{D}$, then the scattered field u^{sc} satisfies

$$(1.3) \quad \begin{cases} (\nabla \cdot A(x)\nabla + \kappa^2\rho(x))u^{\mathrm{sc}} = -(\nabla \cdot (A(x) - \mathrm{Id})\nabla + \kappa^2(\rho(x) - 1))u^{\mathrm{inc}} & \text{in } D, \\ u^{\mathrm{sc}} = 0, \quad \nu \cdot A\nabla u^{\mathrm{sc}} = \nu \cdot (\mathrm{Id} - A)\nabla u^{\mathrm{inc}} & \text{on } \partial D, \end{cases}$$

where ν is the inward pointing unit normal vector to ∂D. One could then hope to show that D cannot have boundary singularities. In the earlier works [CVX23, KSS24], such results were established for some anisotropic scatterers by using the free boundary techniques developed in [CV23, SS21] for nonscattering problems (see also [AI96] for the Calderón problem). In the present article we continue this line of research by covering cases not included in [KSS24], based on the new methods introduced in [SS25].

1.2. **Definitions and terminology.** Before we proceed, let us recall the key definitions, assumptions, and formalism pertaining to corner domains, as referenced in the paper's title. Suppose that $x_0 \in \partial D$ is the boundary point of interest. In some of our results, we assume that the contrast

$$(1.4) \qquad\qquad h(x) := \kappa^2(\rho(x) - 1)\chi_D$$

satisfies the following non-degeneracy condition at $x_0 \in \partial D$:

$$(1.5) \qquad \text{there is } r > 0 \text{ such that } h \in C^\alpha(\overline{D} \cap B_r(x_0)) \text{ and } h(x_0) \neq 0.$$

Recall the following definition for $n = 2$.

Definition 1.2. Let $k \in \mathbb{N} \cup \{\infty\}$. An open set D in $\mathbb{R}^2$ is said to have a *piecewise C^k boundary* if, for any x_0, there exist $\delta > 0$ and a rigid motion $\Phi : B_\delta(x_0) \to V$ into an open set $V \subset \mathbb{R}^2$ such that $\Phi(x_0) = 0$ and

$$\Phi(D \cap B_\delta(x_0)) = \{x_2 > \eta(x_1)\} \cap B_r(0),$$

where $\eta \in C([-r, r])$ and both $\eta|_{[-r,0]}$ and $\eta|_{[0,r]}$ are C^k-functions up to the endpoints of the respective closed intervals.

We also recall the following definition for $n \geq 3$.

Definition 1.3. Let D be an open set in $\mathbb{R}^n$ with $n \geq 3$. A boundary point $x_0 \in \partial D$ is said to be an *edge point* if there is $\delta > 0$ and a C^1 diffeomorphism $\Phi : B_\delta(x_0) \to V$ onto an open set $V \subset \mathbb{R}^n$ such that $\Phi(x_0) = 0$, $(\nabla \otimes \Phi)(x_0)$ is a rotation matrix, and

$$\Phi(\overline{D} \cap B_\delta(x_0)) = (S \times \mathbb{R}^{n-2}) \cap \Phi(B_\delta(x_0)),$$

where $(\nabla \otimes \Phi)_{ij} = \partial_i \Phi_j$, $S = \{(r\cos\theta, r\sin\theta) : r > 0, 0 \leq \theta \leq \theta_0\}$ is a closed sector with angle $\theta_0 \in (0, 2\pi) \setminus \{\pi\}$. In this case, we also refer to the edge point as having angle θ_0.

1.3. **Methodology.** As noted in [KSS24], the anisotropic nonscattering problem (1.3) can be reformulated as a Bernoulli-type free boundary problem of the form

$$(1.6\text{a}) \qquad (\nabla \cdot A(x)\nabla + \kappa^2 \rho(x))u^{\text{sc}} = f\mathcal{L}^n\lfloor D + g\mathcal{H}^{n-1}\lfloor \partial D \text{ in } \mathbb{R}^n, \quad u^{\text{sc}}|_{\mathbb{R}^n \setminus \overline{D}} = 0,$$

where

$$(1.6\text{b}) \qquad f = -(\nabla \cdot (A(x) - \text{Id})\nabla + h)u^{\text{inc}}, \quad g = \nu \cdot (A - \text{Id})\nabla u^{\text{inc}}$$

and $h(x) := \kappa^2(\rho(x) - 1)\chi_D$, see (2.1) below. Hereafter, we denote $\mathcal{L}^n\lfloor D \equiv \chi_D$ the n-dimensional Lebesgue measure restricted to D, and by $\mathcal{H}^{n-1}\lfloor \Gamma$ the $(n-1)$-dimensional Hausdorff measure restricted to Γ. Our main strategy is to consider the blowup limit of problem (1.6). To do this, we suppose that near a boundary point $x_0 = 0$ (say) the quantities u^{inc}, $A - \text{Id}$, and $\rho - 1$ have in $\overline{D}$ the expansions

$$u^{\text{inc}} = H + R^{(m)},$$

$$A - \text{Id} = B + R^{(b)},$$

$$\rho - 1 = \eta + R^{(p)},$$

where H, B, η are homogeneous polynomials of degree m, b and p, respectively, and H is harmonic (since u^{inc} solves $(\Delta + \kappa^2)u^{\text{inc}} = 0$). The remainder terms are assumed to vanish at 0 faster than the main terms, together with relevant derivative bounds.

In the discussion below we will assume that $m \geq 2$ to exclude certain special cases. It follows that near 0, one has in $\overline{D}$

$$f = -\nabla \cdot B\nabla H - \kappa^2 \eta H + R^{(f)},$$

$$g = \nu \cdot B\nabla H + R^{(g)},$$

where $R^{(f)}$ and $R^{(g)}$ vanish at 0 faster than the other terms in f and g, respectively. Note that $\nabla \cdot B\nabla H$ has degree $m+b-2$, ηH has degree $m+p$, and $B\nabla H$ has degree $m+b-1$. We classify the blowup equation corresponding to (1.6) into two cases: $b \geq p+2$ and $0 \leq b < p+2$. Theorems for the case $b \geq p+2$ are stated in Section 1.4.1, followed by the case $0 \leq b < p+2$ in Section 1.4.2.

When $b \geq p+2$, we define the blowup sequence $u_r(x) = u^{\text{sc}}(rx)/r^{m+p+2}$ at 0 (when it exists). The corresponding blowup equation is

$$\Delta u = -\kappa^2 \eta H \chi_{\{u \neq 0\}}.$$

In this scenario, only the contribution from $\rho - 1$ survives in the blowup, while the Bernoulli condition vanishes.

In the case $0 \leq b < p+2$, we define the blowup sequence $u_r(x) = u^{\text{sc}}(rx)/r^{m+b}$ at 0 (when it exists) that leads to the blowup equation

$$\Delta u = -(\nabla \cdot B\nabla H)\mathcal{L}^n\lfloor D_0 + (B\nabla H \cdot \nu_0)\mathcal{H}^{n-1}\lfloor \partial D_0, \qquad u|_{\mathbb{R}^n \setminus \overline{D}_0} = 0,$$

where D_0 is the blowup of D at $x_0 = 0$ (assuming it exists), and ν_0 is the unit outer normal to ∂D_0. In this case the contribution from $\rho - 1$ disappears in the blowup, but the Bernoulli condition survives.

1.4. **Statements of the main results and related work.** The discussion in Section 1.3 indicates that the results will be divided into several cases according to the behavior of A, ρ and u^{inc} at x_0. Before presenting the formal statements of the main results, we briefly highlight them as follows. We say that $A(x_0) \cong \mathrm{Id}$ if $|A(x) - \mathrm{Id}| \leq C|x - x_0|^{2+\alpha}$ and $|\nabla A(x)| \leq C|x - x_0|^{1+\alpha}$, near x_0 in $\overline{D}$, for some $\alpha > 0$.

- If $A(x_0) \cong \mathrm{Id}$ and $\rho(x_0) \neq 1$, any corner with opening angle $\neq \pi$ scatters every incident wave nontrivially (Theorem 1.4).
- If $A(x_0) \cong \mathrm{Id}$ but $\rho(x_0) = 1$, corners of any angle may be invisible to certain incident waves (Example 1.5).
- If $A(x_0) \neq \mathrm{Id}$ and $\rho(x_0) = 1$, corners with angles $\notin \mathbb{Q}\pi$ scatter every incident wave nontrivially (Theorems 1.7 and 1.9), whereas corners with angles $\in \mathbb{Q}\pi$ may be invisible for some incident waves ([KSS24, Example 1.5]).
- The special case $A = a\mathrm{Id}$ and $\rho = a$ with $a \neq 1$, which satisfies $A(x_0) \neq \mathrm{Id}$ and $\rho(x_0) \neq 1$, was studied in [CHLX25], showing that the scattering behavior is linked to the interior transmission eigenvalue problem.

We now proceed to the formal statements of the main results.

1.4.1. *Vanishing Bernoulli condition.* Our first theorem states that convex[1] or any piecewise C^1 planar obstacle whose boundary has a singular (non-C^1) point always scatter:

Theorem 1.4. *Let D be a bounded open set in $\mathbb{R}^2$, and suppose that*
- *either D is convex, or*
- *D is simply connected and has piecewise C^1 boundary.*

Let $\kappa > 0$, let $\rho \in L^\infty(D)$ be a positive real-valued function, and let $A \in (C^1(\overline{D}))^{n \times n}$ be a real symmetric matrix-valued function, satisfying the condition of uniform ellipticity (1.1). Suppose that there exists $x_0 \in \partial D$ such that

(i) *the contrast h given in (1.4) satisfies the non-degeneracy condition (1.5) at x_0, and*
(ii) *the coefficient A satisfies $|A(x) - \mathrm{Id}| \leq C|x - x_0|^{2+\alpha}$ and $|\nabla A(x)| \leq C|x - x_0|^{1+\alpha}$.*

If ∂D is not C^1 near x_0, then the anisotropic medium (D, A, ρ) scatters every incident wave nontrivially.

It is worth remarking that it seems plausible that the assumptions $|A(x) - \mathrm{Id}| \leq C|x - x_0|^{2+\alpha}$ and $|\nabla A(x)| \leq C|x - x_0|^{1+\alpha}$, can be relaxed to

$$(1.7) \qquad |A(x) - \mathrm{Id}| = \omega(|x - x_0|)|x - x_0|^2, \qquad |\nabla A(x)| = \omega(|x - x_0|)|x - x_0|^1$$

where $\omega(r)$ is double-Dini, which is used for the monotonicity formula in this paper. Since this would be a more technical result, we leave it out in this paper.

Example 1.5 (Necessity of condition (1.5)). In the trivial example where $A \equiv \mathrm{Id}$ and $\rho \equiv 1$, the obstacle D is not present and no incident wave produces scattering effects (i.e. $u^{\mathrm{sc}} \equiv 0$ for any incident wave). In particular, D can have corners of any type. This shows that it is not possible to drop the assumption (1.5) in the theorem above.

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[1]We recall that bounded convex domains always have Lipschitz boundary [Gri11, Corollary 1.2.2.3].

