INVERSE SCATTERING FOR SINGLE PHOTONS IN QUANTUM OPTICS

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Abstract. We study inverse problems for a time-harmonic one-photon model describing the interaction of a single photon with a medium of stationary two-level atoms. After time-harmonic reduction, the unknown compactly supported atomic density appears as a frequency-dependent potential in a scattering equation for the half Laplacian. We prove high-frequency uniqueness results for three types of intensity data: source-driven measurements, renormalized far-field intensity measurements, and phaseless far-field measurements obtained from coherent superpositions of incident plane waves. In each case, the corresponding data, given at all sufficiently large frequencies, determine the atomic density uniquely; the source-driven result requires a geometric visibility condition on the source and observation sets.

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1. Introduction

1.1. **Physical motivation and the model.** This paper is concerned with inverse problems for a quantum model of light–matter interactions. The physical setting is a single photon propagating through a medium of stationary two-level atoms, described by an atomic number density $\rho(x)$ supported in a bounded region of $\mathbb{R}^n$. The photon may be absorbed, exciting an atom, and may subsequently be re-emitted. The resulting cooperative dynamics is a basic many-body process in quantum optics and is closely related to phenomena such as single-photon superradiance and subradiance [6, 12, 21].

A continuum real-space formulation of this problem was introduced in [21]. In the one-photon sector, the photon and atomic amplitudes satisfy the coupled system

$$\begin{cases} i\partial_t \psi = c(-\Delta)^{1/2}\psi + g\,\rho\,a, \\ i\partial_t a = g\,\psi + \Omega\,a. \end{cases} \tag{1.1}$$

Here $\psi(x,t)$ is the photon amplitude, $a(x,t)$ is the atomic excitation amplitude, $\Omega > 0$ is the atomic resonance frequency, $g > 0$ is the atom–field coupling constant, and $c > 0$ is the speed of light. The operator $(-\Delta)^{1/2}$ reflects the linear photon dispersion relation $\omega_{\mathbf{k}} = c|\mathbf{k}|$. The atomic density ρ enters the system through the coupling term and is the unknown coefficient in the inverse problems considered below. The present work complements our earlier study of a time-dependent inverse problem for entangled two-photon states [22].

1.2. **Inverse problems and main results.** The purpose of the present paper is to initiate the study of inverse problems for the time-harmonic form of (1.1). We set $c = 1$, which amounts to a rescaling of time, and look for monochromatic solutions of the form

$$\psi(x,t) = e^{-ikt}\psi(x), \qquad a(x,t) = e^{-ikt}a(x),$$

where $k > 0$ is the frequency. This gives the stationary system

$$\begin{cases} k\psi = (-\Delta)^{1/2}\psi + g\rho a, \\ ka = g\psi + \Omega a. \end{cases} \tag{1.2}$$

For $k \neq \Omega$, the second equation determines a algebraically in terms of ψ, and substitution into the first one yields the equation

$$P_k\psi := \left((-\Delta)^{1/2} - k + \frac{g^2}{k-\Omega}\rho\right)\psi = 0. \tag{1.3}$$

Thus the unknown atomic density ρ enters as an effective potential in a fractional equation. We assume throughout that $\rho \in C_c^\infty(\mathbb{R}^n)$ is non-negative and not identically zero, and we write $\Sigma = \operatorname{supp}(\rho)$.

The inverse problem we consider is to determine ρ from measurements of the photon field ψ. A distinctive feature of the present model is that the physically natural data

are of *intensity-type*. Photodetection gives access to probability densities such as $|\psi|^2$, but the complex phase of ψ is not directly measurable. The inverse problems we consider are therefore phaseless: the unknown ρ must be reconstructed from squared moduli of the photon amplitude.

We study three types of intensity measurements, described informally in the following. The first is a near-field measurement generated by localized sources. Sources are placed outside the support of ρ, and the resulting intensity is observed in another exterior region. The question is whether these source-to-intensity data determine the atomic density.

The second and third measurements are taken in the far-field. In the second case, for each illumination the medium is probed by a single incident plane wave, with the incident direction varying over $\mathbb{S}^{n-1}$, and one measures a suitably renormalized intensity of the total field at infinity. Although the total intensity tends to the background value at large distance, its next-order asymptotic term contains information about the scattering amplitude.

In the third case, the incident field is a coherent superposition of two plane waves with several prescribed relative phases. The corresponding phaseless far-field measurements allow us to recover certain mixed products of scattering amplitudes, which are sufficient for uniqueness.

The precise definitions of these three data sets are given in Section 2. Our main results, stated there as Theorems 2.3, 2.4, and 2.5, show that each data set determines ρ uniquely at all sufficiently large frequencies. The source-driven result requires a geometric visibility condition on the source and observation sets; the two far-field results do not.

1.3. **Far-field measurements.** A common feature of the three inverse problems is that the data are intensities: photodetection measures $|\psi|^2$, and not the phase of ψ directly. It is therefore natural, in a far-field scattering setting, to compare the present data with the usual phaseless far-field pattern $|A_k(\theta,\alpha)|^2$. In that formulation the phase of the scattering amplitude is lost, and uniqueness typically requires additional structure, such as reference scatterers, superpositions of incident waves, or measurements at many frequencies. The second inverse problem in this paper is based on a different observation. The measured quantity is the intensity of the total field

$$\psi^{\mathrm{in}}_{k,\alpha} + \psi^{\mathrm{sc}}_{k,\alpha},$$

where the incident plane wave $\psi^{\mathrm{in}}_{k,\alpha}(x) = e^{ik\alpha\cdot x}$ with $\alpha \in \mathbb{S}^{n-1}$ is known. Thus

$$\left|\psi^{\mathrm{in}}_{k,\alpha} + \psi^{\mathrm{sc}}_{k,\alpha}\right|^2 = 1 + 2\,\Re\left(\overline{\psi^{\mathrm{in}}_{k,\alpha}}\,\psi^{\mathrm{sc}}_{k,\alpha}\right) + \left|\psi^{\mathrm{sc}}_{k,\alpha}\right|^2.$$

The cross term is an interference term between the known incident wave and the outgoing scattered wave. After subtracting the background intensity and applying the natural far-field renormalization, which we encode in a renormalized total-field intensity $\mathcal{I}_\rho$ (defined in Section 2), this interference term produces an oscillatory quantity, oscillating in the radial variable at a frequency fixed by the incident and observation directions. More precisely, for non-forward directions $\theta \neq \alpha$, we prove that

$$A_k(\theta,\alpha) = \lim_{R\to\infty} \frac{1}{R}\int_R^{2R} \mathcal{I}_\rho(\alpha,k,\theta,r)\, e^{-ikr(1-\alpha\cdot\theta)}\, dr.$$

Thus, the asymptotic behaviour of the renormalized total-field intensity determines the full complex scattering amplitude $A_k(\theta, \alpha)$. In this sense, the second inverse problem is closer to an interferometric measurement model than to the standard phaseless far-field pattern problem: the known incident wave serves as a reference field, and no separate phase-retrieval step is needed once the renormalized asymptotic intensity is known. The third inverse problem, by contrast, uses genuinely phaseless far-field data. There the measured quantities are squared moduli $|A_k^{(a)}|^2$ of far-field amplitudes corresponding to superpositions of two incident plane waves. Combining the four relative phases $a \in \{1, -1, i, -i\}$ recovers the mixed products

$$A_k(\theta, \alpha_2)\overline{A_k(\theta, \alpha_1)},$$

which are then used, through their high-frequency asymptotics, to determine ρ.

1.4. **Related work.** While inverse problems for the one-photon model (1.1) have not been considered before (to the best of our knowledge), the half-Laplacian and, more generally, the fractional Laplacians $(-\Delta)^s$ with $s \in (0, 1)$ have received considerable attention in the inverse problems literature.

In the Calderón-type setting, Ghosh, Salo, and Uhlmann [11] proved that exterior Dirichlet-to-Neumann data determine the potential in

$$(-\Delta)^s u + qu = 0, \qquad 0 < s < 1.$$

This line of work relies on the unique continuation and Runge approximation properties of the fractional Laplacian, and has since been developed in many directions; see, for example, [25, 3, 4, 10]. The present paper is different both in setting and in data: we work on all of $\mathbb{R}^n$ in a scattering framework, and the data are high-frequency intensity measurements rather than exterior Dirichlet-to-Neumann data.

A closer scattering-theoretic direction concerns the half Laplacian and its fractional generalizations. For the time-dependent problem, scattering and inverse scattering have been studied, for example, in [18, 13, 14]; see also [17] for the related Dirac equation. For further recent results on fractional Helmholtz equations, we refer to [5, 30]. In the stationary setting, Uhlmann and Wang [28] recover the asymptotic behaviour of polyhomogeneous potentials at infinity from fixed-energy scattering data. In contrast, we recover the compactly supported coefficient ρ, which enters through the frequency-dependent potential $V_k = g^2\rho/(k - \Omega)$, from intensity data at all sufficiently large frequencies.

Classical inverse scattering for Schrödinger operators provides another relevant context. For $-\Delta+V$, high-frequency Born approximation methods show that the leading term of the scattering amplitude determines the Fourier transform of the potential; see, for instance, Saito [26]. In phaseless inverse scattering, uniqueness from phaseless data often requires additional information such as reference sources, background scatterers, superpositions of incident waves, or varying-frequency measurements; see, for example, [19, 20, 23, 24, 15]. The superposition mechanism is particularly close to our third inverse problem: by using two incident plane waves with four prescribed relative phases, we recover mixed products of scattering amplitudes, which suffice to determine ρ.

1.5. **Organization of the paper.** The paper is organized as follows. In Section 2 we give the precise formulation of the three inverse problems and state the main uniqueness results. Section 3 develops the direct theory for P_k, including the uniform free resolvent estimate, the limiting absorption principle for the perturbed operator,

and the far-field expansion. In Section 4 we prove the source-driven uniqueness result by reducing the problem to the recovery of line integrals of ρ and then to the injectivity of the X-ray transform. Section 5 proves the two far-field uniqueness results, first for renormalized far-field intensity data and then for phaseless superposition data. The appendix contains the technical estimates used in the geometric-optics approximation.

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2. Formulation of the inverse problems and main results

Let $n \geq 2$ and let $\rho \in C_c^\infty(\mathbb{R}^n)$ be a non-negative, non-zero function with

$$\Sigma = \operatorname{supp}(\rho) \subset \mathbb{R}^n.$$

Let $g, \Omega > 0$ be fixed constants and let $k > 0$ satisfy $k \neq \Omega$. Eliminating a from the stationary one-photon system

$$\begin{cases} k\psi = (-\Delta)^{1/2}\psi + g\rho a, \\ ka = g\psi + \Omega a, \end{cases}$$

yields a single equation for ψ:

$$P_k\psi := \left((-\Delta)^{1/2} - k + \frac{g^2}{k-\Omega}\rho\right)\psi = 0. \tag{2.1}$$

We also use the notation

$$P_k = P_k^0 + V_k, \qquad P_k^0 = (-\Delta)^{1/2} - k, \qquad V_k = \frac{g^2}{k-\Omega}\rho.$$

In this paper we consider three inverse problems associated with (2.1). The first one is a source-driven near-field problem. The second and third ones are far-field problems formulated in terms of intensity-type data.

2.1. Inverse Problem I: Source-driven near-field data.

Let $S \subset \mathbb{R}^n$ be a non-empty open set such that $S \cap \Sigma = \varnothing$. Given $f \in C_c^\infty(S)$, we consider the forced problem

$$P_k\psi = f \quad \text{in } \mathbb{R}^n. \tag{2.2}$$

By Lemma 3.5, there exists $k_\rho > 0$ such that, for each $k > k_\rho$, problem (2.2) has a unique outgoing solution in the sense of the limiting absorption principle. We denote this solution by ψ^f.

Let $W \subset \mathbb{R}^n$ be a non-empty open set with $W \cap \Sigma = \varnothing$. For each $k > k_\rho$, the corresponding measurement map is

$$\Lambda_k^{\mathrm{src}} : C_c^\infty(S) \to C^0(W),$$
$$\Lambda_k^{\mathrm{src}}(f) = \left|\psi^f\right|_W^2.$$

The first inverse problem is to determine the density ρ from the knowledge of the family of measurement maps $\{\Lambda_k^{\mathrm{src}}\}$ for all sufficiently large k.

Remark 2.1. The assumptions $S \cap \Sigma = \varnothing$ and $W \cap \Sigma = \varnothing$ are made only for convenience: since Σ is closed, one may replace S and W by the open sets $S \setminus \Sigma$ and $W \setminus \Sigma$, and all the results below remain valid provided the latter sets satisfy the conditions imposed on S and W.

2.2. **Inverse Problem II: Renormalized far-field intensity data.** For $\alpha \in \mathbb{S}^{n-1}$, we consider the incident plane wave

$$\psi_{k,\alpha}^{\mathrm{in}}(x) = e^{ik\alpha\cdot x}.$$

The corresponding total field is written as

$$\psi_{k,\alpha} = \psi_{k,\alpha}^{\mathrm{in}} + \psi_{k,\alpha}^{\mathrm{sc}}, \tag{2.3}$$

where the scattered field $\psi_{k,\alpha}^{\mathrm{sc}}$ is the unique outgoing solution of

$$P_k\psi_{k,\alpha}^{\mathrm{sc}} = -V_k\psi_{k,\alpha}^{\mathrm{in}} \quad \text{in } \mathbb{R}^n. \tag{2.4}$$

The right-hand side belongs to $C_c^\infty(\mathbb{R}^n)$. By Lemma 3.5, for every admissible density ρ there exists $k_\rho > 0$ such that, for all $k > k_\rho$, equation (2.4) has a unique outgoing solution in the sense of the limiting absorption principle. Consequently, the total field $\psi_{k,\alpha}$ satisfies

$$P_k\psi_{k,\alpha} = 0 \quad \text{in } \mathbb{R}^n.$$

As shown later in Lemma 3.7, the total field admits the far-field expansion

$$\psi_{k,\alpha}(r\theta) = e^{ik\alpha\cdot(r\theta)} + r^{-(n-1)/2}e^{ikr}A_k(\theta,\alpha) + o(r^{-(n-1)/2}), \qquad r \to \infty, \tag{2.5}$$

where $\theta \in \mathbb{S}^{n-1}$. The coefficient $A_k(\theta,\alpha)$ is the scattering amplitude.

In the far-field regime, the directly measurable quantity is the intensity $|\psi_{k,\alpha}(r\theta)|^2$. Using (2.5), we obtain

$$|\psi_{k,\alpha}(r\theta)|^2 = 1 + 2r^{-(n-1)/2}\Re\Big(e^{ikr(1-\alpha\cdot\theta)}A_k(\theta,\alpha)\Big) + o\big(r^{-(n-1)/2}\big), \qquad r \to \infty. \tag{2.6}$$

In particular, $|\psi_{k,\alpha}(r\theta)|^2 \to 1$ as $r \to \infty$, so the limit itself carries no information about ρ. The relevant object is the asymptotic behaviour at the next order, namely

$$2r^{-(n-1)/2}\Re\Big(e^{ikr(1-\alpha\cdot\theta)}A_k(\theta,\alpha)\Big).$$

Accordingly, we introduce the renormalized intensity

$$\mathcal{I}_\rho(\alpha,k,\theta,r) := r^{(n-1)/2}\Big(|\psi_{k,\alpha}(r\theta)|^2 - 1\Big), \tag{2.7}$$

which by (2.6) satisfies

$$\mathcal{I}_\rho(\alpha,k,\theta,r) = 2\Re\Big(e^{ikr(1-\alpha\cdot\theta)}A_k(\theta,\alpha)\Big) + o(1), \qquad r \to \infty. \tag{2.8}$$

The data in the second inverse problem are not the values of $\mathcal{I}_\rho$ at finite radii. Rather, they are its asymptotic behaviour as $r \to \infty$, modulo terms that vanish at infinity.