Another related example is given in [CVX23, Section 3] or [KSS24, Example 1.6] based on diffeomorphism invariance. Let D be a bounded domain in $\mathbb{R}^2$, with piecewise C^∞ boundary, with a corner $x_0 \in \partial D$. Let $\Phi : D \to D$ be a diffeomorphism such that $\Phi \in C^\infty(\overline{D})$, $\Phi^{-1} \in C^\infty(\overline{D})$, $\Phi(x) = x$ for all $x \in \partial D$ and $(\nabla \otimes \Phi)(x_0) = \mathrm{Id}$. Let

$$A = \Phi_*(\mathrm{Id}) = \frac{(\nabla \otimes \Phi)(\nabla \otimes \Phi)^\intercal}{|\det(\nabla \otimes \Phi)|} \circ \Phi^{-1} \in (C^\infty(\overline{D}))^{2\times 2}$$

and

$$\rho = \Phi_*(1) = \frac{1}{|\det(\nabla \otimes \Phi)|} \circ \Phi^{-1} \in C^\infty(\overline{D})$$

be the pushforwards by Φ, which satisfy $A(x_0) = \mathrm{Id}$ and $\rho(x_0) = 1$. Let $u^{\mathrm{inc}} \not\equiv 0$ solves (1.2). Choosing $v = u^{\mathrm{inc}}|_D$ and $u^{\mathrm{to}} = \Phi_* v$ gives a pair $(u^{\mathrm{to}}, u^{\mathrm{inc}})$ satisfying

$$\begin{cases} (\nabla \cdot A\nabla + \kappa^2 \rho)u^{\mathrm{to}} = 0, \quad (\Delta + \kappa^2)u^{\mathrm{inc}} = 0 & \text{in } D, \\ u^{\mathrm{to}} = u^{\mathrm{inc}}, \quad \nu \cdot A\nabla u^{\mathrm{to}} = \partial_\nu u^{\mathrm{inc}} & \text{on } \partial D, \end{cases}$$

therefore the function $u^{\mathrm{sc}} := u^{\mathrm{to}} - u^{\mathrm{inc}}$ verifies (1.3), i.e., the anisotropic medium (D, A, ρ) is nonscattering with respect to such incident field u^{inc}. This example does not contradict Theorem 1.4 since the contrast h defined in (1.4) fails to satisfy the non-degeneracy condition (1.5) at $x_0 \in \partial D$.

An analogous result for domains in $\mathbb{R}^n$ with $n \geq 3$ that contain edge singularities is the following.

Theorem 1.6. *Let D be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 3$ such that $\mathbb{R}^n \setminus \overline{D}$ is connected. Let $\kappa > 0$, let $\rho \in L^\infty(D)$ be a positive real-valued function, let $A \in (C^1(\overline{D}))^{n\times n}$ be a real symmetric matrix-valued function, satisfying the condition of uniform ellipticity (1.1). Assume that there exists $x_0 \in \partial D$ such that conditions (i) and (ii) in Theorem 1.4 are satisfied. If ∂D contains an edge point $x_0 \in \partial D$, then the anisotropic medium (D, A, ρ) scatters every incident wave nontrivially.*

As in Theorem 1.4, it seems plausible to relax the decay conditions to those in (1.7).

The assumptions in our main theorems ensure the following degeneracy

$$\nu \cdot A(x)\nabla u^{\mathrm{sc}}(x) = \nu \cdot (\mathrm{Id} - A(x))\nabla u^{\mathrm{inc}}(x) \to 0 \quad \text{as } x \to x_0,$$

which was not addressed in our earlier work [KSS24]. Here, we also allow $\nu \cdot (\mathrm{Id} - A)\nabla u^{\mathrm{inc}}$ to change sign. In [KSS24], we focused instead on the scattering behavior under non-degeneracy conditions

(1.8) $$\begin{cases} \nu \cdot (\mathrm{Id} - A)\nabla u^{\mathrm{inc}}(x) \geq c > 0 \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial D \text{ near } x_0; \text{ or} \\ \nu \cdot (\mathrm{Id} - A)\nabla u^{\mathrm{inc}}(x) \leq -c < 0 \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial D \text{ near } x_0. \end{cases}$$

Although it is often possible to construct global solutions of the Helmholtz equation that are positive in a given set (see [KSS23]), the real-valued functions u^{inc} and $\partial_j u^{\mathrm{inc}}$ typically exhibit numerous zeros, therefore the assumption (1.8) is not guaranteed to hold in many cases of interest.

In view of condition (i), it is natural to compare our results (Theorems 1.4 and 1.6) with those of [BPS14], whose proof relies on suitable complex geometric optics (CGO) solutions; see also [Blå18, BL17, BL21a, BL21b, HSV16, PSV17, VX21] for further refinements of similar results based on CGO or other harmonic-exponential solutions. This approach was

extended in [CX21] to scattering problems for general divergence-form operators, modeling *isotropic* media rather than the Laplacian. We also refer to recent works [HV25a, VX25] for further refinements in the case $A = \mathrm{Id}$, based on the asymptotic behavior of suitable integral expressions. Our results (Theorems 1.4 and 1.6) further extend these findings to general divergence-form operators modeling anisotropic media.

Other approaches to this type of problem include [EH15, EH18], which use direct expansions of solutions to the Helmholtz equation, and [SS21, KSS24], which build on results related to free boundaries. It is also interesting to mention that the analysis in [CV23] is based on the observation that a nonscattering domain satisfies the assumptions of the Pompeiu conjecture [Pom29], and hence its boundary must be analytic [Wil76, Wil81]. Although this may be beyond our scope, we note that the Pompeiu conjecture admits an equivalent formulation in terms of k-quadrature domains [KLSS24].

1.4.2. *Nonvanishing Bernoulli condition.* The assumption (ii) in Theorems 1.4 and 1.6 enables us to establish our results by following the approach of [SS25]. A natural question, however, is what happens when the condition (ii) in Theorem 1.4 fails. We address this in the following theorem.

Theorem 1.7. *Let $D \subset \mathbb{R}^2$ be a bounded, simply connected open set with piecewise C^1 boundary. Let $\kappa > 0$, let $\rho \in L^\infty(D)$ be a positive real-valued function, let $A \in (C^1(\overline{D}))^{n \times n}$ be a real symmetric matrix-valued function, satisfying the condition of uniform ellipticity (1.1). Suppose that there exists $x_0 \in \partial D$ such that*

- *(i) the contrast h given in (1.4) satisfies $|h(x)| \leq C|x - x_0|^\alpha$, and*
- *(ii) $A(x) - \mathrm{Id} = c_0 B^{-1} + \tilde{R}(x - x_0)$ for some real orthogonal matrix B and for some nonzero constant c_0 with $|\tilde{R}| \leq C|x|^\alpha$ and $|\nabla \tilde{R}| \leq C|x|^{\alpha-1}$.*

If ∂D has a corner at x_0 with angle $\theta_0 \notin \pi\mathbb{Q}$, then the anisotropic medium (D, A, ρ) scatters every incident wave $u^{\mathrm{inc}} \not\equiv 0$ nontrivially.

Remark 1.8. One can expand $u^{\mathrm{inc}}(Bx) - u^{\mathrm{inc}}(0) = H(x) + R(x)$, where $H \not\equiv 0$ is a harmonic homogeneous polynomial of order $m \geq 1$ and $|R(x)| \leq C|x|^{m+1}$. The results of Theorem 1.7 remain valid for any angle θ_0 such that $\sin(m\theta_0) \neq 0$. We also explain the difficulties of replacing B^{-1} by a more general matrix in Remark 3.2.

Adapting Federer's dimension reduction argument, as in the proof of [SS25, Theorem 1.10] (compare [Vel23, Lemma 10.9] and [Wei99]), we can conclude an analogous result for domains in $\mathbb{R}^3$ with $n \geq 3$:

Theorem 1.9. *Let D be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 3$ such that $\mathbb{R}^n \setminus \overline{D}$ is connected. Let $\kappa > 0$, let $\rho \in L^\infty(D)$ be a positive real-valued function, let $A \in (C^1(\overline{D}))^{n \times n}$ be a real symmetric matrix-valued function, satisfying the condition of uniform ellipticity (1.1). If $x_0 \in \partial D$ is an edge point with angle $\theta_0 \notin \pi\mathbb{Q}$, and conditions (i) and (ii) in Theorem 1.7 hold, then the anisotropic medium (D, A, ρ) scatters every incident wave $u^{\mathrm{inc}} \not\equiv 0$ nontrivially.*

Remark 1.10. Let m be the integer appearing in Remark 1.8. The results of Theorem 1.9 remain valid for any angle θ_0 such that $\sin(k\theta_0) \neq 0$ for all $k = 1, \cdots, m$.

In our previous work [KSS24, Example 1.5], we showed that nonscattering domains may have corners with opening angles $\ell\pi/m$ for integers $m \geq 2$ and $1 \leq \ell < 2m - 1$, highlighting the necessity of the condition $\theta_0 \notin \pi\mathbb{Q}$. See also [CHLX25] for additional examples of

nonscattering inhomogeneities. It is also worth mentioning that [CDLZ21, CDLZ22] studied an inverse problem in $\mathbb{R}^3$:

$$\begin{cases} (\Delta + \kappa^2)u^{\text{to}} = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ u^{\text{to}} = u^{\text{inc}} + u^{\text{sc}} & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ \partial_\nu u^{\text{to}} + \eta u^{\text{to}} = 0 & \text{on } \partial D, \\ u^{\text{sc}} \text{ is outgoing}, \end{cases}$$

obtaining results for polyhedral domains D with vertices having angles $\theta_0 \notin \pi\mathbb{Q}$. Although their analysis was restricted to three dimensions, the results can be readily extended to arbitrary dimensions $d \geq 2$ using d-dimensional spherical harmonics [EF14]. They proceed by examining the vanishing order of general solutions to the Helmholtz equation, which they refer to as "generalized Laplacian eigenfunctions". This approach relies on explicit expansions of Helmholtz solutions and is therefore highly specific to the Laplace operator. It is worth noting that the analysis in [HV25b] also relies on the observation that a nonscattering domain satisfies the assumptions of the Pompeiu conjecture [Pom29], incorporating the moving plane technique from [BK82], and is therefore highly specific to the Laplace operator. In contrast, our findings show that blowup analysis can be naturally extended to general elliptic operators beyond the Laplacian.