We say that two renormalized far-field intensities $\mathcal{I}_{\rho_1}$ and $\mathcal{I}_{\rho_2}$ are equivalent at infinity, and write

$$\mathcal{I}_{\rho_1} \sim_\infty \mathcal{I}_{\rho_2},$$

if there exists $K > 0$ such that

$$\mathcal{I}_{\rho_1}(\alpha, k, \theta, r) - \mathcal{I}_{\rho_2}(\alpha, k, \theta, r) \to 0 \qquad \text{as } r \to \infty \tag{2.9}$$

for every $k > K$ and every $\alpha, \theta \in \mathbb{S}^{n-1}$ with $\theta \neq \alpha$. We denote the corresponding asymptotic equivalence class by $[\mathcal{I}_\rho]_\infty$.

The second inverse problem is to determine the density ρ from the knowledge of the asymptotic equivalence class $[\mathcal{I}_\rho]_\infty$ for all sufficiently large k.

2.3. **Inverse Problem III: Phaseless far-field data from superposed waves.** In the third inverse problem, we consider phaseless far-field measurements arising from coherent superpositions of two plane waves.

Let $k > k_\rho$, $\alpha_1, \alpha_2, \theta \in \mathbb{S}^{n-1}$, and $a \in \mathbb{C}$. We consider the incident field

$$\psi^{\mathrm{in},a}_{k,\alpha_1,\alpha_2}(x) = e^{ik\alpha_1\cdot x} + a\, e^{ik\alpha_2\cdot x}.$$

Let $\psi^a_{k,\alpha_1,\alpha_2}$ denote the corresponding total field, and let $A^{(a)}_k(\theta, \alpha_1, \alpha_2)$ denote the corresponding scattering amplitude defined via the analogue of (2.5). Since the scattering problem is linear in the incident field,

$$A^{(a)}_k(\theta, \alpha_1, \alpha_2) = A_k(\theta, \alpha_1) + a\, A_k(\theta, \alpha_2). \tag{2.10}$$

The measurement maps are

$$\mathcal{M}_k : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \{1, -1, i, -i\} \times \mathbb{S}^{n-1} \to \mathbb{R},$$

$$\mathcal{M}_k(\alpha_1, \alpha_2, a, \theta) = \left| A^{(a)}_k(\theta, \alpha_1, \alpha_2) \right|^2.$$

The third inverse problem is to determine the density ρ from the knowledge of the measurement maps $\mathcal{M}_k$ for all sufficiently large k.

2.4. **Main results.** We now state the main uniqueness results proved in this paper. The first result concerns the source-driven near-field data. It requires the following geometric condition on the source set, the observation set, and the support of the density.

For $z, p \in \mathbb{R}^n$, we denote by $[z, p]$ the closed line segment joining z to p. For $z \in \mathbb{R}^n$ and a set $A \subset \mathbb{R}^n$, we define

$$(z; A] := \bigcup_{p \in A} [z, p] \setminus \{z\}.$$

Condition 2.2. There exist points $x, y \in S$ and a non-empty open set $W' \subset W$ such that

$$\Sigma \subset (x; W'] \qquad \text{and} \qquad \Sigma \cap (y; W'] = \varnothing.$$

Geometrically, W' acts as a screen behind Σ as viewed from x, while from the reference source y the same screen is reached without crossing Σ.

Theorem 2.3 (Determination from source-to-solution data)**.** *Assume that S is open, $\Sigma \cap (S \cup W) = \varnothing$, and that Condition 2.2 holds. Then the source-to-solution intensity data $\{\Lambda^{\mathrm{src}}_k\}$ for all sufficiently large k uniquely determine ρ.*

The second result concerns the renormalized far-field intensity data. Here the data are the asymptotic equivalence classes $[\mathcal{I}_\rho]_\infty$ introduced above.

Theorem 2.4 (Determination from renormalized far-field intensity data)**.** *The asymptotic equivalence class* $[\mathcal{I}_\rho]_\infty$ *uniquely determines* ρ*.*

Finally, we consider the phaseless far-field data obtained from coherent superpositions of two incident plane waves.

Theorem 2.5 (Determination from phaseless superposition data)**.** *The phaseless superposition data* $\{\mathcal{M}_k\}$ *for all sufficiently large* k *uniquely determine* ρ*.*

3. Direct problem

In this section, we study the well-posedness of the direct problem associated with the operator P_k and establish the corresponding limiting absorption principle.

We work with weighted Sobolev spaces. For $l, \delta \in \mathbb{R}$, the space $H^l_\delta(\mathbb{R}^n)$ is defined as the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{H^l_\delta(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \langle x \rangle^{2\delta} |(1-\Delta)^{l/2} u(x)|^2 dx \right)^{1/2}.$$

Here $\langle x \rangle = (1+|x|^2)^{1/2}$, and the operator $(1-\Delta)^{l/2}$ is defined via the Fourier transform by

$$\mathcal{F}[(1-\Delta)^{l/2} u](\xi) = (1+|\xi|^2)^{l/2} \hat{u}(\xi).$$

We define $L^2_\delta(\mathbb{R}^n) := H^0_\delta(\mathbb{R}^n)$.

Remark 3.1. Throughout the paper the dimension $n \geq 2$, the atomic resonance frequency $\Omega > 0$, and the atom-field coupling constant $g > 0$ are fixed. Accordingly, the dependence of constants on n, Ω, and g is suppressed unless explicitly indicated otherwise, and we write C for a positive constant that may change from line to line. Dependence on all other parameters is retained explicitly. For $x \in \mathbb{R}$, we denote by $[x]$ the integer part of x.

3.1. Free fractional resolvent estimate.

We first establish a uniform resolvent bound for

$$P_k^0 = (-\Delta)^{1/2} - k$$

by reducing the problem to classical high-frequency resolvent estimates for the Laplacian.

Recall the standard limiting absorption estimate for the Laplacian in $\mathbb{R}^n$ (see [1], [29, Section 7.1]): if $\delta > 1/2$ and $\lambda \geq 1$, then the limit

$$(-\Delta - \lambda \pm i0)^{-1} := \lim_{\varepsilon \to 0^+} (-\Delta - \lambda \pm i\varepsilon)^{-1}$$

is well defined as a bounded operator from $L^2_\delta(\mathbb{R}^n)$ to $L^2_{-\delta}(\mathbb{R}^n)$, and for any multi-index α with $|\alpha| \leq 1$,

$$\|\partial^\alpha (-\Delta - \lambda \pm i0)^{-1} f\|_{L^2_{-\delta}} \leq C \lambda^{\frac{|\alpha|-1}{2}} \|f\|_{L^2_\delta}, \tag{3.1}$$

where C is independent of $\lambda \geq 1$ and $f \in L^2_\delta(\mathbb{R}^n)$.

On the other hand, by the limiting absorption principle for the free fractional Laplacian established in [2], for every fixed $k > 0$ the boundary values

$$((-\Delta)^{1/2} - k \pm i0)^{-1} := \lim_{\varepsilon \to 0^+} ((-\Delta)^{1/2} - k \pm i\varepsilon)^{-1} \tag{3.2}$$

exist as bounded operators from $L^2_\delta(\mathbb{R}^n)$ to $L^2_{-\delta}(\mathbb{R}^n)$, provided $\delta > 1/2$. The next lemma shows that the corresponding operator norms can be chosen uniformly for $k \geq 1$.

Lemma 3.2. *Let $n \geq 2$ and $\delta > \frac{1}{2}$. Then there exists $C_\delta > 0$ such that for all $k \geq 1$, $\varepsilon \geq 0$, and $f \in L^2_\delta(\mathbb{R}^n)$,*

$$\|((-\Delta)^{1/2} - k - i\varepsilon)^{-1} f\|_{L^2_{-\delta}} \leq C_\delta \|f\|_{L^2_\delta}. \tag{3.3}$$

For $\varepsilon = 0$, the operator in (3.3) *is understood as the outgoing boundary value defined in* (3.2).

Proof. We first prove the estimate for $\varepsilon > 0$. Set $w = k + i\varepsilon$, and note that $|t - w| \geq \varepsilon$ and $|t + w| \geq k$ for $t \geq 0$, while $\Im w^2 = 2k\varepsilon > 0$, so that w^2 lies outside the spectrum $[0, \infty)$ of $-\Delta$. As Fourier multipliers, the elementary identities

$$\frac{1}{t - w} = \frac{t + w}{t^2 - w^2} = \frac{1}{t + w} + \frac{2w}{t^2 - w^2}, \qquad t \geq 0,$$

give

$$\big((-\Delta)^{1/2} - k - i\varepsilon\big)^{-1} = \big((-\Delta)^{1/2} + w\big)\big(-\Delta - w^2\big)^{-1} \tag{3.4}$$

and

$$\big((-\Delta)^{1/2} - k - i\varepsilon\big)^{-1} = \big((-\Delta)^{1/2} + w\big)^{-1} + 2w\big(-\Delta - w^2\big)^{-1}. \tag{3.5}$$

The factorization (3.4) will be needed later; here we use (3.5). The multiplier of the first operator in (3.5) is bounded by $1/k \leq 1$, so

$$\|((-\Delta)^{1/2} + w)^{-1} f\|_{L^2_{-\delta}} \leq \|((-\Delta)^{1/2} + w)^{-1} f\|_{L^2} \leq \|f\|_{L^2} \leq \|f\|_{L^2_\delta}.$$

For the second operator, since $w^2 \in \mathbb{C} \setminus [0, \infty)$ and $|w^2| = k^2 + \varepsilon^2 \geq 1$, the Agmon estimate for complex spectral parameter (see [8, Theorem 7.1.1], whose constant depends only on n, δ, and a lower bound for the modulus of the spectral parameter, here equal to 1) gives

$$2|w| \, \|(-\Delta - w^2)^{-1} f\|_{L^2_{-\delta}} \leq 2C_\delta \, |w| \, |w^2|^{-1/2} \, \|f\|_{L^2_\delta} = 2C_\delta \, \|f\|_{L^2_\delta}.$$

Combining the two bounds with (3.5) proves (3.3) for $\varepsilon > 0$.

Let $f \in L^2_\delta(\mathbb{R}^n)$. By the triangle inequality,

$$\begin{aligned} \| \big((-\Delta)^{1/2} - k - i0\big)^{-1} f\|_{L^2_{-\delta}} \leq{}& \| \big((-\Delta)^{1/2} - k - i\varepsilon\big)^{-1} f\|_{L^2_{-\delta}} \\ &+ \| \big((-\Delta)^{1/2} - k - i0\big)^{-1} f - \big((-\Delta)^{1/2} - k - i\varepsilon\big)^{-1} f\|_{L^2_{-\delta}}. \end{aligned}$$

By (3.3) for $\varepsilon > 0$, the first term on the right hand side is bounded from above by $C\|f\|_{L^2_\delta}$. Then, by taking the limit $\varepsilon \to 0$ and using (3.2), we obtain (3.3) for $\varepsilon = 0$. □

For $\varepsilon > 0$ we set

$$R^\varepsilon_k := ((-\Delta)^{1/2} - k - i\varepsilon)^{-1} = (P^0_k - i\varepsilon)^{-1}, \qquad R^0_k := ((-\Delta)^{1/2} - k - i0)^{-1}.$$

As an immediate consequence, we obtain the corresponding estimate in weighted Sobolev spaces.

Corollary 3.3. *Let $n \geq 2$, let $\delta > 1/2$, and let $l \geq 0$ be an integer. Then there exists $C = C_{\delta,l} > 0$ such that, for all $k \geq 1$, $\varepsilon \geq 0$, and $f \in H^l_\delta(\mathbb{R}^n)$,*

$$\|R^\varepsilon_k f\|_{H^l_{-\delta}} \leq C_{\delta,l} \|f\|_{H^l_\delta}. \tag{3.6}$$

For $\varepsilon = 0$, the operator R_k^ε in (3.6) is understood as the outgoing boundary value R_k^0. Moreover, for every fixed $k \geq 1$ and every $f \in H^l_\delta(\mathbb{R}^n)$,

$$R_k^\varepsilon f \to R_k^0 f \quad \text{in } H^l_{-\delta}(\mathbb{R}^n) \quad \text{as } \varepsilon \to 0^+. \tag{3.7}$$

Proof. We first consider $\varepsilon > 0$. Since R_k^ε and $(1-\Delta)^{l/2}$ are Fourier multipliers, they commute. Therefore, by Lemma 3.2,

$$\|R_k^\varepsilon f\|_{H^l_{-\delta}} = \|R_k^\varepsilon (1-\Delta)^{l/2} f\|_{L^2_{-\delta}} \leq C\|(1-\Delta)^{l/2} f\|_{L^2_\delta} = C\|f\|_{H^l_\delta}.$$

This proves (3.6) for $\varepsilon > 0$.

For $\varepsilon = 0$, the same triangle-inequality argument as in the proof of Lemma 3.2, together with (3.2), gives the estimate for the outgoing boundary value. Thus (3.6) also holds for $\varepsilon = 0$.

It remains to prove (3.7). Since R_k^ε and $(1-\Delta)^{l/2}$ commute,

$$\|R_k^\varepsilon f - R_k^0 f\|_{H^l_{-\delta}} = \|R_k^\varepsilon (1-\Delta)^{l/2} f - R_k^0 (1-\Delta)^{l/2} f\|_{L^2_{-\delta}}.$$

Since $(1-\Delta)^{l/2} f \in L^2_\delta(\mathbb{R}^n)$, the convergence (3.2) implies that the right-hand side tends to zero as $\varepsilon \to 0^+$. This proves (3.7). □

Corollary 3.4. *If $f \in C_c^\infty(\mathbb{R}^n)$, then $R_k^\varepsilon f \in C^\infty(\mathbb{R}^n)$ for every $k \geq 1$ and $\varepsilon \geq 0$.*

Proof. Follows from Corollary 3.3. □

3.2. **Limiting absorption principle for the perturbed operator.** We now establish the limiting absorption principle for the perturbed operator P_k. The argument is based on the resolvent identity and a Neumann series expansion, using the uniform bounds for the free resolvent obtained above.

Lemma 3.5. *Let $n \geq 2$. Fix $\delta > 1/2$ and an integer $l \geq 0$. There exists $k_\rho = k_{\rho,l,\delta} > 0$ and a constant $C_{\rho,l,\delta} > 0$ such that for every $k > k_\rho$ and every $f \in H^l_\delta(\mathbb{R}^n)$, the limit*

$$\psi^f := \lim_{\varepsilon \to 0^+} (P_k - i\varepsilon)^{-1} f \tag{3.8}$$

exists in $H^l_{-\delta}(\mathbb{R}^n)$, and

$$\|\psi^f\|_{H^l_{-\delta}} \leq C_{\rho,l,\delta} \|f\|_{H^l_\delta}. \tag{3.9}$$

Moreover, ψ^f admits the Neumann series representation

$$\psi^f = \sum_{j=0}^{\infty} (-R_k^0 V_k)^j R_k^0 f \quad \text{in } H^l_{-\delta}(\mathbb{R}^n). \tag{3.10}$$

Proof. For $\varepsilon > 0$, set $\psi_\varepsilon = (P_k - i\varepsilon)^{-1} f$. Applying R_k^ε to the identity $(P_k^0 - i\varepsilon)\psi_\varepsilon = f - V_k \psi_\varepsilon$ gives

$$\psi_\varepsilon = R_k^\varepsilon f - R_k^\varepsilon (V_k \psi_\varepsilon).$$

Next, we estimate the potential term. Recall that

$$V_k = \frac{g^2}{k - \Omega} \rho,$$

where $\rho \in C_c^\infty(\mathbb{R}^n)$. Note that multiplication by ρ is a bounded operator from $H^l_{-\delta}(\mathbb{R}^n)$ to $H^l_\delta(\mathbb{R}^n)$. Indeed, ρ is smooth with compact support, so it preserves the

Sobolev order, and the weight $\langle x\rangle^{\pm\delta}$ is bounded on the support of ρ. Consequently, for $k > \Omega$,

$$\|V_k u\|_{H^l_\delta} \leq C_{\rho,l,\delta}\frac{g^2}{k-\Omega}\|u\|_{H^l_{-\delta}}. \tag{3.11}$$

Hence, by Corollary 3.3, for any $u \in H^l_{-\delta}(\mathbb{R}^n)$

$$\|R^\varepsilon_k V_k u\|_{H^l_{-\delta}} \leq C_{l,\delta}\|V_k u\|_{H^l_\delta} \leq C_{\rho,l,\delta}\frac{g^2}{k-\Omega}\|u\|_{H^l_{-\delta}},$$

uniformly in $\varepsilon \geq 0$. Therefore, there exists $k_\rho = k_{\rho,l,\delta} > 0$ such that for any $k > k_\rho$ the following estimates hold

$$\|R^\varepsilon_k V_k\|_{H^l_{-\delta}\to H^l_{-\delta}} \leq \frac{C_{\rho,l,\delta}}{k} < 1, \tag{3.12}$$

uniformly in $\varepsilon \geq 0$. Therefore, the Neumann series

$$\psi_\varepsilon = \sum_{j=0}^{\infty}(-R^\varepsilon_k V_k)^j R^\varepsilon_k f \tag{3.13}$$

converges in $H^l_{-\delta}(\mathbb{R}^n)$ for all $\varepsilon > 0$ and sufficiently large $k > k_\rho$. Since (3.12) holds also for $\varepsilon = 0$, we can set

$$\psi_0 = \sum_{j=0}^{\infty}(-R^0_k V_k)^j R^0_k f.$$

Using (3.12), we estimate

$$\|\psi_\varepsilon\|_{H^l_{-\delta}} \leq \frac{C_{\delta,l}}{1 - C_{\rho,l,\delta}/k}\|f\|_{H^l_\delta} \leq C_{\rho,l,\delta}\|f\|_{H^l_\delta}, \tag{3.14}$$

where $C_{\rho,l,\delta} > 0$ is independent of $k > k_\rho$ and $\varepsilon \geq 0$.

It remains to show that ψ_ε converges to ψ_0 in $H^l_{-\delta}(\mathbb{R}^n)$ as $\varepsilon \to 0^+$. Let $\tau > 0$. Since (3.12) holds uniformly in $\varepsilon \geq 0$, the Neumann series converges in operator norm uniformly with respect to ε. Hence we may choose N such that

$$\sum_{j>N}\|(-R^\varepsilon_k V_k)^j R^\varepsilon_k f\|_{H^l_{-\delta}} < \frac{\tau}{3} \tag{3.15}$$

uniformly for $\varepsilon \geq 0$. Therefore, for $\varepsilon > 0$, using the Neumann series representation we obtain

$$\|\psi_\varepsilon - \psi_0\|_{H^l_{-\delta}} \leq \sum_{j=0}^{N}\left\|(-R^\varepsilon_k V_k)^j R^\varepsilon_k f - (-R^0_k V_k)^j R^0_k f\right\|_{H^l_{-\delta}} + \frac{2\tau}{3}. \tag{3.16}$$

We show, by induction, that for each fixed $j \geq 0$,

$$(-R^\varepsilon_k V_k)^j R^\varepsilon_k f \to (-R^0_k V_k)^j R^0_k f \quad \text{in } H^l_{-\delta}(\mathbb{R}^n) \quad \text{as } \varepsilon \to 0^+. \tag{3.17}$$

The case $j = 0$ follows from Corollary 3.3. Assume that the claim holds for $j-1$ and set

$$u_0 = (-R^0_k V_k)^{j-1} R^0_k f, \qquad u_\varepsilon = (-R^\varepsilon_k V_k)^{j-1} R^\varepsilon_k f.$$

By the induction hypothesis, $u_\varepsilon \to u_0$ in $H^l_{-\delta}(\mathbb{R}^n)$ as $\varepsilon \to 0^+$. Next, we estimate

$$\begin{aligned}\left\|(-R^\varepsilon_k V_k)^j R^\varepsilon_k f - (-R^0_k V_k)^j R^0_k f\right\|_{H^l_{-\delta}} &= \left\|[R^\varepsilon_k V_k]u_\varepsilon - [R^0_k V_k]u_0\right\|_{H^l_{-\delta}} \\ &\leq \|R^\varepsilon_k V_k\|_{H^l_{-\delta}\to H^l_{-\delta}}\|u_\varepsilon - u_0\|_{H^l_{-\delta}} + \|[R^\varepsilon_k - R^0_k](V_k u_0)\|_{H^l_{-\delta}}\end{aligned}$$

By (3.12) and the induction hypothesis, the first term tends to zero as $\varepsilon \to 0^+$. By Corollary 3.3, the second term vanishes as $\varepsilon \to 0^+$. Therefore, (3.17) holds, and consequently, for sufficiently small $\varepsilon > 0$, the first term on the right-hand side of (3.16) is bounded from above by $\tau/3$. This means that ψ_ε converges to ψ_0 in $H^l_{-\delta}(\mathbb{R}^n)$ as $\varepsilon \to 0^+$. Hence the limit in (3.8) exists and

$$\psi^f = \psi_0.$$

By the definition of ψ_0, we therefore obtain

$$\psi^f = \sum_{j=0}^{\infty} (-R_k^0 V_k)^j R_k^0 f \quad \text{in } H^l_{-\delta}(\mathbb{R}^n),$$

which is precisely (3.10). Moreover, since (3.14) holds for $\varepsilon = 0$, we get (3.9). This completes the proof. □

3.3. **Far-field asymptotics and the scattering amplitude.** We begin with the far-field expansion of the outgoing Laplacian resolvent applied to compactly supported data. It follows by integrating the asymptotic expansion of the outgoing Green's function of the Helmholtz operator over the support of h; see [8, Eq. (7.3.2)].

Lemma 3.6. *Let $n \geq 2$, $k > 0$, and $h \in L^2(\mathbb{R}^n)$ with compact support. Then*

$$(-\Delta - k^2 - i0)^{-1} h\,(r\theta) = d_n\, k^{(n-3)/2}\, r^{-(n-1)/2}\, e^{ikr}\, \hat{h}(k\theta) + O(r^{-(n+1)/2}), \tag{3.18}$$

as $r \to \infty$, where $\theta \in \mathbb{S}^{n-1}$ and $d_n \in \mathbb{C}$ is a non-zero constant depending only on n. The remainder is uniform for $\theta \in \mathbb{S}^{n-1}$, and the implicit constant may depend on n, k, and h.

We now derive the far-field expansion and amplitude formula for the fractional operator.

Lemma 3.7. *Let $n \geq 2$ and k_ρ be the threshold from Lemma 3.5 with $l = [n/2] + 1$ and $\delta = 3/4$. Then, for $k > k_\rho$ and $\alpha \in \mathbb{S}^{n-1}$, the total field $\psi_{k,\alpha}$ defined by (2.3) satisfies*

$$\psi_{k,\alpha}(r\theta) = e^{ik\alpha\cdot(r\theta)} + r^{-(n-1)/2}\, e^{ikr}\, A_k(\theta,\alpha) + o(r^{-(n-1)/2}), \qquad r \to \infty, \tag{3.19}$$

where the scattering amplitude is given by

$$A_k(\theta,\alpha) = -2d_n k^{(n-1)/2} \int_{\mathbb{R}^n} e^{-ik\theta\cdot y}\, V_k(y)\, \psi_{k,\alpha}(y)\, dy. \tag{3.20}$$

Proof. By (2.4) and (2.3), the scattered field satisfies

$$P_k^0\, \psi_{k,\alpha}^{\mathrm{sc}} = -V_k \psi_{k,\alpha}^{\mathrm{in}} - V_k \psi_{k,\alpha}^{\mathrm{sc}} = -V_k\, \psi_{k,\alpha}.$$

Since $\psi_{k,\alpha}^{\mathrm{sc}}$ is outgoing, we have

$$\psi_{k,\alpha}^{\mathrm{sc}} = -R_k^0 h,$$

where $h = V_k\, \psi_{k,\alpha}$ is a compactly supported function.

By letting $\varepsilon \to 0^+$ in (3.4), in the sense of outgoing boundary values, and using (3.2) together with (3.1), we obtain

$$R_k^0 = ((-\Delta)^{1/2} + k)\,(-\Delta - k^2 - i0)^{-1}.$$

Set

$$u = (-\Delta - k^2 - i0)^{-1} h,$$

so that

$$\psi^{\mathrm{sc}}_{k,\alpha} = -((-\Delta)^{1/2} + k)\, u = -2ku - ((-\Delta)^{1/2} - k)u. \tag{3.21}$$

The first term has known asymptotics from Lemma 3.6:

$$u(r\theta) = d_n\, k^{(n-3)/2}\, r^{-(n-1)/2}\, e^{ikr}\, \hat{h}(k\theta) + O(r^{-(n+1)/2}). \tag{3.22}$$

It remains to determine the far-field behaviour of $w := ((-\Delta)^{1/2} - k)u$. Since

$$((-\Delta) - k^2)u = h,$$

we obtain

$$w = ((-\Delta)^{1/2} + k)^{-1} h.$$

Since $-(-\Delta)^{1/2}$ is a non-positive operator, Corollary 3.6 and Theorem 1.10 in [7] imply that

$$w = \int_0^\infty e^{-kt} e^{-t(-\Delta)^{1/2}} h\, dt.$$

Now the semigroup $e^{-t(-\Delta)^{1/2}}$ is given by convolution with the Poisson kernel,

$$e^{-t(-\Delta)^{1/2}} h(x) = c_n \int_{\mathbb{R}^n} \frac{t}{(t^2 + |x-y|^2)^{(n+1)/2}}\, h(y)\, dy, \qquad c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}},$$

see [27, Chapter III, §2, Proposition 5] for the Poisson kernel and [9, Proposition 16.4] for its identification with the semigroup $e^{-t(-\Delta)^{1/2}}$. Therefore,

$$w(x) = c_n \int_0^\infty e^{-kt} \int_{\mathbb{R}^n} \frac{t}{(t^2 + |x-y|^2)^{(n+1)/2}}\, h(y)\, dy\, dt.$$

Since $h \in L^2_c(\mathbb{R}^n)$, its support is contained in some ball B_R. If $|x| > 2R$ and $y \in \mathrm{supp}(h)$, then

$$|x - y| \geq |x| - |y| \geq \frac{|x|}{2}.$$

Hence

$$\frac{t}{(t^2 + |x-y|^2)^{(n+1)/2}} \leq \frac{t}{|x-y|^{n+1}} \leq \frac{C\, t}{|x|^{n+1}}.$$

It follows that

$$|w(x)| \leq \frac{C}{|x|^{n+1}} \int_0^\infty e^{-kt} t\, dt \int_{\mathbb{R}^n} |h(y)|\, dy \leq \frac{C_k}{|x|^{n+1}}.$$

Thus

$$((-\Delta)^{1/2} - k)u(x) = O(|x|^{-(n+1)}), \qquad |x| \to \infty.$$

Substituting into (3.21) and using (3.22):

$$\psi^{\mathrm{sc}}_{k,\alpha}(r\theta) = -2k\, d_n\, k^{(n-3)/2}\, r^{-(n-1)/2}\, e^{ikr}\, \hat{h}(k\theta) + O(r^{-(n+1)/2}).$$

Recalling that $h = V_k \psi_{k,\alpha}$, we obtain (3.20). □

4. The source-driven inverse problem

4.1. **Geometric optics approximation.** Let $\delta > \frac{1}{2}$ and let $l \geq 0$ be an integer. For $k \geq 1$, $\varepsilon \geq 0$, $\alpha \in \mathbb{S}^{n-1}$, and $h \in H^l_\delta(\mathbb{R}^n)$, we define

$$\widetilde{R}_{k,\varepsilon} h := e^{-ik\alpha\cdot(\cdot)} R^\varepsilon_k \big[e^{ik\alpha\cdot(\cdot)} h\big]. \tag{4.1}$$

We set $\widetilde{R}_k := \widetilde{R}_{k,0}$. For fixed k and α, multiplication by $e^{\pm ik\alpha\cdot x}$ is bounded on the corresponding weighted Sobolev spaces. Hence, by Corollary 3.3, $\widetilde{R}_{k,\varepsilon}$ is a bounded operator from $H^l_\delta(\mathbb{R}^n)$ to $H^l_{-\delta}(\mathbb{R}^n)$. Moreover, Corollary 3.3 implies

$$\widetilde{R}_{k,\varepsilon} h \longrightarrow \widetilde{R}_k h \quad \text{in } H^l_{-\delta}(\mathbb{R}^n) \quad \text{as } \varepsilon \to 0^+. \tag{4.2}$$

In particular, by Corollary 3.4,

$$\widetilde{R}_k : C^\infty_c(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n). \tag{4.3}$$

For $\varepsilon > 0$ and $h \in C^\infty_c(\mathbb{R}^n)$, writing R^ε_k in Fourier variables gives

$$\widetilde{R}_{k,\varepsilon} h(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\eta} \frac{\widehat{h}(\eta)}{|k\alpha+\eta| - k - i\varepsilon}\, d\eta. \tag{4.4}$$

We also define the transport operator: for $h \in C^\infty_c(\mathbb{R}^n)$,

$$T_\alpha h(x) := i \int_{-\infty}^{x\cdot\alpha} h(\tau\alpha + x_\perp)\, d\tau, \qquad x_\perp = x - (x\cdot\alpha)\,\alpha. \tag{4.5}$$

As shown in Lemma A.1, see Appendix A,

$$T_\alpha h(x) = \lim_{\varepsilon\to 0^+} \mathcal{F}^{-1}\left[\frac{\widehat{h}(\cdot)}{\alpha\cdot(\cdot) - i\varepsilon}\right](x).$$

For $M \in \mathbb{N}$ and φ in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, we write

$$\|\varphi\|_{\mathcal{S}_M} := \max_{|\beta|\leq M} \sup_{y\in\mathbb{R}^n} \langle y\rangle^M\, |\partial^\beta \varphi(y)|.$$

The following result, proved in Appendix A, provides a pointwise geometric-optics approximation for $\widetilde{R}_k$.