2. PROOF OF THEOREMS 1.4 AND 1.6

Suppose that (1.3) holds. For each $\psi \in C_c^\infty(\mathbb{R}^n)$, one computes that (we slightly abuse some notations here)

$$\int_{\mathbb{R}^n} \psi \left(-(\nabla \cdot (A(x) - \text{Id})\nabla + \kappa^2(\rho(x) - 1))u^{\text{inc}} \right) \mathcal{L}^n \lfloor D \, dx$$

$$= \int_D \psi \left(-(\nabla \cdot (A(x) - \text{Id})\nabla + \kappa^2(\rho(x) - 1))u^{\text{inc}} \right) dx$$

$$= \int_D \psi(\nabla \cdot A(x)\nabla + \kappa^2\rho(x))u^{\text{sc}} \, dx$$

$$= \int_D \left(-A(x)\nabla\psi \cdot \nabla u^{\text{sc}} + \kappa^2\rho(x)u^{\text{sc}} \right) dx + \int_{\partial D} \psi\nu \cdot A(x)\nabla u^{\text{sc}} \, dS$$

$$= \int_{\mathbb{R}^n} \left(-A(x)\nabla\psi \cdot \nabla u^{\text{sc}} + \kappa^2\rho(x)u^{\text{sc}} \right) dx + \int_{\partial D} \psi \left(\nu \cdot (\text{Id} - A)\nabla u^{\text{inc}} \right) dS$$

$$= \int_{\mathbb{R}^n} \psi(\nabla \cdot A(x)\nabla + \kappa^2\rho(x))u^{\text{sc}} \, dx + \int_{\mathbb{R}^n} \psi \left(\nu \cdot (\text{Id} - A)\nabla u^{\text{inc}} \right) \mathcal{H}^{n-1} \lfloor \partial D \, dx.$$

This shows that (1.3) is equivalent to

(2.1a) $\qquad (\nabla \cdot A(x)\nabla + \kappa^2\rho(x))u^{\text{sc}} = f\mathcal{L}^n\lfloor D + g\mathcal{H}^{n-1}\lfloor \partial D \text{ in } \mathbb{R}^n, \quad u^{\text{sc}}|_{\mathbb{R}^n\setminus\overline{D}} = 0$

in distribution sense, where

(2.1b) $\qquad f = -(\nabla \cdot (A(x) - \text{Id})\nabla + h)u^{\text{inc}}, \quad g = \nu \cdot (A - \text{Id})\nabla u^{\text{inc}}$

and $h(x) := \kappa^2(\rho(x) - 1)\chi_D$.

The above observations allow us to establish Theorems 1.4 and 1.6 by following the approach of [SS25]. In Section 2.1 we prove the $C^{0,1}$ regularity and decay rates for solutions of a certain class of PDE. In Section 2.2, we study blowup limits and prove their homogeneity by using a modified balanced energy functional, and also recall some results related to blowup

solutions. In Section 2.3, we show non-degeneracy and weak flatness properties and regularity of the free boundary. Finally, we prove Theorems 1.4 and 1.6 in Section 2.4.

2.1. Lipschitz regularity and optimal decay rate.
In this section we study the regularity and vanishing order for solutions of the equation

$$(2.2) \qquad (\nabla \cdot A(x)\nabla + q)u = f\mathcal{L}^n \lfloor B_2 + g\mathcal{H}^{n-1} \lfloor \partial D \text{ in } B_2, \quad u|_{B_2 \setminus \overline{D}} = 0,$$

as follows:

Lemma 2.1. *Let D be a bounded Lipschitz domain in $\mathbb{R}^n$ with $0 \in \partial D$, let $A \in (C^{0,1}(\overline{D}))^{n \times n}$ be a real symmetric matrix-valued function, satisfying the condition of uniform ellipticity (1.1), let $q \in L^\infty(B_2)$, let $m \geq 0$ be an integer and suppose that $u \in H^1(B_2)$ solves (2.2) with $|f(x)| \leq C|x|^m$ a.e. in B_2 and $|g(x)| \leq C|x|^{m+1}$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial D \cap B_2$. Then*

$$|u(x)| + |x||\nabla u(x)| \leq C|x|^{m+2} \quad \text{in } B_1.$$

Remark. We will show in Lemma 2.4 below that this decay rate is optimal.

Proof of Lemma 2.1. First, by [KSS24, Lemma 2.2], we obtain that $u \in C^{0.1}(B_1)$ and we see that $S_r := \|u\|_{L^\infty(B_r)} + r\|\nabla u\|_{L^\infty(B_r)}$ is monotone non-decreasing, then the set $\mathscr{D}$ of points, at which it is discontinuous, is at most infinitely countable. We consider the set $\phi(\mathscr{D})$, where $\phi(r) = r^{-m-2}S_r$, which is at most infinitely countable. We want to prove that $S_r \leq Cr^{m+2}$. Suppose the contrary: for any increasing positive sequence $\{\rho_j\}$ with $\{\rho_j\} \cap \phi(\mathscr{D}) = \emptyset$ and $\rho_j \to \infty$ as $j \to \infty$, there is $\tilde{r}_j \in (0,1]$ such that $S_{\tilde{r}_j} > \rho_j \tilde{r}_j^{m+2}$. Choosing $r_j \in (0,1]$ to be the supremum of all $\tilde{r}_j \in (0,1]$ with this property, we have

$$S_{r_j} = \rho_j r_j^{m+2}, \quad S_r \leq \rho_j r^{m+2} \quad \text{for } r \geq r_j,$$

and the sequence $\{r_j\}$ is nonincreasing with $r_j \to 0$ as $j \to \infty$.

If we define the rescaled functions

$$u_j(x) := \frac{u(2r_j x)}{\rho_j r_j^{m+2}} \quad \text{for all } x \in B_{1/r_j},$$

then

$$(2.3) \qquad \|u_j\|_{L^\infty(B_{1/2})} + \|\nabla u_j\|_{L^\infty(B_{1/2})} = \frac{S_{r_j}}{\rho_j r_j^{m+2}} = 1$$

and

$$\|u_j\|_{L^\infty(B_1)} + \|\nabla u_j\|_{L^\infty(B_1)} = \frac{S_{2r_j}}{\rho_j r_j^{m+2}} \leq \frac{\rho_j (2r_j)^{m+2}}{\rho_j r_j^{m+2}} = 2^{m+2}.$$

Since (u_j) is uniformly bounded in $C^{0,1}(\overline{B_{1/2}})$, and by Banach-Alaoglu theorem there is a subsequence[2], still denoted by (u_j), converging to some v in $C^{0,1}(\overline{B_{1/2}})$ weak-$\star$.

On the other hand,

$$\nabla_x \cdot (A(2r_j x)\nabla u_j(x)) + \frac{4}{\rho_j}(qu)(2r_j x) = \frac{4}{\rho_j}f(2r_j x)\mathcal{L}^n\lfloor B_2 + \frac{4}{\rho_j r_j}g(2r_j x)\mathcal{H}^{n-1}\lfloor \partial D$$

---

[2]By Arzela-Ascoli theorem, the sequence also converges strongly to v in $C^0(\overline{B_{1/2}})$.

in B_1, more precisely,

$$-\int_{B_1} A(2r_jx)\nabla u_j(x) \cdot \nabla\phi(x)\,\mathrm{d}x$$
$$= \frac{4}{\rho_j}\int_{B_1} f(2r_jx)\phi(x)\,\mathrm{d}x + \frac{4}{\rho_jr_j}\int_{\partial D} g(2r_jx)\phi(x)\,\mathrm{d}S_x - \frac{4}{\rho_j}\int_{B_1}(qu)(2r_jx)\phi(x)\,\mathrm{d}x$$

for all $\phi \in C_c^\infty(B_1)$. It is not difficult to see that

$$\left|\frac{4}{\rho_j}\int_{B_1} f(2r_jx)\phi(x)\,\mathrm{d}x + \frac{4}{\rho_jr_j}\int_{\partial D} g(2r_jx)\phi(x)\,\mathrm{d}S_x\right| \leq C\|\phi\|_{L^\infty(B_1)}\rho_j^{-1}$$

and

$$\left|\frac{4}{\rho_j}\int_{B_1}(qu)(2r_jx)\phi(x)\,\mathrm{d}x\right| \leq C\|\phi\|_{L^\infty(B_1)}r_j^2,$$

therefore

$$(2.4) \qquad \left|\int_{B_1} A(0)\nabla u_j(x) \cdot \nabla\phi(x)\,\mathrm{d}x\right|$$
$$\leq \left|\int_{B_1}(A(2r_jx) - A(0))\nabla u_j(x) \cdot \nabla\phi(x)\,\mathrm{d}x\right| + \left|\int_{B_1} A(2r_jx)\nabla u_j(x) \cdot \nabla\phi(x)\,\mathrm{d}x\right|$$
$$\leq C\|\nabla\phi\|_{L^\infty(B_1)}r_j + C\|\phi\|_{L^\infty(B_1)}(\rho_j^{-1} + r_j^2).$$

Applying the weak-$\star$ convergence of (u_j) to v in (2.3) and (2.4), we obtain

$$\|v\|_{L^\infty(B_{1/2})} + \|\nabla v\|_{L^\infty(B_{1/2})} \geq 1, \quad \nabla \cdot A(0)\nabla v = 0 \text{ in } B_1 \text{ (distribution sense)}.$$

Moreover, since D is a Lipschitz domain with $0 \in \partial D$ and $u|_{B_2 \setminus \overline{D}} = 0$, it follows that there is an open cone C in $\mathbb{R}^n$ so that each u_j and hence v vanish in $C \cap B_{1/2}$. By the unique continuation principle for the elliptic operator $\nabla \cdot A(0)\nabla$, we conclude $v = 0$ in $B_{1/2}$. This contradicts the condition $\|v\|_{L^\infty(B_{1/2})} + \|\nabla v\|_{L^\infty(B_{1/2})} \geq 1$. $\qquad\square$

2.2. Blowup solutions.

Throughout this section, we will make the following standing assumptions: Let D be a bounded Lipschitz domain with $0 \in \partial D$, let $q \in L^\infty(B_2)$ and let $u \in C^{0,1}_{\mathrm{loc}}(B_2)$ be a solution to

$$(2.5a) \qquad (\nabla \cdot A(x)\nabla + q)u = f\mathcal{L}^n\lfloor\{u \neq 0\} + g\mathcal{H}^{n-1}\lfloor\partial D,$$

satisfying

$$(2.5b) \qquad |u(x)| + |x||\nabla u(x)| \leq C|x|^{m+2}.$$

Note that the function $u_r(x) := u(rx)/r^{m+2}$ also satisfies the estimate (2.5b) and solves the equation

$$\nabla \cdot A(rx)\nabla u_r(x) = (r^{-m}f(rx) - r^2q(rx)u_r(x))\mathcal{L}^n\lfloor\{u_r \neq 0\}$$
$$+ r^{-m-1}(g\mathcal{H}^{n-1}\lfloor\partial D)(rx).$$

Similar to [SS25, Lemma 2.5], we now assume that

$$(2.5c) \qquad \begin{aligned} &f = H + R, \text{ where } H \text{ is homogeneous polynomial of degree } m \\ &\text{and } |R(x)| \leq C|x|^{m+\alpha} \text{ for some } \alpha > 0, \end{aligned}$$

as well as

$$(2.5d) \qquad |g(x)| \leq C|x|^{m+1+\alpha} \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial D \cap B_2.$$

We also assume that $A \in (C^{0,1}(\overline{D}))^{n \times n}$ is a real symmetric matrix-valued function, satisfying the condition of uniform ellipticity (1.1), as well as

(2.5e) $|A(x) - \mathrm{Id}| \leq C|x|, \quad |\nabla A(x)| \leq C.$

In view of Banach-Alaoglu theorem, we say that v is a *blowup limit of u of order $m+2$ at 0* if there is a sequence $r_j \to 0$ so that $u_{r_j} \to v$ in $C^{0,1}(\overline{B_1})$ weak-$\star$.