Proposition 4.1. *For every compact set $K \subset \mathbb{R}^n$ and multi-index β there exist $C_{K,\beta} > 0$ and $M_\beta \in \mathbb{N}$ such that, for every $\alpha \in \mathbb{S}^{n-1}$, $h \in C^\infty_c(\mathbb{R}^n)$, and $k > 1$,*

$$\sup_{x\in K} \Big|\partial^\beta_x \Big(\widetilde{R}_k\, h(x) - T_\alpha h(x)\Big)\Big| \leq \frac{C_{K,\beta}}{k} \|\widehat{h}\|_{\mathcal{S}_{M_\beta}}. \tag{4.6}$$

Let $\alpha \in \mathbb{S}^{n-1}$ and $\chi \in C^\infty_c(\mathbb{R}^n)$. Define

$$b_0(x) := T_\alpha \chi(x) = i \int_{-\infty}^{x\cdot\alpha} \chi(\tau\alpha + x_\perp)\, d\tau, \qquad x_\perp = x - (x\cdot\alpha)\alpha. \tag{4.7}$$

Note that $b_0 \in C^\infty(\mathbb{R}^n)$.

The uniform resolvent estimates of Section 3 transfer to the conjugated operators. Let $\widetilde{P}^0_k$ denote the Fourier multiplier with symbol $|k\alpha+\eta| - k$, so that

$$\widetilde{P}^0_k = e^{-ik\alpha\cdot(\cdot)}\, P^0_k\, e^{ik\alpha\cdot(\cdot)} \qquad \text{and} \qquad \widetilde{R}_{k,\varepsilon} = (\widetilde{P}^0_k - i\varepsilon)^{-1} \tag{4.8}$$

for $\varepsilon > 0$. We also set $\widetilde{P}_k := \widetilde{P}^0_k + V_k$.

Lemma 4.2. *Let $n \geq 2$, $\delta > \frac{1}{2}$, and let $l \geq 0$ be an integer.*

(i) For all $k \geq 1$, $\varepsilon \geq 0$, $\alpha \in \mathbb{S}^{n-1}$, and $f \in H^l_\delta(\mathbb{R}^n)$,

$$\|\widetilde{R}_{k,\varepsilon} f\|_{H^l_{-\delta}} \leq C_{\delta,l}\, \|f\|_{H^l_\delta}, \tag{4.9}$$

where $\widetilde{R}_{k,0} := \widetilde{R}_k$ and $C_{\delta,l}$ is the constant of Corollary 3.3; in particular, it does not depend on k, ε, or α.

(ii) There exists $k_{\rho,\delta,l} > 0$ such that for $k > k_{\rho,\delta,l}$ and $f \in H^l_\delta(\mathbb{R}^n)$, the limit

$$\widetilde{G}_k f := \lim_{\varepsilon \to 0^+} (\widetilde{P}_k - i\varepsilon)^{-1} f$$

exists in $H^l_{-\delta}(\mathbb{R}^n)$ and satisfies

$$\|\widetilde{G}_k f\|_{H^l_{-\delta}} \leq C_{\rho,l,\delta}\, \|f\|_{H^l_\delta}, \tag{4.10}$$

with $C_{\rho,l,\delta}$ and $k_{\rho,\delta,l}$ independent of α. Moreover,

$$e^{ik\alpha\cdot(\cdot)}\, \widetilde{G}_k f = (P_k - i0)^{-1}\big(e^{ik\alpha\cdot(\cdot)} f\big). \tag{4.11}$$

Proof. (i) Let $\varepsilon > 0$. Since $|e^{\pm ik\alpha\cdot x}| = 1$, multiplication by $e^{\pm ik\alpha\cdot x}$ is an isometry of $L^2_{\pm\delta}(\mathbb{R}^n)$. Therefore, by (4.1), we conclude that

$$\|\widetilde{R}_{k,\varepsilon}\|_{L^2_\delta \to L^2_{-\delta}} = \|R^\varepsilon_k\|_{L^2_\delta \to L^2_{-\delta}}.$$

Thus, Lemma 3.2 holds for $\widetilde{R}_{k,\varepsilon}$. Since $\widetilde{R}_{k,\varepsilon}$ is a Fourier multiplier, it commutes with $(1-\Delta)^{l/2}$. Hence, by repeating steps of the proof of Corollary 3.3, we obtain (4.9) for $\varepsilon > 0$. Letting $\varepsilon \to 0^+$ in (4.9) and using (4.2), we obtain (4.9) for $\varepsilon = 0$.

(ii) Due to (4.8), we have

$$\widetilde{P}_k - i\varepsilon = (\widetilde{P}^0_k - i\varepsilon)\big(I + \widetilde{R}_{k,\varepsilon} V_k\big). \tag{4.12}$$

By part (i) and (3.11), for sufficiently large $k > 0$,

$$\|\widetilde{R}_{k,\varepsilon} V_k\|_{H^l_{-\delta} \to H^l_{-\delta}} \leq \frac{C_{\rho,l,\delta}}{k} < 1,$$

uniformly in $\varepsilon \geq 0$ and $\alpha \in \mathbb{S}^{n-1}$.

Therefore, for $\varepsilon > 0$ and sufficiently large $k > 0$, the operator $I + \widetilde{R}_{k,\varepsilon} V_k$ is invertible on $H^l_{-\delta}(\mathbb{R}^n)$, with

$$\big(I + \widetilde{R}_{k,\varepsilon} V_k\big)^{-1} = \sum_{j=0}^{\infty} (-\widetilde{R}_{k,\varepsilon} V_k)^j.$$

Consequently, by (4.12) and (4.8),

$$(\widetilde{P}_k - i\varepsilon)^{-1} f = \sum_{j=0}^{\infty} (-\widetilde{R}_{k,\varepsilon} V_k)^j \widetilde{R}_{k,\varepsilon} f. \tag{4.13}$$

The series converges in $H^l_{-\delta}(\mathbb{R}^n)$, and part (i) gives

$$\|(\widetilde{P}_k - i\varepsilon)^{-1} f\|_{H^l_{-\delta}} \leq \frac{C_{\delta,l}}{1 - C_{\rho,l,\delta}/k} \|f\|_{H^l_\delta} \leq C_{\rho,l,\delta} \|f\|_{H^l_\delta},$$

for sufficiently large $k > 0$.

By (4.2), one can prove inductively that, for every fixed $j \geq 0$,

$$(-\widetilde{R}_{k,\varepsilon} V_k)^j \widetilde{R}_{k,\varepsilon} f \longrightarrow (-\widetilde{R}_k V_k)^j \widetilde{R}_k f$$

in $H^l_{-\delta}(\mathbb{R}^n)$ as $\varepsilon \to 0^+$. Since the Neumann series in (4.13) has a geometric tail bounded uniformly in ε, we may pass to the limit and obtain

$$\widetilde{G}_k f := \lim_{\varepsilon \to 0^+} (\widetilde{P}_k - i\varepsilon)^{-1} f = \sum_{j=0}^{\infty} (-\widetilde{R}_k V_k)^j \widetilde{R}_k f$$

in $H^l_{-\delta}(\mathbb{R}^n)$. The preceding uniform estimate also gives (4.10).

Finally, by recalling the definitions of $\widetilde{P}_k$ and $\widetilde{P}_k^0$, we write

$$\widetilde{P}_k = e^{-ik\alpha\cdot(\cdot)}\, P_k^0\, e^{ik\alpha\cdot(\cdot)} + e^{-ik\alpha\cdot(\cdot)}\, V_k\, e^{ik\alpha\cdot(\cdot)} = e^{-ik\alpha\cdot(\cdot)}\, P_k\, e^{ik\alpha\cdot(\cdot)}.$$

Thus, for every $\varepsilon > 0$,

$$e^{ik\alpha\cdot(\cdot)}(\widetilde{P}_k - i\varepsilon)^{-1} = (P_k - i\varepsilon)^{-1} e^{ik\alpha\cdot(\cdot)}.$$

Applying this identity to f and letting $\varepsilon \to 0^+$ gives (4.11). □

Lemma 4.3. *Let $\alpha \in \mathbb{S}^{n-1}$, $\chi \in C_c^\infty(\mathbb{R}^n)$, $f = e^{ik\alpha\cdot(\cdot)}\chi$, and $x \in \mathbb{R}^n$. Then, there exists $k_\rho > 0$ such that for $k > k_\rho$, the outgoing solution ψ^f satisfies*

$$\psi^f(x) = \psi_0(x) - e^{ik\alpha\cdot x}\, \frac{g^2}{k-\Omega}\, T_\alpha(\rho\, b_0)(x) + O_{\rho,\chi,x}(k^{-2}), \tag{4.14}$$

where $\psi_0 = R_k^0 f$ and b_0 is defined by (4.7).

Proof. Fix $x \in \mathbb{R}^n$. We set $\delta = 3/4$ and $l = [n/2] + 1$. Let $k_\rho \geq \max\{1, 2\Omega\}$ be sufficiently large so that the conclusions of Lemmas 3.5 and 4.2 hold for $k > k_\rho$.

Now, we assume that $k > k_\rho$. By Lemma 3.5,

$$\psi^f = \sum_{m=0}^{\infty} (-R_k^0 V_k)^m \psi_0 \quad \text{in } H^l_{-\delta}(\mathbb{R}^n). \tag{4.15}$$

Fix a closed ball B with $x \in B$. Since $l > n/2$, the Sobolev embedding $H^l_{-\delta}(\mathbb{R}^n) \hookrightarrow C^0(B)$ holds, so the series (4.15) converges in $C^0(B)$. As each term is continuous, the identity below may be evaluated pointwise at x, and any tail is controlled in $C^0(B)$ by its $H^l_{-\delta}$-norm.

Split the series

$$\psi^f = \psi_0 - R_k^0 V_k \psi_0 + \sum_{m=2}^{\infty} (-1)^m (R_k^0 V_k)^m \psi_0. \tag{4.16}$$

We recall (4.1) and the definition of ψ_0 to conclude that

$$R_k^0 V_k \psi_0 = e^{ik\alpha\cdot(\cdot)} \frac{g^2}{k-\Omega}\, \widetilde{R}_k\, \rho\, \widetilde{R}_k\, \chi \tag{4.17}$$

By Proposition 4.1 applied with $h = \rho\widetilde{R}_k\chi$ and $\beta = 0$, we obtain

$$\widetilde{R}_k \left(\rho\, \widetilde{R}_k\, \chi\right)(x) = T_\alpha \left(\rho\, \widetilde{R}_k \chi\right)(x) + O\left(k^{-1} \|\widehat{\rho \widetilde{R}_k \chi}\|_{\mathbb{S}_M}\right).$$

for some $M \in \mathbb{N}$. Since $\rho\widetilde{R}_k\chi$ is smooth and supported in Σ,

$$\|\widehat{\rho \widetilde{R}_k \chi}\|_{\mathbb{S}_M} \leq C_{\rho,M} \|\rho \widetilde{R}_k \chi\|_{C^M(\Sigma)} \leq C_{\rho,M} \|\widetilde{R}_k \chi\|_{C^M(\Sigma)}.$$

By Sobolev embedding and Lemma 4.2(i), applied with a sufficiently large Sobolev order, we obtain $\|\widetilde{R}_k\chi\|_{C^M(\Sigma)} \le C_{\chi,M}$, uniformly in k and α. Therefore,

$$\widetilde{R}_k\left(\rho\,\widetilde{R}_k\,\chi\right)(x) = T_\alpha\left(\rho\,\widetilde{R}_k\chi\right)(x) + O_{\rho,\chi,x}(k^{-1}).$$

Moreover, since ρ is supported in Σ,

$$\begin{aligned}\left|T_\alpha\left(\rho\,\widetilde{R}_k\chi - \rho b_0\right)(x)\right| &= \left|T_\alpha\left(\rho\,\widetilde{R}_k\chi - \rho\,T_\alpha\chi\right)(x)\right| \\ &\le \int_{\mathbb{R}}\left|\rho(x+t\alpha)\right|dt\ \sup_{\Sigma}\left|\widetilde{R}_k\chi - T_\alpha\chi\right| = O_{\rho,\chi}(k^{-1}).\end{aligned}$$

Combining the last two estimates and (4.17) gives

$$R_k^0 V_k\psi_0(x) = e^{ik\alpha\cdot x}\,\frac{g^2}{k-\Omega}\,T_\alpha(\rho\,b_0)(x) + O_{\rho,\chi,x}(k^{-2}). \tag{4.18}$$

Next, we estimate the remainder. We write

$$r := \sum_{m=2}^{\infty}(-1)^m(R_k^0V_k)^m\psi_0 = \sum_{m=0}^{\infty}(-1)^m(R_k^0V_k)^mR_k^0\left(V_kR_k^0V_kR_k^0f\right).$$

We denote $\psi_1 = R_k^0V_kR_k^0f$. Applying (3.8) and (3.10) with f replaced by $V_k\psi_1$ gives

$$r = (P_k - i0)^{-1}(V_k\psi_1). \tag{4.19}$$

Therefore, by (4.11),

$$r = e^{ik\alpha\cdot x}\,\widetilde{G}_k(V_k\,e^{-ik\alpha\cdot(\cdot)}\,\psi_1).$$

By (4.10) and (3.11), we estimate

$$\|\widetilde{G}_k(V_k\,e^{-ik\alpha\cdot(\cdot)}\,\psi_1)\|_{H^l_{-\delta}} \le C_\rho\,\|V_k\,e^{-ik\alpha\cdot(\cdot)}\,\psi_1\|_{H^l_\delta} \le \frac{C_\rho}{k}\,\|e^{-ik\alpha\cdot(\cdot)}\,\psi_1\|_{H^l_{-\delta}},$$

By (4.9) and (3.11),

$$\|e^{-ik\alpha\cdot(\cdot)}\,\psi_1\|_{H^l_{-\delta}} = \|e^{-ik\alpha\cdot(\cdot)}\,R_k^0\,V_k\,R_k^0\,e^{ik\alpha\cdot(\cdot)}\,\chi\|_{H^l_{-\delta}} = \|\widetilde{R}_k\,V_k\,\widetilde{R}_k\,\chi\|_{H^l_{-\delta}} \le \frac{C_\rho}{k}\,\|\chi\|_{H^l_\delta}$$

Since $l > n/2$, the Sobolev embedding gives

$$|r(x)| = |\widetilde{G}_k(V_k\,e^{-ik\alpha\cdot(\cdot)}\,\psi_1)(x)| \le C_x\,\|\widetilde{G}_k(V_k\,e^{-ik\alpha\cdot(\cdot)}\,\psi_1)\|_{H^l_{-\delta}} = O_{\rho,\chi,x}(k^{-2}).$$

Substituting this and (4.18) into (4.16) completes the proof. □

4.2. **Recovery of the X-ray transform.** Let us fix $z \in S$ and $\alpha \in \mathbb{S}^{n-1}$. For $x \in \mathbb{R}^n$, we write

$$x = s\alpha + x_\perp,$$

where

$$s = x\cdot\alpha \in \mathbb{R}, \qquad x_\perp = x - (x\cdot\alpha)\alpha.$$

Since $z \in S$ and S is open, there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that

$$\mathcal{T}_{\rm src} = \{z + s\alpha + x_\perp : |s| < \varepsilon_0,\ |x_\perp| < 2\delta_0\} \subset S. \tag{4.20}$$

Let $\chi_\parallel \in C_c^\infty(\mathbb{R})$ and $\chi_\perp \in C_c^\infty(\alpha^\perp)$ be non-negative functions such that $\chi_\parallel(0) = \chi_\perp(0) = 1$, and

$$\operatorname{supp}(\chi_\parallel) \subset (-\varepsilon_0,\varepsilon_0), \qquad \operatorname{supp}(\chi_\perp) \subset \{|y| < 2\delta_0\}.$$

Define

$$\chi(x) = \chi_\parallel(x\cdot\alpha)\,\chi_\perp(x_\perp), \qquad \chi_{z,\alpha}(x) = \chi(x-z), \tag{4.21}$$

and set

$$f^{\alpha}_{k,z}(x) = e^{ik\alpha\cdot x}\,\chi_{z,\alpha}(x). \tag{4.22}$$

Note that $f^{\alpha}_{k,z} \in C^{\infty}_{c}(S)$.