Similarly as in [SS25, Section 3.1], we introduce a balanced energy functional (see [CSY18, Lemma 22] or [PSU12, Section 3.5]):

$$W_A(r, u) := \frac{1}{r^{2m+n+2}} \int_{B_r} (A(x)\nabla u \cdot \nabla u + 2Hu)\, \mathrm{d}x - \frac{m+2}{r^{2m+n+3}} \int_{\partial B_r} u^2\, \mathrm{d}S$$

$$= \int_{B_1} (A(rx)\nabla u_r \cdot \nabla u_r + 2Hu_r)\, \mathrm{d}x - (m+2)\int_{\partial B_1} u_r^2\, \mathrm{d}S.$$

We compute its derivative (we slightly abuse the notations in the computations below)

$$\frac{1}{2}\partial_r W_A(r, u) = \int_{B_1} (A(rx)\nabla u_r \cdot \nabla \partial_r u_r + H\partial_r u_r)\, \mathrm{d}x + \frac{1}{2}\int_{B_1} \partial_r(A(rx))\nabla u_r \cdot \nabla u_r\, \mathrm{d}x$$

$$- (m+2)\int_{\partial B_1} u_r \partial_r u_r\, \mathrm{d}S$$

$$= \int_{B_1} (-\nabla \cdot (A(rx)\nabla u_r) + H)\, \partial_r u_r\, \mathrm{d}x + \frac{1}{2}\int_{B_1} \partial_r(A(rx))\nabla u_r \cdot \nabla u_r\, \mathrm{d}x$$

$$+ \int_{\partial B_1} \overbrace{(\partial_\nu u_r - (m+2)u_r)}^{= r\partial_r u_r} \partial_r u_r\, \mathrm{d}S + \int_{\partial B_1} \nu \cdot (A(rx) - \mathrm{Id})\nabla u_r \partial_r u_r\, \mathrm{d}S$$

$$= \int_{\partial B_1} r(\partial_r u_r)^2\, \mathrm{d}S - \int_{B_1} r^{-m} R(rx)\partial_r u_r\, \mathrm{d}x + \int_{B_1} r^2 q(rx) u_r \partial_r u_r\, \mathrm{d}x$$

$$+ \frac{1}{2}\int_{B_1} \partial_r(A(rx))\nabla u_r \cdot \nabla u_r\, \mathrm{d}x$$

$$- r^{-m-1}\int_{B_1 \cap (r^{-1}\partial D)} g(rx)\partial_r u_r\, \mathrm{d}S + \int_{\partial B_1} \nu \cdot (A(rx) - \mathrm{Id})\nabla u_r \partial_r u_r\, \mathrm{d}S.$$

We introduce the functional

$$F_A(r, u) = 2\int_0^r \int_{B_1} \tau^{-m} R(\tau x)\partial_\tau u_\tau\, \mathrm{d}x\, \mathrm{d}\tau - 2\int_0^r \int_{B_1} \tau^2 q(\tau x) u_\tau \partial_\tau u_\tau\, \mathrm{d}x\, \mathrm{d}\tau$$

$$+ \int_0^r \int_{B_1} \partial_\tau(A(\tau x))\nabla u_\tau \cdot \nabla u_\tau\, \mathrm{d}x\, \mathrm{d}\tau$$

$$+ 2\int_0^r \int_{B_1 \cap (\tau^{-1}\partial D)} \tau^{-m-1} g(\tau x)\partial_\tau u_\tau\, \mathrm{d}S\, \mathrm{d}\tau$$

$$- 2\int_0^r \int_{\partial B_1} \nu \cdot (A(\tau x) - \mathrm{Id})\nabla u_\tau \partial_\tau u_\tau\, \mathrm{d}S\, \mathrm{d}\tau.$$

in order to extract the following non-negative quantity:

$$\partial_r (W_A(r, u) - F_A(r, u)) = 2\int_{\partial B_1} r(\partial_r u_r)^2\, \mathrm{d}S \geq 0.$$

We denote
$$W(r, u) := W_{\mathrm{Id}}(r, u) = \frac{1}{r^{2m+n+2}} \int_{B_r} \left(|\nabla u|^2 + 2Hu \right) \mathrm{d}x - \frac{m+2}{r^{2m+n+3}} \int_{\partial B_r} u^2 \,\mathrm{d}S.$$

From (2.5b) and (2.5e) we see that

$$(2.6) \qquad |W(r, u) - W_A(r, u)| = \frac{1}{r^{2m+n+2}} \left| \int_{B_r} (A(x) - \mathrm{Id}) \nabla u \cdot \nabla u \,\mathrm{d}x \right| \le Cr.$$

Since u_r satisfies the estimates (2.5b) and
$$|\partial_r u_r(x)| = |-(m+2) r^{-m-3} u(rx) + r^{-m-2} x \cdot \nabla u(sx)| \le Cr^{-1}|x|^{m+2},$$
then

$$\begin{aligned}
|F_A(r, u)| &\le C \int_0^r \int_{B_1} (\tau^\alpha |x|^{m+\alpha} + \tau^2 |u_\tau(x)|) |\partial_\tau u_\tau(x)| \,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + C \int_0^r \int_{B_1} \tau |\nabla u_\tau|^2 \,\mathrm{d}x\,\mathrm{d}\tau + C \int_0^r \int_{B_1 \cap (\tau^{-1}\partial D)} \tau^\alpha |x|^{m+1+\alpha} |\partial_\tau u_\tau(x)| \,\mathrm{d}S\,\mathrm{d}\tau \\
&\quad + C \int_0^r \int_{\partial B_1} \tau |x| |\nabla u_\tau| |\partial_\tau u_\tau| \,\mathrm{d}S\,\mathrm{d}\tau \\
&\le Cr^\alpha
\end{aligned}$$

It is easy to see that
$$W_A(r, u) \ge -C.$$
Thus, the non-decreasing quantity $r \mapsto W_A(r, u) + F_A(r, u)$ has a finite limit as $r \to 0$, and this limit equals $W_A(0+, u)$. Now by (2.6) we see that the quantity $r \mapsto W(r, u)$ has a finite limit as $r \to 0$, and this limit equals $W(0+, u) = W_A(0+, u)$. Since
$$W(rs, u) = W(s, u_r) \quad \text{for all } r, s \in (0, 1],$$
we can use the standard argument in [Yer16, Lemma 16] (we omit the details here) to conclude the following lemma:

Lemma 2.2. *Suppose $u \in C^{0,1}_{\mathrm{loc}}(B_2)$ satisfies (2.5). Any blowup limit v (of order $m+2$) of u is homogeneous of degree $m+2$ solves the equation*
$$\Delta v = H\mathcal{L}^n \lfloor \{v \neq 0\} \quad \text{in } B_1.$$
One has $W(s, v) = W(0+, u)$ for $0 < s \le 1$. Moreover, the homogeneous degree $m+2$ extension of v (still denoted by v) solves the equation
$$(2.7) \qquad \Delta v = H\mathcal{L}^n \lfloor \{v \neq 0\} \quad \text{in } \mathbb{R}^n.$$

We recall the following result, which characterizes all blowup limits in two dimensions:

Lemma 2.3 ([SS25, Theorem 3.12])**.** *Let $n = 2$. Suppose that $v \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ is nontrivial, homogeneous of degree $m+2$ and solves the equation*
$$\Delta v = H\mathcal{L}^n \lfloor \{v \neq 0\} \quad \text{in } \mathbb{R}^2$$
where H is a harmonic homogeneous polynomial of degree m. Then one of the following holds:

(a) $\mathrm{supp}\,(v)$ is a half space and v is a polynomial[3] of order $m+2$.

---

[3] After a rotation, the form given in [SS25, Lemma 3.3 or Lemma 3.5].

(b) $\mathrm{supp}\,(v) = \mathbb{R}^2$ *and* $v = \frac{1}{4m+4}|x|^2 H + w$ *where* w *is a harmonic homogeneous polynomial of degree* $m + 2$.

2.3. Non-degeneracy and weak flatness.

Throughout this section, we will make the following standing assumptions: Let D be a bounded Lipschitz domain with $0 \in \partial D$, let $q \in C^{\alpha}_{\mathrm{loc}}(B_2)$ and let $u \in C^{0,1}_{\mathrm{loc}}(B_2)$ be a solution to

$$(2.8a) \qquad (\nabla \cdot A(x)\nabla + q)u = f\mathcal{L}^n \lfloor D + g\mathcal{H}^{n-1}\lfloor \partial D, \quad u|_{B_2 \setminus \overline{D}} = 0.$$

We assume that

$$(2.8b) \qquad \begin{aligned} &f = P + R, \text{ where } P \not\equiv 0 \text{ is a homogeneous polynomial of order } m \geq 0 \\ &\text{and } |R(x)| \leq C|x|^{m+\alpha} \text{ for some } \alpha > 0 \end{aligned}$$

We also assume that g satisfies (2.5d) and A satisfies (2.5e). In this setting, we say that v is a blowup limit of u at 0 if there is a sequence $r_j \to 0$ so that $u_{r_j} \to v$ in $C^{0,1}(\overline{B_1})$ weak-$\star$, where $u_r(x) := u(rx)/r^{m+2}$.

On the other hand, since D is a Lipschitz domain, there are $\delta > 0$ and $r_0 > 0$ so that we have the following consequence of the interior cone property:

$$(2.9) \qquad \text{If } x \in \overline{D} \cap \overline{B_{r_0}}, \text{ then } B_r(x) \cap \overline{D} \text{ contains a ball of radius } \delta r \text{ whenever } r \leq r_0.$$

Based on the above observations, as in [SS25, Lemma 4.1], we first prove a non-degeneracy result showing that u cannot vanish faster than a certain order in $\mathrm{supp}\,(u)$:

Lemma 2.4. *Suppose* $u \in C^{0,1}_{\mathrm{loc}}(B_2)$ *satisfies* (2.8). *For any* $\epsilon \in (0,1)$, *there exists a pair of positive numbers* (r_ϵ, c_ϵ) *such that*

$$\|u\|_{L^\infty(B_{\epsilon|x|}(x))} \geq c_\epsilon |x|^{m+2} \quad \text{for all } x \in \overline{D} \cap \overline{B_{r_\epsilon}}.$$

Remark. If we further assume that

$$(2.10) \qquad \overline{B_{r_0}} \cap \overline{D} \subset \mathrm{supp}\,(f) \text{ for some } r_0 > 0,$$

then Lemma 2.4 remains valid with r_ϵ replaced by a constant r_0 that is independent of ϵ. The condition (2.10) was valid in the setting of [SS25]. However, the choice made in (2.1b) does not guarantee (2.10), unless the coefficient A is analytic in a neighborhood of $x_0 = 0$. Thus the statement and proof of Lemma 2.4 are slightly different from [SS25, Lemma 4.1].

Proof of Lemma 2.4. Suppose, to the contrary, that there exists some $\epsilon \in (0,1)$ for which no such pair of positive numbers (r_ϵ, c_ϵ) can be found. That is, for any pair of positive numbers (r, c), there exists $x \in \overline{D} \cap \overline{B_r}$ such that

$$\|u\|_{L^\infty(B_{\epsilon|x|}(x))} < c|x|^{m+2}.$$

For each $j \geq 1$, by choosing the pair $(r, c) = (1/j, 1/j)$, there exists $x_j \in \overline{D} \cap \overline{B_{1/j}}$ such that

$$(2.11) \qquad \|u\|_{L^\infty(B_{\epsilon|x_j|}(x_j))} < \frac{1}{j}|x_j|^{m+2}.$$

There exists a subsequence, still denoted as (x_j), such that $x_j \to 0$. Let $r_j = |x_j|$ and the blowup sequence $u_{r_j}(x) := u(r_j x)/r_j^{m+2}$. Now using (2.9) with $x = x_j$ and $r = \epsilon r_j$, for each sufficiently large $j \geq 1$ there is y_j so that

$$(2.12) \qquad B_{\delta \epsilon r_j}(y_j) \subset B_{\epsilon r_j}(x_j) \cap \overline{D}.$$

It also follows that $(1 - \epsilon)r_j \leq |y_j| \leq (1 + \epsilon)r_j$. Then by (2.11) we have

$$\|u\|_{L^\infty(B_{\delta\epsilon r_j}(y_j))} < \frac{1}{j} r_j^{m+2}.$$

Writing $z_j = y_j/r_j$, this means that

$$\|u_{r_j}\|_{L^\infty(B_{\delta\epsilon}(z_j))} < \frac{1}{j}.$$

By considering a suitable subsequence, we may assume that z_j converges to some z with $1 - \epsilon \leq |z| \leq 1 + \epsilon$. Thus we have

(2.13) $$u_{r_j} \to 0 \quad \text{in } L^\infty(B_{\delta\epsilon/2}(z)).$$

On the other hand, since

$$\nabla \cdot A(r_j x)\nabla u_{r_j}(x) = -r_j^2 q(r_j x)u_{r_j}(x) + r_j^{-m}(f\mathcal{L}^n\lfloor D)(r_j x)$$
$$+ r_j^{-m-1}(g\mathcal{H}^{n-1}\lfloor\partial D)(r_j x),$$

and since $|g(x)| \leq C|x|^{m+1+\alpha}$, then we have

$$\nabla \cdot A(r_j x)\nabla u_{r_j}(x) = -r_j^2 q(r_j x)u_{r_j}(x) + r_j^{-m}(f\mathcal{L}^n\lfloor D)(r_j x) + O(r_j^\alpha)(\mathcal{H}^{n-1}\lfloor\partial D)(r_j x)$$

uniformly on $x \in \overline{B_1}$. Now (2.12) implies that

$$\nabla \cdot A(r_j x)\nabla u_{r_j}(x) = -r_j^2 q(r_j x)u_{r_j}(x) + P(x) + r_j^{-m}R(r_j x) + O(r_j^\alpha)(\mathcal{H}^{n-1}\lfloor\partial D)(r_j x) \text{ in } B_{\delta\epsilon}(z_j).$$

Since $r_j \to 0$ and $|u_{r_j}(x)| \leq C$, $|R(x)| \leq C|x|^{m+\alpha}$, we have

$$\nabla \cdot A(r_j x)\nabla u_{r_j} \to P \quad \text{in the distributional sense on } B_{\delta\epsilon/2}(z).$$

Since P does not vanish on any open subset of $\mathcal{S}^{n-1}$, this contradicts (2.13) by uniqueness of distributional limits, completing the proof by contradiction. $\square$

As shown in Lemma 2.2, the blowup limit u_0 in this work is identical to the one in [SS25, Section 4]. With Lemma 2.4 at hand, we can establish some weak flatness properties for ∂D at 0, using the exact same argument in [SS25, Lemma 4.3]:

Lemma 2.5. *Suppose* $u \in C^{0,1}_{\text{loc}}(B_2)$ *satisfies* (2.8) *and suppose that* u_0 *is a blowup limit of such* u *at 0 with* $\text{supp}\,(u_0) = \overline{\mathbb{R}^n_+}$. *Then for any* $\delta > 0$ *there is* $r > 0$ *so that* $\partial D \cap B_r \subset \{|x_n| \leq \delta r\}$.

2.4. **Conclusions.** To prepare for the main results, we summarize the preceding discussion in the following proposition.

Proposition 2.6 (Properties of blowups: I)**.** *Let* $D \subset \mathbb{R}^n$ *be a Lipschitz domain such that* $0 \in \partial D$, *let* $q \in C^\alpha_{\text{loc}}(B_2)$, *let* $f \in C^\alpha_{\text{loc}}(B_2)$, *let* $A \in (C^{0,1}(\overline{D}))^{n\times n}$ *be a real symmetric matrix-valued function, satisfying the condition of uniform ellipticity* (1.1) *and*

$$|A(x) - \text{Id}| \leq C|x|, \quad |\nabla A(x)| \leq C.$$

Suppose that $u \in H^1_{\text{loc}}(B_2)$ *solves*

$$(\nabla \cdot A(x)\nabla + q)u = f\mathcal{L}^n\lfloor D + g\mathcal{H}^{n-1}\lfloor\partial D, \quad u|_{B_2\backslash\overline{D}} = 0.$$

Let $m \geq 0$ *be an integer. Assume that* $|g(x)| \leq C|x|^{m+1+\alpha}$ *for* $\mathcal{H}^{n-1}$-*a.e.* $x \in \partial D \cap B_2$ *and* $f = H + R$, *where* $H \not\equiv 0$ *is a homogeneous polynomial of degree* m *and* $|R(x)| \leq C|x|^{m+\alpha}$. *Then* u *has the following properties:*

(a) $u \in C^{0,1}_{\text{loc}}(B_2)$ *with* $|u(x)| + |x||\nabla u(x)| \le C|x|^{m+2}$.

(b) *If v is any blowup limit of $u_r(x) := u(rx)/r^{m+2}$, then v is homogeneous of degree $m + 2$ and its homogeneous degree $m + 2$ extension, still denoted by v, solves $\Delta v = H\mathcal{L}^n\lfloor\{v \ne 0\}$ in $\mathbb{R}^n$.*

(c) *If v is any blowup limit of $u_r(x) := u(rx)/r^{m+2}$, then $\operatorname{supp}(v) \ne \emptyset$ and for any $\epsilon \in (0,1)$ there is $c_\epsilon > 0$ so that*

$$\|v\|_{L^\infty(B_{\epsilon|x|}(x))} \ge c_\epsilon|x|^{m+2} \quad \text{for all } x \in \operatorname{supp}(v) \cap \overline{B_{1/2}}.$$

(d) *If there exists a blowup limit v of $u_r(x) := u(rx)/r^{m+2}$ such that its support is the half-space $x \cdot e \ge 0$, then for any $\delta > 0$ there is $r > 0$ such that $\partial D \cap B_r \subset \{|x \cdot e| \le \delta r\}$.*

Proof. This follows by combining [SS25, Lemma 2.5], Lemmas 2.1, 2.2, 2.4 and 2.5. □

We refine the above result for our purpose:

Proposition 2.7 (Properties of blowups: II). *Suppose that all assumptions in Proposition 2.6 hold. If we additionally assume that H is harmonic and $n = 2$, then the support of any blowup limit v is a half space[4].*

Proof. Since D has Lipschitz boundary, and v vanishes in an exterior cone, it follows that $\operatorname{supp}(v) \ne \mathbb{R}^2$. On the other hand, by using Proposition 2.6(c), we see that $\operatorname{supp}(v) \ne \emptyset$. Finally, our proposition immediately follows from Lemma 2.3. □

We can now prove the free boundary regularity in two dimensions.

Proposition 2.8. *Suppose that all assumptions in Proposition 2.7 hold.*

(a) *If ∂D is piecewise C^1, then ∂D is C^1 near 0.*

(b) *If D is convex, then ∂D is C^1 near 0.*

Proof. (a) Let v be any blowup limit of $u_r(x) := u(rx)/r^{m+2}$. By using Proposition 2.7, we know that $\operatorname{supp}(v) = \{x \cdot e \ge 0\}$ for some unit vector e. Then the weak flatness property in Proposition 2.6(d) ensures that for any $\delta > 0$ there exists $r > 0$ such that $\partial D \cap B_r \subset \{|x \cdot e| \le \delta r\}$. Since ∂D is piecewise C^1, then near 0, ∂D is the union of two C^1 arcs meeting at 0 at some angle $\theta \in (0, 2\pi)$. If $\theta \ne \pi$, one gets a contradiction with the weak flatness property above. This implies that $\theta = \pi$ and the tangent vectors of the two C^1 arcs are parallel at 0, which concludes that ∂D is the graph of a C^1 function near 0. This proves case (a).

(b) As in the proof of case (a) we already obtain the weak flatness property: for any $\delta > 0$ there exists $r > 0$ such that $\partial D \cap B_r \subset \{|x \cdot e| \le \delta r\}$. Since D is convex, then for each boundary point near 0, there is a unique supporting plane. Finally, [SS25, Lemma 4.4] implies that ∂D is C^1 near 0. □

We also prove the result for edge points in dimension $n \ge 3$.

Proposition 2.9. *Suppose that all assumptions in Proposition 2.6 hold. If we additionally assume that H is harmonic, then $0 \in \partial D$ is not an edge point.*

---

[4]In this case, the blowup limits have explicit form depending on H, see [SS25, Lemma 3.5].

Proof. Suppose the contrary: $0 \in \partial D$ is an edge point. After a rotation, we may arrange that the blowup of D at 0, i.e., the limit of $r^{-1}(D \cap B_r)$ is $(S \times \mathbb{R}^{n-2}) \cap B_1$ as $r \to 0$, where S is a closed sector in $\mathbb{R}^2$ with angle $\neq \pi$. Let v be any blowup limit of u_r, and we see that $C := \mathrm{supp}\,(v) \subset S \times \mathbb{R}^{n-2}$, and the non-degeneracy statement in Proposition 2.6(c) implies that $C = S \times \mathbb{R}^{n-2}$. Following exactly the same argument as in the proof of [SS25, Theorem 1.10], which utilized Federer's reduction argument as in [Wei99] or [Vel23, Lemma 10.9], we can conclude that ∂D is C^1 near 0, which contradicts with the assumption that $0 \in \partial D$ is an edge point. $\qquad\square$

We finally prove our main results.