Theorem 4.4. *Let $n \geq 2$ and k_ρ be the threshold from Lemma 3.5 with $l = [n/2]+1$ and $\delta = 3/4$. Assume that*

$$\Sigma \cap (S \cup W) = \varnothing.$$

Assume that $z_1, z_2 \in S$ and $x^* \in W$ satisfy

$$[z_2, x^*] \cap \Sigma = \varnothing.$$

Then the data $\{\Lambda^{\mathrm{src}}_k\}_{k>k_\rho}$ determines the integral

$$\int_0^1 \rho\big(z_1 + s(x^* - z_1)\big)\, ds.$$

Proof. If $x^* = z_1$, the integral in question equals $\rho(z_1) = 0$, since $z_1 \in S$ and $S \cap \Sigma = \varnothing$, and there is nothing to prove. If $x^* = z_2$, then, since S is open, $x^* \notin \Sigma$, and Σ is closed, we may replace z_2 by a nearby point $z_2' \in S \setminus \{x^*\}$ satisfying $[z_2', x^*] \cap \Sigma = \varnothing$. Hence we may assume $x^* \neq z_1, z_2$.

Let $\alpha_j = (x^* - z_j)/|x^* - z_j|$ and $t_j^* = |x^* - z_j|$ for $j = 1, 2$, so that $z_j + t_j^*\alpha_j = x^*$. Let $\chi_1, \chi_2 \in C^{\infty}_{c}(S)$ be cutoff functions as in (4.21), corresponding to the directions α_1, α_2 and points z_1, z_2, with parameters $\varepsilon_{0,1}, \varepsilon_{0,2} > 0$ and corresponding tubes $\mathcal{T}_{\mathrm{src},1}, \mathcal{T}_{\mathrm{src},2} \subset S$ as in (4.20), respectively. We choose $\varepsilon_{0,j} < t_j^*$, $j = 1, 2$. Define

$$f_j = f^{\alpha_j}_{k,z_j} = e^{ik\alpha_j\cdot x}\chi_j(x), \qquad j = 1, 2,$$

and let ψ^{f_j} denote the corresponding outgoing solutions. In the argument below we take $k > k_*$, where $k_* \geq k_\rho$ is chosen large enough so that Lemma 4.3 applies to both sources f_1 and f_2.

The polarization identity gives

$$\psi^{f_1}(w)\,\overline{\psi^{f_2}(w)} = \frac{1}{4}\sum_{j=0}^{3} i^j\,\Lambda^{\mathrm{src}}_k(f_1 + i^j f_2)(w), \qquad w \in W. \tag{4.23}$$

Therefore the data determines $\psi^{f_1}(w)\,\overline{\psi^{f_2}(w)}$ for $w \in W$.

We now evaluate at $w = x^*$. By Lemma 4.3, since χ_j is supported in S and x^* lies beyond the support of each χ_j in the α_j-direction:

$$b_{0,j}(x^*) = T_{\alpha_j}\chi_j(x^*) = i\int_{-\infty}^{\infty} \chi_j(\tau\alpha_j + (x^*)_{\perp,j})\, d\tau = iC_{\chi_j}, \qquad C_{\chi_j} = \|\chi_{\parallel,j}\|_{L^1},$$

where $(x^*)_{\perp,j} = x^* - (x^*\cdot\alpha_j)\alpha_j$. Since $x^* - z_1 = t_1^*\alpha_1$, along the ray $z_1 + \tau\alpha_1$ we have

$$b_{0,1}(z_1 + \tau\alpha_1) = i\int_{-\infty}^{\tau} \chi_{\parallel,1}(s)\, ds.$$

Hence

$$b_{0,1}(z_1 + \tau\alpha_1) = 0, \qquad \tau \leq -\varepsilon_{0,1},$$

whereas, for $-\varepsilon_{0,1} < \tau \leq 0$, the point $z_1 + \tau\alpha_1$ belongs to $\mathcal{T}_{\mathrm{src},1} \subset S$. Since $S \cap \Sigma = \emptyset$, we have

$$\rho(z_1 + \tau\alpha_1) = 0, \qquad -\varepsilon_{0,1} < \tau \leq 0.$$

Therefore,

$$\rho(z_1 + \tau\alpha_1)b_{0,1}(z_1 + \tau\alpha_1) = 0, \qquad \tau \leq 0.$$

Moreover, since $\varepsilon_{0,1} < t_1^*$,

$$b_{0,1}(z_1 + \tau\alpha_1) = iC_{\chi_1}, \qquad \varepsilon_{0,1} \le \tau \le t_1^*,$$

and for $0 \le \tau < \varepsilon_{0,1}$ we have $\rho(z_1 + \tau\alpha_1) = 0$. Consequently,

$$\rho(z_1 + \tau\alpha_1) b_{0,1}(z_1 + \tau\alpha_1) = iC_{\chi_1}\rho(z_1 + \tau\alpha_1), \qquad 0 \le \tau \le t_1^*.$$

Hence, by the definition of T_{α_1} and $V_k = \frac{g^2}{k-\Omega}\rho$,

$$\begin{aligned} T_{\alpha_1}(V_k b_{0,1})(x^*) &= i\int_{-\infty}^{t_1^*} V_k(z_1 + \tau\alpha_1)\, b_{0,1}(z_1 + \tau\alpha_1)\, d\tau \\ &= i\,(iC_{\chi_1})\,\frac{g^2}{k-\Omega}\int_0^{t_1^*} \rho(z_1 + \tau\alpha_1)\, d\tau = -C_{\chi_1}\frac{g^2}{k-\Omega}\int_0^{t_1^*} \rho(z_1 + \tau\alpha_1)\, d\tau. \end{aligned}$$

Writing $x^* = z_2 + t_2^*\alpha_2$, we have

$$T_{\alpha_2}(V_k b_{0,2})(x^*) = i\int_{-\infty}^{t_2^*} V_k(z_2 + s\alpha_2) b_{0,2}(z_2 + s\alpha_2)\, ds.$$

The integrand vanishes on the whole interval $(-\infty, t_2^*]$: for $s \le -\varepsilon_0$ this follows from the support of $\chi_{\parallel,2}$; for $|s| < \varepsilon_0$ the point $z_2 + s\alpha_2$ lies in $\mathcal{T}_{\mathrm{src},2} \subset S$, where $\rho = 0$; and for $s \in [\varepsilon_0, t_2^*]$ the point lies on $[z_2, x^*]$, which is disjoint from Σ by hypothesis. Hence,

$$T_{\alpha_2}(V_k b_{0,2})(x^*) = 0.$$

Applying (4.14):

$$\begin{aligned} \psi^{f_1}(x^*) &= e^{ik\alpha_1\cdot x^*}\left(\widetilde{R}_k\chi_1(x^*) + C_{\chi_1}\frac{g^2}{k-\Omega}\int_0^{t_1^*}\rho(z_1 + \tau\alpha_1)\, d\tau + O(k^{-2})\right), \\ \psi^{f_2}(x^*) &= e^{ik\alpha_2\cdot x^*}\big(\widetilde{R}_k\chi_2(x^*) + O(k^{-2})\big). \end{aligned}$$

Therefore,

$$\begin{aligned} \psi^{f_1}(x^*)\overline{\psi^{f_2}(x^*)} = e^{ik(\alpha_1-\alpha_2)\cdot x^*}\Bigg(&\widetilde{R}_k\chi_1(x^*)\overline{\widetilde{R}_k\chi_2(x^*)} \\ &+ C_{\chi_1}\overline{\widetilde{R}_k\chi_2(x^*)}\frac{g^2}{k-\Omega}\int_0^{t_1^*}\rho(z_1 + \tau\alpha_1)\, d\tau + O(k^{-2})\Bigg). \end{aligned} \tag{4.24}$$

The left-hand side is known from the data via (4.23). The factor $e^{ik(\alpha_1-\alpha_2)\cdot x^*}$ has modulus one and depends only on k, α_1, α_2, and x^*, hence is known. The first term $\widetilde{R}_k\chi_1(x^*)\overline{\widetilde{R}_k\chi_2(x^*)}$ is independent of ρ and known. Therefore, by (4.24), the data determine

$$\begin{aligned} \frac{k-\Omega}{g^2}\Big(e^{-ik(\alpha_1-\alpha_2)\cdot x^*}\,\psi^{f_1}(x^*)\overline{\psi^{f_2}(x^*)} - \widetilde{R}_k\chi_1(x^*)\overline{\widetilde{R}_k\chi_2(x^*)}\Big) \\ = C_{\chi_1}\overline{\widetilde{R}_k\chi_2(x^*)}\int_0^{t_1^*}\rho(z_1 + \tau\alpha_1)\, d\tau + O(k^{-1}). \end{aligned}$$

By Proposition 4.1, $\widetilde{R}_k\chi_2(x^*) \to T_{\alpha_2}\chi_2(x^*) = b_{0,2}(x^*) = iC_{\chi_2}$ as $k \to \infty$, so

$$C_{\chi_1}\overline{\widetilde{R}_k\chi_2(x^*)} \to -iC_{\chi_1}C_{\chi_2} \neq 0.$$

Letting $k \to \infty$ therefore gives

$$\int_0^{t_1^*} \rho(z_1 + \tau\alpha_1)\, d\tau$$
$$= \frac{1}{-iC_{\chi_1}C_{\chi_2}} \lim_{k\to\infty} \frac{k-\Omega}{g^2} \left(e^{-ik(\alpha_1-\alpha_2)\cdot x^*}\, \psi^{f_1}(x^*)\overline{\psi^{f_2}(x^*)} - \widetilde{R}_k\chi_1(x^*)\overline{\widetilde{R}_k\chi_2(x^*)} \right).$$

Since the right-hand side is computed entirely from the data, the integral is determined.

Finally, since

$$x^* - z_1 = t_1^*\alpha_1,$$

the change of variables $\tau = st_1^*$ gives

$$\int_0^{t_1^*} \rho(z_1 + \tau\alpha_1)d\tau = t_1^* \int_0^1 \rho(z_1 + s(x^* - z_1))ds.$$

The number $t_1^* = |x^* - z_1|$ is known. Hence the data determine

$$\int_0^1 \rho(z_1 + s(x^* - z_1))ds,$$

as claimed. □

4.3. **Determination from the source-to-solution map.** We now prove Theorem 2.3. Theorem 4.4 shows that the source-to-solution data $\{\Lambda_k^{\mathrm{src}}\}_{k>k_\rho}$ determine the integral of ρ along a segment $[z_1, x^*]$ which intersects Σ, provided that there is a reference source $z_2 \in S$ such that the segment $[z_2, x^*]$ avoids Σ. Condition 2.2 ensures that such line integrals are available for a sufficiently rich family of rays. We use this family of line integrals to determine ρ by injectivity of the X-ray transform.

We will use the following elementary auxiliary lemma.

Lemma 4.5. *Let $A \subset \mathbb{R}^n$ be compact, let $B \subset \mathbb{R}^n$ be open, and let $x \in \mathbb{R}^n$. Assume that $A \subset (x; B]$. Then there exists $\varepsilon > 0$ such that $A \subset (q; B]$ for every $q \in B_\varepsilon(x)$.*

Proof. For $q, a \in \mathbb{R}^n$, $a \in (q; B]$ if and only if $(1-s)q + sa \in B$ for some $s \geq 1$. Indeed, this follows by setting $s = 1/t$ in $a = (1-t)q + tb$.

For each $a \in A$, choose $s_a \geq 1$ such that

$$(1 - s_a)x + s_a a \in B,$$

and define

$$F_a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \qquad F_a(q, a') = (1 - s_a)q + s_a a'.$$

The map F_a is continuous, and $F_a(x, a) \in B$. Since B is open, there exist neighborhoods V_a of x and U_a of a such that $F_a(V_a \times U_a) \subset B$. Thus, for every $q \in V_a$ and $a' \in U_a$, we have $a' \in (q; B]$.

Since A is compact, finitely many of the sets U_a cover A. Let $U_{a_1}, \ldots, U_{a_N}$ be such a finite subcover and set $V = \bigcap_{j=1}^N V_{a_j}$. Then $A \subset (q; B]$ for every $q \in V$. Choosing $\varepsilon > 0$ such that $B_\varepsilon(x) \subset V$ proves the claim. □

Now we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. Since Σ is compact and $\Sigma \subset (x; W']$, Lemma 4.5 implies that there exists $\varepsilon > 0$ such that

$$\Sigma \subset (q; W'] \qquad \text{for all } q \in B_\varepsilon(x). \tag{4.25}$$

Let us fix $\varepsilon > 0$ such that $B_\varepsilon(x) \subset S$ and (4.25) holds. Fix $z \in B_\varepsilon(x)$. By Condition 2.2, $[y,p] \cap \Sigma = \varnothing$ for any $p \in W'$. Then Theorem 4.4, applied with $z_1 = z$, $z_2 = y$, and $x^* = p \in W'$, shows that the data determine

$$\int_0^1 \rho(z + s(p-z))ds.$$

Finally, since this holds for every $z \in B_\varepsilon(x)$, and $B_\varepsilon(x)$ is an open, non-empty set, and since $\Sigma \subset (z; W']$ as we established above, we conclude from Theorem 6.1 from [16] that the given data determine the density ρ. □

5. The far-field inverse problems

5.1. **Determination from renormalized far-field intensity data.** We now prove that the renormalized far-field intensity data determine the density. The proof proceeds in two steps. First, we show that the oscillatory asymptotics of the renormalized intensity determine the scattering amplitude away from the forward direction. Second, we use the high-frequency behaviour of the scattering amplitude to recover the Fourier transform of ρ.

Proposition 5.1. *Let k_ρ be the threshold from Lemma 3.5 with $l = [n/2] + 1$ and $\delta = 3/4$. Let $k > k_\rho$ and let $\alpha, \theta \in \mathbb{S}^{n-1}$ satisfy $\theta \neq \alpha$. Then $\mathcal{I}_\rho(\alpha, k, \theta, \cdot)$ determines the scattering amplitude $A_k(\theta, \alpha)$. More precisely,*

$$A_k(\theta,\alpha) = \lim_{R\to\infty} \frac{1}{R}\int_R^{2R} \mathcal{I}_\rho(\alpha,k,\theta,r)e^{-ikr(1-\alpha\cdot\theta)}\,dr. \tag{5.1}$$

Proof. Fix $k > k_\rho$ and $\alpha, \theta \in \mathbb{S}^{n-1}$ with $\theta \neq \alpha$. Set

$$\phi := k(1 - \alpha\cdot\theta). \tag{5.2}$$

Since $\theta \neq \alpha$, we have $\alpha \cdot \theta < 1$, and therefore $\phi \neq 0$.