Proof of Theorems 1.4 and 1.6. Without loss of generality, we may assume that $x_0 = 0$ and $\Phi \equiv \mathrm{Id}$. The contrast h is C^α near 0 with $h(0) \neq 0$ by (1.5). Since u^{inc} solves $(\Delta + k^2)u^{\mathrm{inc}} = 0$ in $\mathbb{R}^n$, then one has $u^{\mathrm{inc}} = H + R_0$, where H is a harmonic homogeneous polynomial of order m and $|R_0(x)| \leq C|x|^{m+1}$. By using the arguments in [SS25, Lemma 2.5], one can show that

$$-hu^{\mathrm{inc}} = h(0)H + R_1, \quad |R_1(x)| \leq C|x|^{m+\alpha}.$$

Since $|A(x) - \mathrm{Id}| \leq C|x|^{2+\alpha}$ and $|\nabla A(x)| \leq C|x|^{1+\alpha}$, then

$$|\nabla \cdot (A(x) - \mathrm{Id})\nabla u^{\mathrm{inc}}| \leq C|\nabla A(x)||\nabla u^{\mathrm{inc}}| + C|A(x) - \mathrm{Id}||\nabla^{\otimes 2}u^{\mathrm{inc}}| \leq C|x|^{m+\alpha},$$

where $\nabla^{\otimes 2}u^{\mathrm{inc}}$ is the Hessian matrix of u^{inc}. Here we used the facts $|\nabla u^{\mathrm{inc}}| \leq C|x|^{m-1}$ and $|\nabla^{\otimes 2}u^{\mathrm{inc}}| \leq C|x|^{m-2}$. Since $A \in (C^{0,1}(\overline{D}))^{n \times n}$, then we conclude that

$$f = h(0)H + R \text{ with } |R(x)| \leq C|x|^{m+\alpha}.$$

On the other hand, we see that

$$|g(x)| \leq C|A(x) - \mathrm{Id}||\nabla u^{\mathrm{inc}}| \leq C|x|^{m+1+\alpha}.$$

After this reduction, Theorems 1.4 and 1.6 follows from the corresponding free boundary results in Propositions 2.8 and 2.9, respectively. $\qquad\square$

3. Proof of Theorems 1.7 and 1.9

We assume the conditions in Theorem 1.7. Without loss of generality, we may assume that $x_0 = 0$. Let B denote the real orthogonal matrix specified in (ii). Since u^{inc} satisfies $(\Delta + k^2)u^{\mathrm{inc}} = 0$ in $\mathbb{R}^n$, it follows that $u^{\mathrm{inc}} \circ B$ also satisfies the same equation in $\mathbb{R}^n$. Therefore, we can write $u^{\mathrm{inc}}(Bx) - u^{\mathrm{inc}}(0) = H(x) + R_0(x)$, where $H \not\equiv 0$ is a harmonic homogeneous polynomial of order $m \geq 1$ and $|R_0(x)| \leq C|x|^{m+1}$. From (ii) it follows that

$$(A(y) - \mathrm{Id})\nabla_y u^{\mathrm{inc}}\big|_{y=Bx} = c_0 \nabla_x H(x) + \vec{R}(x)$$

with $|\vec{R}(x)| \leq C|x|^{m-1+\alpha}$. In view of (2.1), it therefore suffices, without loss of generality, to prove Theorem 1.7 in the case $B = \mathrm{Id}$.

By a direct application of Lemma 2.1 (with $m + 2$ being replaced by m) we obtain

$$(3.1) \qquad\qquad |u^{\mathrm{sc}}(x)| + |x||\nabla u^{\mathrm{sc}}(x)| \leq C|x|^m,$$

which naturally motivates the scaling function $\tilde{u}_r := u^{\mathrm{sc}}(rx)/r^m$, which also satisfies the estimate (3.1) and solves the equation

$$
\begin{aligned}
(3.2) \qquad \nabla \cdot A(rx)\nabla \tilde{u}_r(x) &= (r^{-(m-2)}f(rx) - r^2\kappa^2\rho(rx)\tilde{u}_r(x))\mathcal{L}^n\lfloor\{\tilde{u}_r \neq 0\} \\
&\quad + r^{-(m-1)}(g\mathcal{H}^{n-1}\lfloor\partial D)(rx).
\end{aligned}
$$

In this setting, we say that v is a blowup limit of u^{sc} of order m at 0 if there is a sequence $r_j \to 0$ so that $\tilde{u}_{r_j} \to v$ in $C^{0,1}(\overline{B_1})$ weak-$\star$.

We begin with the case $n = 2$. Since ∂D is piecewise C^1, there exists $r_0 > 0$ such that $\partial D \cap B_{r_0} = (\Gamma_- \cup \Gamma_+) \cap B_{r_0}$, where the C^1 interfaces $\Gamma_\pm$ intersect at $x_0 = 0$. Without loss of generality, we may further assume that $\Gamma_- \cap B_{r_0} \subset \{x_1 \leq 0\}$ and $\Gamma_+ \cap B_{r_0} \subset \{x_1 \geq 0\}$. Since the normal vector ν is a continuous vector field on each $\Gamma_\pm$, we can define

$$\nu_- := \lim_{\Gamma_1 \ni x \to 0} \nu(x), \quad \nu_+ := \lim_{\Gamma_2 \ni x \to 0} \nu(x),$$

and introduce the straight lines $\tilde{\Gamma}_- \subset \{x_1 \leq 0\}$ and $\tilde{\Gamma}_+ \subset \{x_1 \geq 0\}$, each perpendicular to ν_- and ν_+, respectively. Any blowup limit $v \in C^{0,1}(\overline{B_1})$ of order m of u^{sc} solves the equation

$$(3.3) \qquad \Delta v = c_0 \sum_{\pm} (\nu_\pm \cdot \nabla H) \mathcal{H}^1 \lfloor \tilde{\Gamma}_\pm \quad \text{in } B_1.$$

At this stage, it is not yet known whether v is homogeneous of degree m. However, this fact is not needed for the proof of the results in the case $n = 2$.

First, all half-space solutions can be characterized directly by the same arguments as in [SS25, Lemmas 3.3 and 3.5]:

Lemma 3.1. *There exists a unique blowup limit v that satisfies (3.3) with $\tilde{\Gamma}_\pm = \{x_2 = 0\} \cap \{\pm x_1 \geq 0\}$ and $\mathrm{supp}\,(v) = \{x_2 \geq 0\}$. For any $m \geq 1$, we write $H(re^{\mathbf{i}\theta}) = ar^m e^{\mathbf{i}m\theta} + br^m e^{-\mathbf{i}m\theta}$ for some constants $a, b \in \mathbb{C}$, then the unique solution v is a polynomial of order m given explicitly by*

$$(3.4) \qquad v(re^{\mathbf{i}\theta}) = c_0(a-b)r^m \left(e^{\mathbf{i}m\theta} - e^{-\mathbf{i}m\theta} \right).$$

for all $(x_1, x_2) \cong x_1 + \mathbf{i}x_2 = re^{\mathbf{i}\theta}$ with $0 < \theta < \pi$ and $0 < r < 1$.

Remark. It is natural to extend (3.4) to all $r > 0$, and this extension satisfies

$$(3.5) \qquad \Delta v = c_0 \sum_{\pm} (\nu_\pm \cdot \nabla H) \mathcal{H}^1 \lfloor \tilde{\Gamma}_\pm \quad \text{in } \mathbb{R}^2.$$

Remark 3.2. We now explain the difficulty that arises if we replace B in Theorem 1.7(ii) with a general invertible matrix. Since u^{inc} satisfies $(\Delta + k^2)u^{\mathrm{inc}} = 0$ in $\mathbb{R}^n$, it follows that $u_B^{\mathrm{inc}} := u^{\mathrm{inc}} \circ B$ satisfies

$$\nabla \cdot \left[B^{-1}(B^{-1})^{\mathsf{T}} \nabla u_B^{\mathrm{inc}} \right] + k^2 u_B^{\mathrm{inc}} = 0 \quad \text{in } \mathbb{R}^n.$$

Therefore, we can write $u^{\mathrm{inc}}(Bx) - u^{\mathrm{inc}}(0) = H(x) + R_0(x)$, where $H \not\equiv 0$ is a harmonic homogeneous polynomial of order $m \geq 1$ and $|R_0(x)| \leq C|x|^{m+1}$. From Theorem 1.7(ii) it follows that

$$(A(y) - \mathrm{Id})\nabla_y u^{\mathrm{inc}}\big|_{y=Bx} = c_0 \nabla_x H(x) + \vec{R}(x)$$

with $|\vec{R}(x)| \leq C|x|^{m-1+\alpha}$. Using (2.1), we see that the function $u_B^{\mathrm{sc}} := u^{\mathrm{sc}} \circ B$ satisfies

$$\nabla \cdot \left[B^{-1}A(Bx)(B^{-1})^{\mathsf{T}} \nabla u_B^{\mathrm{sc}} \right] + \rho(Bx)u_B^{\mathrm{sc}} = f_B \mathcal{L}^n \lfloor B^{-1}(D) + g_B \mathcal{H}^{n-1} \lfloor \partial(B^{-1}(D)),$$

where

$$f_B = -\nabla \cdot \left[B^{-1}(A(Bx) - \mathrm{Id})(B^{-1})^{\mathsf{T}} \nabla u_B^{\mathrm{inc}} \right] - h(Bx)u_B^{\mathrm{inc}}$$

and

$$g_B = \nu(Bx) \cdot (A(y) - \mathrm{Id})\nabla_y u^{\mathrm{inc}}\big|_{y=Bx}.$$

However, now $B^{-1}A(Bx)(B^{-1})^\intercal \to B^{-1}(B^{-1})^\intercal$ as $x \to 0$, and (3.3) becomes

$$\nabla \cdot (B^{-1}(B^{-1})^\intercal \nabla v) = c_0 \sum_\pm (\nu_\pm \cdot \nabla H)\mathcal{H}^1 \lfloor \tilde{\Gamma}_\pm \quad \text{in } B_1,$$

which makes the subsequent computations in Lemma 3.1 considerably more complicated.

Proof of Lemma 3.1. We verify that v is a harmonic polynomial of degree m of the form $v = cr^m e^{im\theta} + dr^m e^{-im\theta}$. Imposing the Dirichlet boundary condition at $\theta = 0$ (or $\theta = \pi$) gives $d = -c$, so

$$v(re^{i\theta}) = cr^m \left(e^{im\theta} - e^{-im\theta}\right).$$

Finally, imposing the Neumann boundary condition at $\theta = 0$ (or $\theta = \pi$) gives

$$\mathbf{i}mcr^m = \mathbf{i}mcr^m \left(e^{im\theta} + e^{-im\theta}\right)\big|_{\theta=0} = \partial_\theta v(re^{i\theta})\big|_{\theta=0}$$
$$= c_0 \, \partial_\theta H(re^{i\theta})\big|_{\theta=0} = c_0 \, \mathbf{i}m \left(ar^m e^{im\theta} - br^m e^{-im\theta}\right)\big|_{\theta=0} = c_0 \mathbf{i}mr^m(a - b),$$

which conclude our lemma. $\square$

It is important to note that the blowup limit v satisfies $\Delta v = g\lfloor \tilde{\Gamma}_\pm$ with the special property that g can be expressed in terms harmonic homogeneous polynomial. This structure allows us to rule out blowup solutions supported in sectors of angle $\neq \pi$ and is crucial for their characterization, using the same ideas as in [SS25, Lemma 3.9]:

Lemma 3.3. *Let $\theta_0 \in (0, 2\pi) \setminus \pi\mathbb{Q}$, let $C = \{re^{i\theta} : r > 0, \theta \in (0, \theta_0)\}$, and suppose that w solves*

$$\Delta w = 0 \text{ in } C, \quad w|_{\theta=0} = 0, \quad w|_{\theta=\theta_0} = 0$$

as well as the Bernoulli condition

$$\partial_\theta w|_{\theta=0} = c_0 \, \partial_\theta H(re^{i\theta})\big|_{\theta=0}, \quad \partial_\theta w|_{\theta=\theta_0} = c_0 \, \partial_\theta H(re^{i\theta})\big|_{\theta=\theta_0},$$

where $H(re^{i\theta}) = ar^m e^{im\theta} + br^m e^{-im\theta}$ is a harmonic homogeneous polynomial of degree $m \geq 1$ on $\mathbb{R}^2$. Then $w \equiv 0$ and $H \equiv 0$.

Proof. Note that the solution in Lemma 3.1 also extends to $0 < \theta < 2\pi$, corresponding to an antipodally even or odd extension when m is even or odd, respectively. By unique continuation property, the antipodal extension of the solution v in Lemma 3.1 is the unique solution to

$$(3.6) \qquad \Delta w = 0 \text{ in } C, \quad w|_{\theta=0} = 0, \quad \partial_\theta w|_{\theta=0} = c_0 \, \partial_\theta H(re^{i\theta})\big|_{\theta=0}.$$

From the condition $w|_{\theta=\theta_0} = 0$, we see that

$$(3.7a) \qquad (a - b)\left(e^{im\theta_0} - e^{-im\theta_0}\right) = 0 \quad (\text{choosing } r = 1),$$

and from the condition $\partial_\theta w|_{\theta=\theta_0} = c_0 \, \partial_\theta H(re^{i\theta})\big|_{\theta=\theta_0}$, we see that

$$(a - b)\left(e^{im\theta_0} + e^{-im\theta_0}\right) = ae^{im\theta_0} - be^{-im\theta_0} \quad (\text{choosing } r = 1),$$

equivalently,

$$(3.7b) \qquad -be^{im\theta_0} + ae^{-im\theta_0} = 0.$$

The equations in (3.7) can be written in matrix form as

$$(3.8) \qquad \begin{pmatrix} e^{im\theta_0} - e^{-im\theta_0} & -e^{im\theta_0} + e^{-im\theta_0} \\ e^{-im\theta_0} & -e^{im\theta_0} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0.$$

We compute that

$$\det \begin{pmatrix} e^{\mathbf{i}m\theta_0} - e^{-\mathbf{i}m\theta_0} & -e^{\mathbf{i}m\theta_0} + e^{-\mathbf{i}m\theta_0} \\ e^{-\mathbf{i}m\theta_0} & -e^{\mathbf{i}m\theta_0} \end{pmatrix}$$

$$= -e^{2\mathbf{i}m\theta_0} + 2 - e^{-2\mathbf{i}m\theta_0} = -(e^{\mathbf{i}m\theta_0} - e^{-\mathbf{i}m\theta_0})^2$$

$$= -(2\mathbf{i}\sin(m\theta_0))^2 = 4\sin^2(m\theta_0) \neq 0$$

since $\theta_0 \in (0, 2\pi) \setminus \pi\mathbb{Q}$, therefore we conclude that $a = b = 0$, which is our lemma. $\qquad \square$

We now ready to prove our main result.

Proof of Theorem 1.7. Suppose, to the contrary, that the medium does not scatter for some incident wave $u^{\mathrm{inc}} \not\equiv 0$. Then, by Lemma 3.3, the blowup limit v of u^{sc} of order m at 0 must vanish identically. From (3.5), it follows that

$$(\nu_\pm \cdot \nabla H)\mathcal{H}^{n-1}\lfloor \tilde{\Gamma}_\pm = 0.$$

Without loss of generality, we may assume $x_0 = 0$, $\tilde{\Gamma}_+ = \{re^{\mathbf{i}\theta} : r > 0, \theta = 0\}$ and $\tilde{\Gamma}_- = \{re^{\mathbf{i}\theta} : r > 0, \theta = \theta_0\}$. In this case,

$$\partial_\theta H(re^{\mathbf{i}\theta})\big|_{\theta=0} = \partial_\theta H(re^{\mathbf{i}\theta})\big|_{\theta=\theta_0} = 0.$$

Any harmonic homogeneous polynomial of order $m \geq 1$ can be written as $H(re^{\mathbf{i}\theta}) = ar^m e^{\mathbf{i}m\theta} + br^m e^{-\mathbf{i}m\theta}$ for some constants $a, b \in \mathbb{C}$. The condition $\partial_\theta H(re^{\mathbf{i}\theta})\big|_{\theta=0} = 0$ gives $b = a$, so

$$H(re^{\mathbf{i}\theta}) = a(r^m e^{\mathbf{i}m\theta} + r^m e^{-\mathbf{i}m\theta}).$$

Then the condition $\partial_\theta H(re^{\mathbf{i}\theta})\big|_{\theta=\theta_0} = 0$ yields

$$0 = a\left(e^{\mathbf{i}m\theta_0} - e^{-\mathbf{i}m\theta_0}\right) = 2\mathbf{i}a\sin(m\theta_0).$$

Since $\theta_0 \notin \pi\mathbb{Q}$, we must have $a = 0$, hence $H \equiv 0$, contradicting the fact that H is a harmonic homogeneous polynomial of order $m \geq 1$. $\qquad \square$

Next, we consider the case $n \geq 3$. In an earlier work by two of the current authors, the homogeneity of the blowup limit was needed to apply the well-known Federer's dimension reduction argument. This required the introduction of a balanced energy functional and its monotonicity. In the Bernoulli case this is slightly more complicated, and would need a technical computation; one such possible monotonicity formula can be found in [CSY18]. Since the situation in our case (and also in the work of [SS25]) is somewhat easier, one can avoid such a monotonicity functional, and circumvent the issue using the simple geometric situation.[5] The idea is to use the unique continuation property, adapting the ideas in [SS25, Lemma 3.3]:

Lemma 3.4. *Let $n \geq 3$ and let*

$$\tilde{\Gamma}_\pm = \{re^{\mathbf{i}\theta_\pm} \in \mathbb{C} \cong \mathbb{R}^2 : r \geq 0\} \times \mathbb{R}^{n-2}$$

with distinct $\theta_\pm \in [0, 2\pi)$ and $\nu_\pm$ being the corresponding unit normal vectors. Let $w \in C^{0,1}(\overline{B}_1)$ be a solution to

$$(3.9) \qquad \Delta w = c_0 \sum_\pm (\nu_\pm \cdot \nabla P)\mathcal{H}^{n-1}\lfloor \tilde{\Gamma}_\pm \quad \text{in } B_1 \subset \mathbb{R}^n,$$

---

[5]It should be remarked that the use of monotonicity formula may actually enhance the main result and achieve a stronger version of the theorem. We may come back to this in the future, as it is too technical for the current paper.

and assume that $w = 0$ in one connected component of $B_1 \setminus (\tilde{\Gamma}_+ \cup \tilde{\Gamma}_-)$, where P is a homogeneous polynomial of order k. Then w is a homogeneous polynomial of order k.

Proof. Since both the Laplacian and polynomial homogeneity are invariant under orthogonal transformations, it suffices to prove the lemma in the case $\theta_+ = \pi/2$, that is,

$$\tilde{\Gamma}_+ = \{0\} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-2}.$$

Let D and V be the connected components of $B_1 \setminus (\tilde{\Gamma}_+ \cup \tilde{\Gamma}_-)$ such that $w = 0$ in V. Without loss of generality, assume that w vanishes to the left of $\tilde{\Gamma}_+$ (i.e. V contains $B(\frac{1}{2}e_2, \delta) \cap \{x_1 < 0\}$ for some $\delta > 0$). As in the beginning of Section 2, it follows that

(3.10) $$\Delta w = 0 \text{ in } D \cap \{x_1 > 0\}, \quad w|_{B_1 \cap \tilde{\Gamma}_+} = 0, \quad \partial_1 w|_{B_1 \cap \tilde{\Gamma}_+} = -c_0 \partial_1 P|_{B_1 \cap \tilde{\Gamma}_+}.$$

By the unique continuation property, there is at most one solution to (3.10). By the Cauchy-Kowalevski theorem, there exists an analytic solution

$$w(x) = \sum_{\ell=0}^{\infty} \frac{\partial_1^\ell w|_{\tilde{\Gamma}_+}}{\ell!} x_1^\ell = x_1 \partial_1 P|_{\tilde{\Gamma}_+} + \sum_{\ell=2}^{\infty} \frac{\partial_1^\ell w|_{\tilde{\Gamma}_+}}{\ell!} x_1^\ell$$

for all $x \in U$ with $x_1 > 0$. We compute that

$$0 = \Delta_{x'} w + \partial_1^2 w = x_1 \Delta_{x'} \partial_1 P|_{\tilde{\Gamma}_+} + \sum_{\ell=2}^{\infty} \frac{\Delta_{x'} \partial_1^\ell w|_{\tilde{\Gamma}_+}}{\ell!} x_1^\ell + \sum_{\ell=0}^{\infty} \frac{\partial_1^{\ell+2} w|_{\tilde{\Gamma}_+}}{\ell!} x_1^\ell$$

$$= \partial_1^2 w|_{\tilde{\Gamma}_+} + \left(\Delta_{x'} \partial_1 P|_{\tilde{\Gamma}_+} + \partial_1^3 w|_{\tilde{\Gamma}_+} \right) x_1 + \sum_{\ell=2}^{\infty} \frac{(\Delta_{x'} \partial_1^\ell w + \partial_1^{\ell+2} w)|_{\tilde{\Gamma}_+}}{\ell!} x_1^\ell$$

for all $x \in U$ with $x_1 > 0$. Comparing coefficients, we obtain

$$\partial_1^{2\ell} w|_{\tilde{\Gamma}_+} = 0, \quad \partial_1^{2\ell+1} w|_{\tilde{\Gamma}_+} = (-1)^\ell \Delta_{x'}^\ell \partial_1 P|_{\tilde{\Gamma}_+} \quad \text{for all } \ell \in \mathbb{N}.$$

Since $\partial_1 P|_{\tilde{\Gamma}_+}$ is a polynomial of order $k-1$, then

$$\partial_1^{2\ell+1} w|_{\tilde{\Gamma}_+} = 0 \quad \text{for all } \ell \in \mathbb{N} \text{ with } 2\ell + 1 > k.$$

Putting the above discussions together, we conclude

(3.11) $$w(x) = -c_0 \sum_{\ell=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-1)^\ell \Delta_{x'}^\ell \partial_1 P|_{B_1 \cap \tilde{\Gamma}_+}}{(2\ell+1)!} x_1^{2\ell+1} \quad \text{in } D \cap \{x_1 > 0\}$$

is the unique solution to (3.10). Again, by the unique continuation property, the solution to (3.9) must coincide, up to an orthogonal transformation, with (3.11) in D, which establishes the result. $\square$

We are now in a position to prove Theorem 1.9.

Proof of Theorem 1.9. Suppose, to the contrary, that the medium does not scatter for some incident wave $u^{\text{inc}} \not\equiv 0$. The first two steps are the same as in the proof of the two-dimensional case (Theorem 1.7):

- First, one obtains the Lipschitz regularity of u^{sc} as in (3.1);

- Next, one sees that any blowup limit v satisfies

$$\Delta v = c_0 \sum_{\pm} (\nu_{\pm} \cdot \nabla H) \mathcal{H}^{n-1} \lfloor \tilde{\Gamma}_{\pm} \quad \text{in } B_1 \subset \mathbb{R}^n,$$

and $v = 0$ in a component of $B_1 \setminus (\tilde{\Gamma}_+ \cup \tilde{\Gamma}_-)$.