By (2.8), there exists a function $\varepsilon(r) = o(1)$ as $r \to \infty$ such that

$$\mathcal{I}_\rho(\alpha,k,\theta,r) = 2\Re\big(e^{i\phi r}A_k(\theta,\alpha)\big) + \varepsilon(r), \qquad r\to\infty. \tag{5.3}$$

Moreover,

$$\Re\big(e^{i\phi r}A_k(\theta,\alpha)\big) = \frac{1}{2}e^{i\phi r}A_k(\theta,\alpha) + \frac{1}{2}e^{-i\phi r}\overline{A_k(\theta,\alpha)}. \tag{5.4}$$

Multiplying by $2e^{-i\phi r}$, we get

$$2\Re\big(e^{i\phi r}A_k(\theta,\alpha)\big)e^{-i\phi r} = A_k(\theta,\alpha) + \overline{A_k(\theta,\alpha)}e^{-2i\phi r}. \tag{5.5}$$

Using (5.3), we obtain

$$\frac{1}{R}\int_R^{2R} \mathcal{I}_\rho(\alpha,k,\theta,r)e^{-i\phi r}\,dr = A_k(\theta,\alpha) + \overline{A_k(\theta,\alpha)}\frac{1}{R}\int_R^{2R} e^{-2i\phi r}\,dr + \frac{1}{R}\int_R^{2R}\varepsilon(r)e^{-i\phi r}\,dr.$$

We now verify that the last two terms tend to zero.

First, since $\phi \neq 0$,

$$\frac{1}{R}\int_R^{2R} e^{-2i\phi r}\,dr = \frac{e^{-4i\phi R} - e^{-2i\phi R}}{-2i\phi R} \to 0, \qquad \text{as } R\to\infty. \tag{5.6}$$

Second, since $\varepsilon(r) \to 0$ as $r \to \infty$, we have

$$\sup_{r\in[R,2R]} |\varepsilon(r)| \to 0 \qquad \text{as } R \to \infty. \tag{5.7}$$

Consequently,

$$\left|\frac{1}{R}\int_R^{2R} \varepsilon(r) e^{-i\phi r}\, dr\right| \le \frac{1}{R}\int_R^{2R} |\varepsilon(r)|\, dr \le \sup_{r\in[R,2R]} |\varepsilon(r)| \to 0. \tag{5.8}$$

Hence the averaged contribution of the $\varepsilon(r)$-term tends to zero as $R \to \infty$. Therefore

$$\lim_{R\to\infty} \frac{1}{R}\int_R^{2R} \mathcal{I}_\rho(\alpha, k, \theta, r) e^{-i\phi r}\, dr = A_k(\theta, \alpha). \tag{5.9}$$

Since $\phi = k(1 - \alpha\cdot\theta)$, this proves (5.1). □

The previous formula depends only on the asymptotic class of $\mathcal{I}_\rho$ modulo terms tending to zero at infinity. We record this consequence explicitly, since this is the form in which the result will be used for the inverse problem.

Corollary 5.2. *Let k_ρ be the threshold from Lemma 3.5 with $l = [n/2] + 1$ and $\delta = 3/4$. Let $k > k_\rho$ and let $\alpha, \theta \in \mathbb{S}^{n-1}$ satisfy $\theta \neq \alpha$. If $\mathcal{I}_{\rho_1}$ and $\mathcal{I}_{\rho_2}$ are equivalent in the sense of* (2.9)*, then*

$$A_k^{\rho_1}(\theta, \alpha) = A_k^{\rho_2}(\theta, \alpha), \tag{5.10}$$

where $A_k^{\rho_j}(\theta, \alpha)$ denotes the scattering amplitude associated with ρ_j, $j = 1, 2$.

In particular, the equivalence class $[\mathcal{I}_\rho]_\infty$ determines $A_k(\theta, \alpha)$ for every $k > k_\rho$ and every non-forward pair $\theta \neq \alpha$.

Proof. Set $\phi := k(1 - \alpha\cdot\theta)$. Since $\theta \neq \alpha$, we have $\phi \neq 0$. Define

$$\delta(r) := \mathcal{I}_{\rho_1}(\alpha, k, \theta, r) - \mathcal{I}_{\rho_2}(\alpha, k, \theta, r).$$

By the equivalence relation (2.9), $\delta(r) \to 0$ as $r \to \infty$.

Applying Proposition 5.1 to ρ_1 and to ρ_2, we obtain

$$A_k^{\rho_j}(\theta, \alpha) = \lim_{R\to\infty} \frac{1}{R}\int_R^{2R} \mathcal{I}_{\rho_j}(\alpha, k, \theta, r)\, e^{-i\phi r}\, dr, \qquad j = 1, 2.$$

Subtracting these two limits,

$$A_k^{\rho_1}(\theta, \alpha) - A_k^{\rho_2}(\theta, \alpha) = \lim_{R\to\infty} \frac{1}{R}\int_R^{2R} \delta(r)\, e^{-i\phi r}\, dr.$$

We estimate

$$\left|\frac{1}{R}\int_R^{2R} \delta(r)\, e^{-i\phi r}\, dr\right| \le \frac{1}{R}\int_R^{2R} |\delta(r)|\, dr \le \sup_{r\in[R,2R]} |\delta(r)|.$$

Since $\delta(r) \to 0$ as $r \to \infty$, we have $\sup_{r\in[R,2R]} |\delta(r)| \to 0$ as $R \to \infty$. Hence

$$A_k^{\rho_1}(\theta, \alpha) - A_k^{\rho_2}(\theta, \alpha) = 0,$$

which proves (5.10). □

It remains to show that knowledge of these scattering amplitudes is enough to determine the density.

Lemma 5.3. *There exists $k_\rho > 0$ such that the scattering amplitudes*

$$\{A_k(\theta,\alpha) : \theta,\alpha \in \mathbb{S}^{n-1},\ \theta \neq \alpha,\ k > k_\rho\}$$

uniquely determine the density ρ.

Proof. By (3.20) and $V_k = \frac{g^2}{k-\Omega}\rho$,

$$A_k(\theta,\alpha) = \frac{-2d_n g^2\, k^{(n-1)/2}}{k-\Omega} \int_{\mathbb{R}^n} e^{-ik\theta\cdot y}\, \rho(y)\, \psi_{k,\alpha}(y)\, dy. \tag{5.11}$$

We substitute $\psi_{k,\alpha} = e^{ik\alpha\cdot y} + \psi^{\mathrm{sc}}_{k,\alpha}$ to obtain

$$A_k(\theta,\alpha) = \frac{-2d_n g^2\, k^{(n-1)/2}}{k-\Omega} \big(\hat\rho(k(\theta-\alpha)) + E_k(\theta,\alpha)\big), \tag{5.12}$$

where

$$E_k(\theta,\alpha) = \int_{\mathbb{R}^n} e^{-ik\theta\cdot y}\, \rho(y)\, \psi^{\mathrm{sc}}_{k,\alpha}(y)\, dy.$$

We estimate the error term. Set $l = [n/2]+1$ and $\delta = 3/4$, and let $k_\rho \geq \max\{1, 2\Omega\}$ be sufficiently large the conclusions of Lemmas 3.5 and 4.2 hold for $k > k_\rho$. By (2.4) and (4.11),

$$e^{-ik\alpha\cdot(\cdot)}\, \psi^{\mathrm{sc}}_{k,\alpha} = -\widetilde{G}_k V_k.$$

Hence, by Sobolev embedding ($l > n/2$) and (4.10), we estimate

$$\begin{aligned}\|\psi^{\mathrm{sc}}_{k,\alpha}\|_{L^\infty(\Sigma)} = \|e^{-ik\alpha\cdot(\cdot)}\psi^{\mathrm{sc}}_{k,\alpha}\|_{L^\infty(\Sigma)} &\leq C\|e^{-ik\alpha\cdot(\cdot)}\psi^{\mathrm{sc}}_{k,\alpha}\|_{H^l_{-\delta}} \\ &= C\|\widetilde{G}_k V_k\|_{H^l_{-\delta}} \leq C_\rho \|V_k\|_{H^l_\delta} \leq \frac{C_\rho}{k}.\end{aligned}$$

Since ρ is supported in Σ, it follows that

$$|E_k(\theta,\alpha)| \leq \|\rho\|_{L^1(\Sigma)} \|\psi^{\mathrm{sc}}_{k,\alpha}\|_{L^\infty(\Sigma)} \leq \frac{C_\rho}{k}. \tag{5.13}$$

We emphasize that the constant in the bound $|E_k(\theta,\alpha)| \leq C_\rho/k$ is independent of θ and α: the factor $e^{-ik\theta\cdot y}$ has modulus one, and the constants in Lemma 4.2 do not depend on $\alpha \in \mathbb{S}^{n-1}$. In particular, the bound remains valid at the k-dependent pair (θ_k, α_k) constructed below.

Let $\xi \in \mathbb{R}^n \setminus \{0\}$ be arbitrary. Since $n \geq 2$, we may choose a unit vector

$$\omega \in \mathbb{S}^{n-1}, \qquad \omega\cdot\xi = 0.$$

For every sufficiently large k with $k > |\xi|/2$, define

$$\theta_k := \frac{\xi}{2k} + \sqrt{1 - \frac{|\xi|^2}{4k^2}}\,\omega, \qquad \alpha_k := -\frac{\xi}{2k} + \sqrt{1 - \frac{|\xi|^2}{4k^2}}\,\omega.$$

Then $\theta_k, \alpha_k \in \mathbb{S}^{n-1}$ and

$$k(\theta_k - \alpha_k) = \xi.$$

Since $\xi \neq 0$, we have $\theta_k - \alpha_k = \xi/k \neq 0$, so the pair (θ_k, α_k) is non-forward and $A_k(\theta_k, \alpha_k)$ is determined by the data for $k > \max\{k_\rho, |\xi|/2\}$. Substituting (θ_k, α_k) into (5.12) and using $E_k(\theta_k, \alpha_k) = O(k^{-1})$, which holds by the θ,α-uniformity just established, we obtain

$$\frac{k-\Omega}{-2d_n g^2 k^{(n-1)/2}}\, A_k(\theta_k,\alpha_k) = \hat\rho(\xi) + O(k^{-1}). \tag{5.14}$$

Therefore,

$$\hat{\rho}(\xi) = \lim_{k\to\infty} \frac{k-\Omega}{-2d_n g^2 k^{(n-1)/2}} A_k(\theta_k, \alpha_k).$$

Since $\xi \in \mathbb{R}^n \setminus \{0\}$ was arbitrary, the scattering amplitudes determine $\widehat{\rho}(\xi)$ for every $\xi \neq 0$. Since $\rho \in C_c^\infty(\mathbb{R}^n)$, the function $\widehat{\rho}$ is continuous, and therefore $\widehat{\rho}(0)$ is also determined by taking the limit $\xi \to 0$. Hence, the data determine ρ uniquely by the Fourier inversion formula. This completes the proof. □

Combining the two steps gives the desired uniqueness result for the original renormalized intensity data.

Proof of Theorem 2.4. By Corollary 5.2, the equivalence class $[\mathcal{I}_\rho]_\infty$ determines the scattering amplitudes $A_k(\theta, \alpha)$ for all $k > k_\rho$ and all $\theta, \alpha \in \mathbb{S}^{n-1}$ with $\theta \neq \alpha$. By Lemma 5.3, these scattering amplitudes determine ρ uniquely. Hence the renormalized far-field intensity data determine ρ uniquely. □

5.2. **Determination from phaseless superposition data.** We now prove the uniqueness result for the phaseless far-field data introduced in Section 2. The key point is that measurements with the four relative phases $a \in \{1, -1, i, -i\}$ determine the mixed products of the scattering amplitudes

$$A_k(\theta, \alpha_2)\overline{A_k(\theta, \alpha_1)}.$$

The high-frequency asymptotics of A_k then allow us to recover $\widehat{\rho}(\xi)$ for every $\xi \in \mathbb{R}^n$.

Proof of Theorem 2.5. By the linearity relation (2.10), we have

$$\left|A_k^{(a)}(\theta, \alpha_1, \alpha_2)\right|^2 = |A_k(\theta, \alpha_1)|^2 + |a|^2|A_k(\theta, \alpha_2)|^2 + 2\Re\Big(a\, A_k(\theta, \alpha_2)\overline{A_k(\theta, \alpha_1)}\Big).$$

Since $a \in \{1, -1, i, -i\}$, we obtain

$$\mathcal{M}_k(\alpha_1, \alpha_2, 1, \theta) - \mathcal{M}_k(\alpha_1, \alpha_2, -1, \theta) = 4\Re\Big(A_k(\theta, \alpha_2)\overline{A_k(\theta, \alpha_1)}\Big), \tag{5.15}$$

$$\mathcal{M}_k(\alpha_1, \alpha_2, i, \theta) - \mathcal{M}_k(\alpha_1, \alpha_2, -i, \theta) = -4\Im\Big(A_k(\theta, \alpha_2)\overline{A_k(\theta, \alpha_1)}\Big). \tag{5.16}$$

Hence the data determine the product

$$A_k(\theta, \alpha_2)\overline{A_k(\theta, \alpha_1)}.$$

Next we use the high-frequency asymptotics of the scattering amplitude (5.12) and (5.13) to obtain

$$\begin{aligned} &A_k(\theta, \alpha_2)\overline{A_k(\theta, \alpha_1)} \\ &\qquad = \left|\frac{-2d_n g^2 k^{(n-1)/2}}{k-\Omega}\right|^2 \widehat{\rho}\big(k(\theta-\alpha_2)\big)\overline{\widehat{\rho}\big(k(\theta-\alpha_1)\big)} + O\left(k^{n-4}\right), \end{aligned} \tag{5.17}$$

uniformly in $\theta, \alpha_1, \alpha_2 \in \mathbb{S}^{n-1}$. Taking $\alpha_1 = \alpha_2 = \theta$ for any $\theta \in \mathbb{S}^{n-1}$ gives

$$|A_k(\theta, \theta)|^2 = \left|\frac{-2d_n g^2 k^{(n-1)/2}}{k-\Omega}\right|^2 |\widehat{\rho}(0)|^2 + O(k^{n-4}),$$

which is determined by the data through (5.15)–(5.16). Hence the data determine $|\widehat{\rho}(0)|^2$. Since $\rho \geq 0$ and $\rho \not\equiv 0$, we have $\widehat{\rho}(0) > 0$. Therefore, the data determine $\widehat{\rho}(0)$.

We now fix $\xi \in \mathbb{R}^n$. Choose k sufficiently large so that $|\xi| \leq 2k$. We choose $\theta_k, \alpha_{1,k} \in \mathbb{S}^{n-1}$ such that

$$k(\theta_k - \alpha_{1,k}) = \xi. \tag{5.18}$$

For example, choose $\theta_k \in \mathbb{S}^{n-1}$ satisfying

$$\theta_k \cdot \xi = \frac{|\xi|^2}{2k},$$

and then set

$$\alpha_{1,k} = \theta_k - \frac{\xi}{k}.$$

A direct computation shows that $|\alpha_{1,k}| = 1$, hence $\alpha_{1,k} \in \mathbb{S}^{n-1}$, and (5.18) holds.