Without loss of generality, assume $\theta_+ = 0$. We now apply Federer's dimension reduction argument, as mentioned earlier. Fix a point $e = e_n \in \Gamma_+ \cap \Gamma_-$. Then H has a zero at e of order $1 \leq k \leq m$. By a direct application of Lemma 2.1,

$$|v(z+e)| + |z||\nabla v(z+e)| \leq C|z|^k,$$

which naturally motivates the scaling function $v_r := v(rz+e)/r^k$. Next, the blowup limit w of v_r satisfies (3.9) for some harmomic homogeneous polynomial of order k satisfying $H(e+z) = P(z) + O(|z|^{k+1})$, and $w = 0$ in a component of $B_1 \setminus (\tilde{\Gamma}_+ \cup \tilde{\Gamma}_-)$. Using Lemma 3.4, w is homogeneous or order k. Following the arguments in the proof of [SS25, Theorem 1.10], one can show that w is independent of z_n, reducing the problem to $\mathbb{R}^{n-1}$. Iterating this procedure eventually reduces the problem to $\mathbb{R}^2$, after which the remaining steps follow from the proof of Theorem 1.7. $\square$

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Declarations

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References

[AI96] G. Alessandrini and V. Isakov. Analyticity and uniqueness for the inverse conductivity problem. *Rend. Istit. Mat. Univ. Trieste*, 28(1–2):351–369, 1996. MR1463923, Zbl:0887.35157, https://rendiconti.dmi.units.it.

[Blå18] E. Blåsten. Nonradiating sources and transmission eigenfunctions vanish at corners and edges. *SIAM J. Math. Anal.*, 50(6):6255–6270, 2018. MR3885754, Zbl:1409.35164, doi:10.1137/18M1182048, arXiv:1803.10917.

[BL17] E. Blåsten and H. Liu. On vanishing near corners of transmission eigenfunctions. *J. Funct. Anal.*, 273(11):3616–3632, 2017. MR3706612, Zbl:1387.35437, doi:10.1016/j.jfa.2017.08.023, arXiv:1701.07957 [Addendum: arXiv:1710.08089].

[BL21a] E. Blåsten and H. Liu. On corners scattering stably and stable shape determination by a single far-field pattern. *Indiana Univ. Math. J.*, 70(3):907–947, 2021. MR4284101, Zbl:1479.35648, doi:10.1512/IUMJ.2021.70.8411, arXiv:1611.03647.

[BL21b] E. Blåsten and H. Liu. Scattering by curvatures, radiationless sources, transmission eigenfunctions, and inverse scattering problems. *SIAM J. Math. Anal.*, 53(4):3801–3837, 2021. MR4283699, Zbl:1479.35838, doi:10.1137/20M1384002, arXiv:1808.01425.

[BPS14]   E. Blåsten, L. Päivärinta, and J. Sylvester. Corners always scatter. *Comm. Math. Phys.*, 331(2):725–753, 2014. MR3238529, Zbl:1298.35214, doi:10.1007/s00220-014-2030-0, arXiv:1211.1848.

[BK82]   L. Brown and J.-P. Kahane. A note on the Pompeiu problem for convex domains. *Math. Ann.*, 259(1):107–110, 1982. MR0656655, Zbl:0464.30035, doi:10.1007/BF01456832.

[CSY18]   L. A. Caffarelli, H. Shahgholian, and K. Yeressian. A minimization problem with free boundary related to a cooperative system. *Duke Math. J.*, 167(10):1825–1882, 2018. MR3827812, Zbl:1395.35226, doi:10.1215/00127094-2018-0007, arXiv:1608.07689.

[CCH23]   F. Cakoni, D. Colton, and H. Haddar. *Inverse scattering theory and transmission eigenvalues*, volume 98 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 2023. MR4539629, Zbl:7647933, doi:10.1137/1.9781611977424.

[CV23]   F. Cakoni and M. S. Vogelius. Singularities almost always scatter: regularity results for non-scattering inhomogeneities. *Comm. Pure Appl. Math.*, 76(12):4022–4047, 2023. MR4655358, Zbl:1526.35249, doi:10.1002/cpa.22117, arXiv:2104.05058.

[CVX23]   F. Cakoni, M. S. Vogelius, and J. Xiao. On the regularity of non-scattering anisotropic inhomogeneities. *Arch. Rational Mech. Anal.*, 247(31), 2023. MR4571295, Zbl:07673750, doi:10.1007/s00205-023-01863-y, arXiv:2208.13231.

[CX21]   F. Cakoni and J. Xiao. On corner scattering for operators of divergence form and applications to inverse scattering. *Comm. Partial Differential Equations*, 46(3):413–441, 2021. MR4232500, Zbl:1469.35164, doi:10.1080/03605302.2020.1843489, arXiv:1905.02558.

[CDLZ21] X. Cao, H. Diao, H. Liu, and J. Zou. On novel geometric structures of laplacian eigenfunctions in $\mathbb{R}^3$ and applications to inverse problems. *SIAM J. Math. Anal.*, 53(2):1263–1294, 2021. MR4223217, Zbl:1461.35167, doi:10.1137/19M1292989, arXiv:1909.10174.

[CDLZ22] X. Cao, H. Diao, H. Liu, and J. Zou. Two single-measurement uniqueness results for inverse scattering problems within polyhedral geometries. *Inverse Probl. Imaging*, 16(6):1501–1528, 2022. MR4520372, Zbl:1512.35655, doi:10.3934/ipi.2022023, arXiv:2111.13886.

[CHLX25] L. Chesnel, H. Haddar, H. Li, and J. Xiao. Examples of non-scattering inhomogeneities. *Inverse Probl. Imaging*, 19(5):940–964, 2025. MR4907043, Zbl:8054736, doi:10.3934/ipi.2025004, arXiv:2406.17527.

[CK19]   D. Colton and R. Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93 of *Applied Mathematical Sciences*. Springer, Cham, fourth edition, 2019. MR3971246, Zbl:1425.35001, doi:10.1007/978-3-030-30351-8.

[EF14]   C. Efthimiou and C. Frye. *Spherical harmonics in p dimensions*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2014. MR3290046, Zbl:1296.33001, doi:10.1142/9134, arXiv:1205.3548.

[EH15]   J. Elschner and G. Hu. Corners and edges always scatter. *Inverse Problems*, 31(1):015003, 17 pp., 2015. MR3302364, Zbl:1319.35140, doi:10.1088/0266-5611/31/1/015003.

[EH18]   J. Elschner and G. Hu. Acoustic scattering from corners, edges and circular cones. *Arch. Ration. Mech. Anal.*, 228(2):653–690, 2018. MR3766986, Zbl:1392.35224, doi:10.1007/s00205-017-1202-4, arXiv:1603.05186.

[Gri11]   P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 69 of *Classics Appl. Math.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. Reprint of the 1985 original. MR3396210, Zbl:1231.35002, doi:10.1137/1.9781611972030.ch1.

[HV25a]   N. Hovsepyan and M. S. Vogelius. Scattering from analytic and piecewise analytic inhomogeneities. *arXiv preprint*, 2025. arXiv:2507.13986.

[HV25b]   N. Hovsepyan and M. S. Vogelius. Scattering of plane waves. *arXiv preprint*, 2025. arXiv:2509.19716.

[HSV16]   G. Hu, M. Salo, and E. V. Vesalainen. Shape identification in inverse medium scattering problems with a single far-field pattern. *SIAM J. Math. Anal.*, 48(1):152–165, 2016. MR3439763, Zbl:1334.35427, doi:10.1137/15M1032958, arXiv:1507.07846.

[KG08]   A. Kirsch and N. Grinberg. *The factorization method for inverse problems*, volume 36 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2008. MR2378253, Zbl:1222.35001, doi:10.1093/acprof:oso/9780199213535.001.0001.

[KLSS24] P.-Z. Kow, S. Larson, M. Salo, and H. Shahgholian. Quadrature domains for the Helmholtz equation with applications to non-scattering phenomena. *Potential Anal.*, 60(1):387–424, 2024. MR4696043, Zbl:7798456 doi:10.1007/s11118-022-10054-5. The results in the appendix are well-known, and the proofs can found at arXiv:2204.13934.

[KSS23] P.-Z. Kow, M. Salo, and H. Shahgholian. On positivity sets for Helmholtz solutions. *Vietnam J. Math.*, 51:985–994, 2023. Special issue dedicated to Carlos E. Kenig on the occation of his 70th birthday. MR4669027, Zbl:7785099, doi:10.1007/s10013-023-00646-y, arXiv:2301.04965.

[KSS24] P.-Z. Kow, M. Salo, and H. Shahgholian. On scattering behavior of corner domains with anisotropic inhomogeneities. *SIAM J. Math. Anal.*, 56(4):4834–4853, 2024. MR4768461, Zbl:1547.35198, doi:10.1137/23M1603029, arXiv:2309.11213.

[PSV17] L. Päivärinta, M. Salo, and E. V. Vesalainen. Strictly convex corners scatter. *Rev. Mat. Iberoam.*, 33(4):1369–1396, 2017. MR3729603, Zbl:1388.35138, doi:10.4171/RMI/975, arXiv:1404.2513.

[PSU12] A. Petrosyan, H. Shahgholian, and N. Uraltseva. *Regularity of free boundaries in obstacle-type problems*, volume 136 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012. MR2962060, Zbl:1254.35001, doi:10.1090/gsm/136.

[Pom29] P. Pompeiu. Sur certains systèmes d'équations linéaires et sur une propriété intégrale des fonctions de plusieurs variables. *Comptes Rendus de l'Académie des Sciences. Série I. Mathématique*, 188:1138–1139, 1929. JFM:55.0139.02.

[SS21] M. Salo and H. Shahgholian. Free boundary methods and non-scattering phenomena. *Res. Math. Sci.*, 8(4):Paper No. 58, 2021. MR4323345, Zbl:1480.35408, doi:10.1007/s40687-021-00294-z, arXiv:2106.15154.

[SS25] M. Salo and H. Shahgholian. A free boundary approach to non-scattering obstacles with vanishing contrast. *arXiv preprint*, 2025. arXiv:2506.22328.

[Vel23] B. Velichkov. *Regularity of the one-phase free boundaries*, volume 28 of *Lect. Notes Unione Mat. Ital.* Springer, Cham, 2023. MR4807210, Zbl:7582511, doi:10.1007/978-3-031-13238-4.

[VX21] M. Vogelius and J. Xiao. Finiteness results concerning nonscattering wave numbers for incident plane and Herglotz waves. *SIAM J. Math. Anal.*, 53(5):5436–5464, 2021. MR4319099, Zbl:1475.35126, doi:10.1137/20M1367854.

[VX25] M. S. Vogelius and J. Xiao. Finiteness results for non-scattering Herglotz waves: The case of inhomogeneities obtained by very general perturbations of disks. *SIAM J. Math. Anal.*, 2025. doi:10.1137/24M1709844, arXiv:2411.07743.

[Wei99] G. S. Weiss. Partial regularity for a minimum problem with free boundary. *J. Geom. Anal.*, 9(2):317–326, 1999. MR1759450, Zbl:0960.49026, doi:10.1007/BF02921941.

[Wil76] S. A. Williams. A partial solution of the Pompeiu problem. *Math. Ann.*, 223(2):183–190, 1976. MR0414904, Zbl:0329.35045, doi:10.1007/BF01360881.

[Wil81] S. A. Williams. Analyticity of the boundary for Lipschitz domains without the Pompeiu property. *Indiana Univ. Math. J.*, 30(3):357–369, 1981. MR0611225, Zbl:0439.35046, doi:10.1512/iumj.1981.30.30028.

[Yer16] K. Yeressian. Obstacle problem with a degenerate force term. *Anal. PDE*, 9(2):397–437, 2016. MR3513139, Zbl:1338.35511, doi:10.2140/apde.2016.9.397.

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