Next we choose $\alpha_{2,k} = \theta_k$. Then

$$k(\theta_k - \alpha_{2,k}) = 0.$$

Substituting these choices into (5.17), we obtain

$$A_k(\theta_k, \alpha_{2,k})\overline{A_k(\theta_k, \alpha_{1,k})} = \left|\frac{-2d_n g^2 k^{(n-1)/2}}{k - \Omega}\right|^2 \widehat{\rho}(0)\,\overline{\widehat{\rho}(\xi)} + O\left(k^{n-4}\right).$$

Equivalently,

$$\overline{\widehat{\rho}(\xi)} = \frac{1}{\widehat{\rho}(0)}\left|\frac{k - \Omega}{-2d_n g^2 k^{(n-1)/2}}\right|^2 A_k(\theta_k, \alpha_{2,k})\overline{A_k(\theta_k, \alpha_{1,k})} + O(k^{-1}).$$

Since the right-hand side is determined by the data, it follows that the data determine $\widehat{\rho}(\xi)$ for every $\xi \in \mathbb{R}^n$ in the limit $k \to \infty$.

Since $\xi \in \mathbb{R}^n$ was arbitrary, the data determine $\widehat{\rho}$ on all of $\mathbb{R}^n$, and therefore determine ρ uniquely by the Fourier inversion formula. □

Appendix A. Auxiliary results

In this appendix we prove Proposition 4.1. We first establish two auxiliary results.

Lemma A.1. *Let $\alpha \in \mathbb{S}^{n-1}$ and $h \in \mathcal{S}(\mathbb{R}^n)$. For $\varepsilon > 0$, define*

$$T_\alpha^\varepsilon h(x) := \mathcal{F}^{-1}\left[\frac{\widehat{h}(\eta)}{\alpha \cdot \eta - i\varepsilon}\right](x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{ix\cdot\eta}\widehat{h}(\eta)}{\alpha \cdot \eta - i\varepsilon} d\eta,$$

$$T_\alpha h(x) := i\int_{-\infty}^{x\cdot\alpha} h(\tau\alpha + x_\perp)d\tau, \qquad x_\perp = x - (x\cdot\alpha)\alpha.$$

Then, for any compact set K,

$$\lim_{\varepsilon\to 0} \|T_\alpha h - T_\alpha^\varepsilon h\|_{C^0(K)} = 0.$$

Proof. We may assume $\alpha = e_1$ and write $x = (x_1, x')$, $\eta = (\eta_1, \eta')$. Then

$$T_{e_1}^\varepsilon h(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{ix_1\eta_1}}{\eta_1 - i\varepsilon}\widetilde{h}(\eta_1, x')d\eta_1,$$

where

$$\widetilde{h}(\eta_1, x') := \mathcal{F}_{\eta'}^{-1}[\widehat{h}(\eta_1, \cdot)](x') = \mathcal{F}_{x_1}[h(\cdot, x')](\eta_1)$$

is the partial Fourier transform of h in x_1 only. Since

$$\mathcal{F}_{\eta_1}^{-1}[(\eta_1 - i\varepsilon)^{-1}](x_1) = ie^{-\varepsilon x_1}\mathbb{H}(x_1)$$

where $\mathbb{H}$ is the Heaviside function. Therefore, the convolution theorem implies

$$T^{\varepsilon}_{e_1} h(x) = i\int_0^\infty e^{-\varepsilon s} h(x_1 - s, x')ds = i\int_{-\infty}^{x_1} e^{-\varepsilon(x_1-\tau)} h(\tau, x')d\tau. \tag{A.1}$$

Note that for $\tau \le x_1$ and $x \in K$,

$$\left|e^{-\varepsilon(x_1-\tau)} - 1\right| = 1 - e^{-\varepsilon(x_1-\tau)} \le \varepsilon(x_1 - \tau) \le \varepsilon(C_K + |\tau|),$$

where $C_K = \sup_{x\in K} |x_1|$. Since $h \in \mathcal{S}(\mathbb{R}^n)$, for any $N \ge 3$ we have

$$|h(\tau, x')| \le \|h\|_{\mathcal{S}_N} \langle(\tau, x')\rangle^{-N}.$$

Therefore,

$$\begin{aligned}\left\|T^{\varepsilon}_{e_1} h - T_{e_1} h\right\|_{C^0(K)} &\le \sup_{x\in K} \int_{-\infty}^{x_1} \left|e^{-\varepsilon(x_1-\tau)} - 1\right| |h(\tau, x')|d\tau \\ &\le \varepsilon\|h\|_{\mathcal{S}_N} \int_{\mathbb{R}} (C_K + |\tau|)\langle\tau\rangle^{-N} d\tau \le C_{K,N}\varepsilon\|h\|_{\mathcal{S}_N},\end{aligned}$$

where the integral converges since $N \ge 3$. Hence

$$\left\|T^{\varepsilon}_{e_1} h - T_{e_1} h\right\|_{C^0(K)} \to 0 \qquad \text{as } \varepsilon \to 0^+.$$

This completes the proof. □

Lemma A.2. *Let $k > 1$. Define*

$$\sigma_k(\eta) := \eta_1 + \frac{|\eta|^2}{2k}, \qquad a_k(\eta) := \frac{|ke_1 + \eta| + k}{2k},$$

and for $\theta, \vartheta \in [0, 1]$ set

$$\sigma^{\vartheta}_k(\eta) := \eta_1 + \frac{\vartheta|\eta|^2}{2k}, \qquad c_\theta(\eta) := \theta a_k(\eta) + (1 - \theta). \tag{A.2}$$

For $\varepsilon > 0$, set

$$X_{\theta,\vartheta}(\eta) := \sigma^{\vartheta}_k(\eta) - i\varepsilon c_\theta(\eta). \tag{A.3}$$

Then there exist constant $C > 0$, depending only on n, such that for all $k > 1$, $\theta, \vartheta \in [0, 1]$, $\varepsilon \in (0, 1]$, and all $\psi \in \mathcal{S}(\mathbb{R}^n)$ supported in $\{|\eta| \le k/2\}$:

$$\left|\int_{\mathbb{R}^n} \frac{\psi(\eta)}{X_{\theta,\vartheta}(\eta)}\, d\eta\right| \le C\|\psi\|_{\mathcal{S}_{n+4}}, \tag{A.4}$$

$$\left|\int_{\mathbb{R}^n} \frac{\psi(\eta)}{X_{\theta,\vartheta}(\eta)^2}\, d\eta\right| \le C\|\psi\|_{\mathcal{S}_{n+5}}. \tag{A.5}$$

Proof. We write $X = X_{\theta,\vartheta}$. Next, we list certain properties of functions σ^{ϑ}_k, c_θ, and X on $\{|\eta| \le k/2\}$.

Properties of σ^{ϑ}_k. We compute

$$\partial_{\eta_1}\sigma^{\vartheta}_k = 1 + \frac{\vartheta\eta_1}{k},$$

so

$$\frac{1}{2} \le \partial_{\eta_1}\sigma^{\vartheta}_k \le \frac{3}{2} \qquad \text{on } \{|\eta_1| \le k/2\}. \tag{A.6}$$

Moreover,

$$\partial_{\eta_i}\partial_{\eta_j}\sigma^{\vartheta}_k = \frac{\vartheta\delta_{ij}}{k} \le \frac{1}{k}$$

and all derivatives of order greater than 2 are zero. Hence,

$$|\partial_\eta^\nu \sigma_k^\vartheta| \le C_\nu \qquad \text{on } \{|\eta| \le k/2\}, \quad \text{for } |\nu| \ge 1, \tag{A.7}$$

uniformly in $k > 1$ and $\vartheta \in [0,1]$.

For $\vartheta = 0$, $\sigma_k^0 = \eta_1$ has the unique zero $\eta_1^* = 0$. For $\vartheta > 0$, the equation $\sigma_k^\vartheta(\eta_1, \eta') = 0$ is quadratic in η_1 with two roots:

$$\eta_1^* = \frac{k}{\vartheta}\Big(-1 + \sqrt{1 - \tfrac{\vartheta^2|\eta'|^2}{k^2}}\Big), \qquad \eta_1^{**} = \frac{k}{\vartheta}\Big(-1 - \sqrt{1 - \tfrac{\vartheta^2|\eta'|^2}{k^2}}\Big).$$

Since $|\eta_1^{**}| \ge k/\vartheta \ge k$, we obtain $\eta_1^{**} \notin \{|\eta_1| \le k/2\}$. For $\eta' \in \{|\eta'| \le k/2\}$, we estimate

$$|\eta_1^*| = \frac{k}{\vartheta}\Big(1 - \sqrt{1 - \tfrac{\vartheta^2|\eta'|^2}{k^2}}\Big) \le \frac{\frac{\vartheta|\eta'|^2}{k}}{1 + \sqrt{1 - \frac{\vartheta^2|\eta'|^2}{k^2}}} \le \frac{|\eta'|^2}{k} \le \frac{k}{4} < \frac{k}{2}. \tag{A.8}$$

Hence, $\eta_1^* \in \{|\eta_1| \le k/2\}$. Using the mean value theorem and (A.6), we obtain the two-sided bound:

$$\tfrac{1}{2}|\eta_1 - \eta_1^*| \le |\sigma_k^\vartheta(\eta_1, \eta')| \le \tfrac{3}{2}|\eta_1 - \eta_1^*|, \qquad \text{for } \eta \in \{|\eta| < k/2\}\,. \tag{A.9}$$

Note that in general (η_1^*, η') is not in $\{|\eta| < k/2\}$.

Properties of c_θ. Note that

$$\frac{k}{2} \le |ke_1 + \eta| \le \frac{3k}{2}, \qquad \text{on } \{|\eta| \le k/2\}$$

Therefore,

$$\frac{3}{4} \le c_\theta \le \frac{5}{4} \qquad \text{on } \{|\eta| \le k/2\}. \tag{A.10}$$

Writing $\xi = ke_1 + \eta$ with $|\xi| \ge k/2$, we have $\partial_{\eta_j} a_k = \xi_j/(2k|\xi|)$, and each further η-derivative brings a factor of $|\xi|^{-1} \le 2/k$. Hence

$$|\partial_\eta^\nu a_k| \le C_\nu k^{-|\nu|}, \qquad \text{for } |\nu| \ge 1 \tag{A.11}$$

and therefore

$$|\partial_\eta^\nu c_\theta| \le C_\nu k^{-|\nu|} \le C_\nu \qquad \text{on } \{|\eta| \le k/2\}, \quad |\nu| \ge 1, \tag{A.12}$$

uniformly in $k > 1$ and $\theta \in [0,1]$.

Properties of X. Due to (A.10),

$$|X| = |\sigma_k^\vartheta - i\varepsilon c_\theta| \ge \varepsilon c_\theta \ge \frac{3}{4}\varepsilon > 0, \qquad \text{on } \{|\eta| \le k/2\}, \tag{A.13}$$

and hence, X does not vanish on $\{|\eta| \le k/2\}$. Also $\operatorname{Im} X = -\varepsilon c_\theta < 0$, so X lies in the open lower half-plane. By (A.6)

$$|\partial_{\eta_1} X| = \sqrt{(\partial_{\eta_1}\sigma_k^\vartheta)^2 + \varepsilon^2(\partial_{\eta_1} c_\theta)^2} \ge |\partial_{\eta_1}\sigma_k^\vartheta| \ge \frac{1}{2}. \tag{A.14}$$

Estimates (A.7) and (A.12) imply that

$$|\partial_\eta^\nu X| \le C_\nu \qquad \text{on } \{|\eta| \le k/2\}, \quad \text{for } |\nu| \ge 1, \tag{A.15}$$

uniformly in $k > 1$, $\theta, \vartheta \in [0,1]$, and $\varepsilon \in (0,1]$.

Combining the last two estimates gives

$$\left|\partial_\eta^\nu \left[\frac{1}{\partial_{\eta_1} X}\right]\right| \le C_\nu \qquad \text{on } \{|\eta| \le k/2\}, \quad \text{for all } \nu. \tag{A.16}$$

Proof of (A.4)**.** As we established, X and $\partial_{\eta_1} X$ do not vanish on $\operatorname{supp}(\psi) \subset \{|\eta| \le k/2\}$. Therefore, by using

$$\frac{1}{X} = \frac{\partial_{\eta_1}[\log X]}{\partial_{\eta_1} X}$$

and integrating by parts, we obtain

$$\int_{\mathbb{R}^n} \frac{\psi}{X}\, d\eta = -\int_{\mathbb{R}^n} \log X(\eta)\, \partial_{\eta_1}\left[\frac{\psi(\eta)}{\partial_{\eta_1} X(\eta)}\right] d\eta. \tag{A.17}$$

Here $\log X$ is the principal-branch logarithm, well-defined since $\operatorname{Im} X < 0$ (so $\arg X \in (-\pi, 0)$). Therefore,

$$|\log X| \le |\log |X|| + |\arg X| \le |\log |X|| + \pi. \tag{A.18}$$

By the definition of X and (A.10),

$$|\sigma_k^\vartheta| \le |X| \le |\sigma_k^\vartheta| + \frac{5}{4}.$$

For $|X| \ge 1$, we estimate

$$\log |X| \le \log\left(|\sigma_k^\vartheta| + \frac{5}{4}\right) \le \begin{cases} \log\left(\frac{9}{4}|\sigma_k^\vartheta|\right) & |\sigma_k^\vartheta| \ge 1, \\ \log\left(\frac{9}{4}\right) & |\sigma_k^\vartheta| \le 1. \end{cases}$$

For $|X| \le 1$,

$$|\log |X|| = -\log |X| \le -\log |\sigma_k^\vartheta| = |\log |\sigma_k^\vartheta||.$$

In both cases,

$$|\log |X|| \le |\log |\sigma_k^\vartheta|| + \log\left(\frac{9}{4}\right).$$

Combining this with (A.9) and (A.18), we obtain

$$|\log X(\eta)| \le |\log |\eta_1 - \eta_1^*(\eta')|| + C \tag{A.19}$$

for some absolute constant C.

Next, we estimate the second term in the integral (A.17). Set $\varphi(\eta) := \partial_{\eta_1}[\psi/\partial_{\eta_1} X]$. Since ψ is smooth and supported in $\{|\eta| \le k/2\}$, (A.16) implies that

$$\|\varphi\|_{\mathbb{s}_m} \le C \|\psi\|_{\mathbb{s}_{m+1}} \tag{A.20}$$

for every m, with $C = C(m, n)$.

Combining (A.17) and (A.19) gives

$$\left|\int_{\mathbb{R}^n} \frac{\psi}{X} d\eta\right| \le C \int_{\mathbb{R}^n} (1 + |\log |\eta_1 - \eta_1^*(\eta')||)\, |\varphi(\eta)| d\eta.$$

We split the inner η_1-integral at $|\eta_1 - \eta_1^*| = 1$:

$$\begin{aligned} \left|\int_{\mathbb{R}^n} \frac{\psi}{X} d\eta\right| \le C \int_{\mathbb{R}^n} |\varphi(\eta)| d\eta + C \int_{\mathbb{R}^{n-1}} \int_{|\eta_1 - \eta_1^*(\eta')| \le 1} |\log |\eta_1 - \eta_1^*(\eta')|| \varphi(\eta)| d\eta \\ + C \int_{\mathbb{R}^{n-1}} \int_{|\eta_1 - \eta_1^*(\eta')| \ge 1} |\log |\eta_1 - \eta_1^*(\eta')|| \varphi(\eta)| d\eta. \end{aligned} \tag{A.21}$$

Since $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$|\varphi(\eta)| \le \|\varphi\|_{\mathbb{s}_{p+1}} (1 + |\eta|)^{-p} \qquad \text{for any } p \in \mathbb{N}.$$

Choosing $p = n + 2$, we estimate the first term of the right-hand side of (A.21)

$$\int_{\mathbb{R}^n} |\varphi(\eta)| d\eta \leq C\|\varphi\|_{s_{n+3}}.$$

To estimate the second term, we note that

$$\int_{|\eta_1-\eta_1^*|\leq 1} \frac{|\log|\eta_1-\eta_1^*||}{(1+|\eta|)^p} d\eta_1 \leq \frac{1}{(1+|\eta'|)^p} \int_{|s|\leq 1} |\log|s||ds = \frac{2}{(1+|\eta'|)^p}.$$

Hence,

$$\int_{\mathbb{R}^{n-1}} \int_{|\eta_1-\eta_1^*(\eta')|\leq 1} |\log|\eta_1-\eta_1^*(\eta')||\varphi(\eta)| d\eta \leq C\|\varphi\|_{s_{n+3}}.$$

To estimate the last term of the of the right-hand side of (A.21), by using (A.8) and $\log|t| \leq |t|$ for $|t| > 1$, we obtain

$$\frac{|\log|\eta_1-\eta_1^*||}{(1+|\eta|)^p} \leq \frac{|\eta_1-\eta_1^*|}{(1+|\eta|)^p} \leq \frac{|\eta_1|+|\eta_1^*|}{(1+|\eta|)^p} \leq \frac{C(1+|\eta|)}{(1+|\eta|)^p} = \frac{C}{(1+|\eta|)^{p-1}}.$$

Therefore,

$$\int_{\mathbb{R}^{n-1}} \int_{|\eta_1-\eta_1^*(\eta')|\geq 1} |\log|\eta_1-\eta_1^*(\eta')||\varphi(\eta)| d\eta \leq C\|\varphi\|_{s_{n+3}}.$$

By combining estimates for the three terms of the right-hand side of (A.21), and then using (A.20),

$$\left|\int_{\mathbb{R}^n} \frac{\psi}{X} d\eta\right| \leq C\|\varphi\|_{s_{n+3}} \leq C\|\psi\|_{s_{n+4}}$$

This proves (A.4).

Proof of (A.5)**.** Since X and $\partial_{\eta_1} X$ do not vanish on $\operatorname{supp}(\psi) \subset \{|\eta| \leq k/2\}$, we can write

$$\frac{1}{X^2} = -\frac{1}{\partial_{\eta_1} X} \partial_{\eta_1} \left[\frac{1}{X}\right]$$

on $\{|\eta| \leq k/2\}$. Then, the integration by parts gives

$$\int_{\mathbb{R}^n} \frac{\psi}{X^2} d\eta = \int_{\mathbb{R}^n} \frac{1}{X} \partial_{\eta_1} \left[\frac{\psi}{\partial_{\eta_1} X}\right] d\eta = \int_{\mathbb{R}^n} \frac{\varphi}{X} d\eta.$$

Recall that φ is a smooth function supported in $\{|\eta| \leq k/2\}$. Therefore, applying (A.4) and (A.20), we obtain

$$\left|\int_{\mathbb{R}^n} \frac{\psi}{X^2} d\eta\right| = \left|\int_{\mathbb{R}^n} \frac{\varphi}{X} d\eta\right| \leq C\|\varphi\|_{s_{n+4}} \leq C\|\psi\|_{s_{n+5}}.$$

This gives (A.5). □

A.1. **Proof of Proposition 4.1.**

Proof. Since ∂_x^β corresponds to multiplication by $(i\eta)^\beta$ in frequency, it suffices to prove the estimate for $\beta = 0$, replacing h by $\partial^\beta h$. We may assume $\alpha = e_1$ and write $\eta = (\eta_1, \eta') \in \mathbb{R} \times \mathbb{R}^{n-1}$. We use the notation σ_k, a_k, σ_k^ϑ, c_θ, $X_{\theta,\vartheta}$ from Lemma A.2.

Set

$$\phi_x(\eta) := e^{ix\cdot\eta}\widehat{h}(\eta).$$

Next, fix $\chi_0 \in C_c^\infty(\mathbb{R}^n)$ such that

$$\chi_0 = 1 \quad \text{on } \{|\eta| \leq 1\}, \qquad \chi_0 = 0 \quad \text{on } \{|\eta| \geq 2\},$$

and define

$$\chi_k(\eta) := \chi_0(4\eta/k).$$

Then

$$\chi_k = 1 \quad \text{on } \{|\eta| \le k/4\}, \qquad \chi_k = 0 \quad \text{on } \{|\eta| \ge k/2\}.$$

We decompose

$$\phi_x = \phi_x^{\text{in}} + \phi_x^{\text{out}}, \qquad \phi_x^{\text{in}} := \chi_k \phi_x, \qquad \phi_x^{\text{out}} := (1-\chi_k)\phi_x.$$

We also define

$$h^{\text{in}} := \mathcal{F}^{-1}[\chi_k \widehat{h}], \qquad h^{\text{out}} := \mathcal{F}^{-1}[(1-\chi_k)\widehat{h}].$$

Then

$$h = h^{\text{in}} + h^{\text{out}},$$

and

$$\phi_x^{\text{in}}(\eta) = e^{ix\cdot\eta}\widehat{h^{\text{in}}}(\eta), \qquad \phi_x^{\text{out}}(\eta) = e^{ix\cdot\eta}\widehat{h^{\text{out}}}(\eta).$$

Next we decompose

$$\widetilde{R}_k h(x) - T_{e_1} h(x) = (\widetilde{R}_k h^{\text{in}}(x) - T_{e_1} h^{\text{in}}(x)) + (\widetilde{R}_k h^{\text{out}}(x) - T_{e_1} h^{\text{out}}(x)),$$

and estimate the two terms on the right-hand side separately.

Exterior contribution. Next we estimate

$$\widetilde{R}_k h^{\text{out}}(x) - T_{e_1} h^{\text{out}}(x)$$

We now estimate $\widetilde{R}_k h^{\text{out}}$. By (4.1) with $\varepsilon = 0$,

$$\widetilde{R}_k h^{\text{out}}(x) = e^{-ike_1\cdot x}\, R_k^0\big(e^{ike_1\cdot(\cdot)} h^{\text{out}}\big)(x).$$

We set $f := e^{ike_1\cdot(\cdot)} h^{\text{out}}$, $l = [n/2]+1$, and $\delta = 3/4$. By Corollary 3.3, for $k \ge 1$, we estimate

$$\begin{aligned} \|\widetilde{R}_k h^{\text{out}}\|_{H^l_{-\delta}} = \|e^{-ike_1\cdot x} R_k^0 f\|_{H^l_{-\delta}} &\le C_l k^l \|R_k^0 f\|_{H^l_{-\delta}} \\ &\le C_l k^l \|f\|_{H^l_\delta} = C_l k^l \|e^{ike_1\cdot(\cdot)} h^{\text{out}}\|_{H^l_\delta} \le C_l k^{2l} \|h^{\text{out}}\|_{H^l_\delta}. \end{aligned} \tag{A.22}$$

Let us bound $\|h^{\text{out}}\|_{H^l_\delta}$. Since $(1-\Delta)^{l/2} h^{\text{out}} \in \mathcal{S}(\mathbb{R}^n)$, for an integer $N > n/2 + \delta$, we estimate

$$\begin{aligned} \|h^{\text{out}}\|^2_{H^l_\delta} = \int_{\mathbb{R}^n} \langle x\rangle^{2\delta} |(1-\Delta)^{l/2} h^{\text{out}}(x)|^2 dx &\le \int_{\mathbb{R}^n} \langle x\rangle^{2\delta} \langle x\rangle^{-2N} \|(1-\Delta)^{l/2} h^{\text{out}}\|^2_{\mathcal{S}_N} dx \\ &\le C_{\delta,N} \|(1-\Delta)^{l/2} h^{\text{out}}\|^2_{\mathcal{S}_N}. \end{aligned}$$

Since $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a topological isomorphism, there exist $C_N > 0$ and $M \in \mathbb{N}$ such that

$$\begin{aligned} \|h^{\text{out}}\|_{H^l_\delta} \le C_{\delta,N} \|(1-\Delta)^{l/2} h^{\text{out}}\|_{\mathcal{S}_N} &\le C_{\delta,N} C_N \|\mathcal{F}\left[(1-\Delta)^{l/2} h^{\text{out}}\right]\|_{\mathcal{S}_M} \\ &\le C_{\delta,N} \|\langle\cdot\rangle^l (1-\chi_k(\cdot))\widehat{h}(\cdot)\|_{\mathcal{S}_M}. \end{aligned}$$

Since $1-\chi_k$ is supported in $\{|\eta| \ge k/4\}$ and $\langle\eta\rangle^{-1} \le C/k$ on $\{|\eta| \ge k/4\}$, we conclude that

$$\|h^{\text{out}}\|_{H^l_\delta} \le \frac{C_{\delta,N,L}}{k^L} \|\langle\cdot\rangle^{l+L} (1-\chi_k(\cdot))\widehat{h}(\cdot)\|_{\mathcal{S}_M}.$$

By the definition of χ_k,

$$\|h^{\text{out}}\|_{H^l_\delta} \le \frac{C_{\delta,N,L}}{k^L} \|\widehat{h}\|_{\mathcal{S}_{M+l+L}}.$$

We put the last estimate into (A.22) to obtain

$$\|\widetilde{R}_k h^{\text{out}}\|_{H^l_{-\delta}} \leq \frac{C_{\delta,l}}{k^{L-2l}} \|\widehat{h}\|_{\mathcal{S}_{M+l+L}}. \tag{A.23}$$

Since $\langle x\rangle^{-\delta}$ is bounded below on K, the norms $H^l_{-\delta}(K)$ and $H^l(K)$ are equivalent. Therefore, by choosing $L = 2l+1$, and using the Sobolev embedding theorem, we obtain

$$\sup_{x\in K} |\widetilde{R}_k h^{\text{out}}(x)| \leq \frac{C_K}{k} \|\widehat{h}\|_{\mathcal{S}_M}. \tag{A.24}$$

for some $M \in \mathbb{N}$.

Next we estimate $T_{e_1} h^{\text{out}}$. By Lemma A.1,

$$T_{e_1} h^{\text{out}}(x) = i \int_{-\infty}^{x_1} h^{\text{out}}(\tau, x') d\tau.$$

Then, we estimate

(A.25)

$$|T_{e_1} h^{\text{out}}(x)| \leq \int_{\mathbb{R}} |h^{\text{out}}(\tau, x')| d\tau \leq \sup_{y\in\mathbb{R}^n} \langle y\rangle^2 |h^{\text{out}}(y)| \int_{\mathbb{R}} \langle(\tau, x')\rangle^{-2} d\tau \leq C \|h^{\text{out}}\|_{\mathcal{S}_2}.$$

Let

$$F_k(\eta) := (1-\chi_k(\eta))\widehat{h}(\eta), \qquad h^{\text{out}} = \mathcal{F}^{-1} F_k.$$

For all multi-indices α, β with $|\alpha|, |\beta| \leq 2$, we have

$$x^\alpha \partial_x^\beta h^{\text{out}}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\eta} i^{-|\alpha|} \partial_\eta^\alpha \left[(i\eta)^\beta F_k(\eta)\right] \, d\eta.$$

Therefore,

$$\|h^{\text{out}}\|_{\mathcal{S}_2} \leq C \sum_{|\alpha|,|\beta|\leq 2} \left\| \partial_\eta^\alpha \left[\eta^\beta (1-\chi_k)\widehat{h}\right] \right\|_{L^1(\mathbb{R}^n)}.$$

Since $1-\chi_k$ is supported in $\{|\eta| \geq k/4\}$, we have

$$1 \leq \frac{4|\eta|}{k} \leq \frac{4\langle\eta\rangle}{k} \qquad \text{on } \operatorname{supp}(1-\chi_k). \tag{A.26}$$

Also, for every multi-index γ,

$$|\partial_\eta^\gamma \chi_k(\eta)| \leq C_\gamma k^{-|\gamma|} \leq C_\gamma, \qquad k \geq 1.$$

Using Leibniz' rule, we obtain, for $|\alpha|, |\beta| \leq 2$,

$$\left| \partial_\eta^\alpha \left[\eta^\beta (1-\chi_k)\widehat{h}\right] \right| \leq C \sum_{|\mu|\leq N_0} \langle\eta\rangle^{N_0} \left| \partial_\eta^\mu \widehat{h}(\eta) \right|$$

for some fixed integer N_0. Since $1-\chi_k$ and all its derivatives are supported where $|\eta| \geq k/4$, (A.26) implies

$$\langle\eta\rangle^{N_0} \leq \frac{C}{k} \langle\eta\rangle^{N_0+1} \qquad \text{on } \operatorname{supp}(1-\chi_k).$$

Thus, after increasing the Schwartz seminorm index, we get

$$\left\| \partial_\eta^\alpha \left[\eta^\beta (1-\chi_k)\widehat{h}\right] \right\|_{L^1(\mathbb{R}^n)} \leq \frac{C}{k} \|\widehat{h}\|_{\mathcal{S}_N}, \qquad |\alpha|, |\beta| \leq 2.$$

Consequently,

$$\|h^{\text{out}}\|_{\mathcal{S}_2} \leq \frac{C}{k} \|\widehat{h}\|_{\mathcal{S}_N},$$

and hence, (A.25) gives

$$\sup_{x\in K}|T_{e_1}h^{\mathrm{out}}(x)|\leq\frac{C}{k}\|\widehat{h}\|_{\mathbb{s}_{M'}}.$$

Combining this with (A.24), we conclude that

$$\sup_{x\in K}\left|\widetilde{R}_k h^{\mathrm{out}}(x)-T_{e_1}h^{\mathrm{out}}(x)\right|\leq\frac{C}{k}\,\|\widehat{h}\|_{\mathbb{s}_{M'}}. \tag{A.27}$$

Interior contribution. From the identity

$$(|ke_1+\eta|-k)(|ke_1+\eta|+k)=2k\eta_1+|\eta|^2,$$

we obtain

$$|ke_1+\eta|-k=\frac{2k\eta_1+|\eta|^2}{|ke_1+\eta|+k}=\frac{2k}{|ke_1+\eta|+k}\cdot\left(\eta_1+\frac{|\eta|^2}{2k}\right)=\frac{\sigma_k(\eta)}{a_k(\eta)}$$

Therefore, for every $\varepsilon>0$,

$$\frac{1}{|ke_1+\eta|-k-i\varepsilon}=\frac{c_1(\eta)}{X_{1,1}(\eta)}. \tag{A.28}$$

Then, we write

$$\begin{aligned}\widetilde{R}_{k,\varepsilon}h^{\mathrm{in}}(x)-T^{\varepsilon}_{e_1}h^{\mathrm{in}}(x)&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left[\frac{1}{|ke_1+\eta|-k-i\varepsilon}-\frac{1}{\eta_1-i\varepsilon}\right]e^{ix\cdot\eta}\widehat{h^{\mathrm{in}}}(\eta)d\eta\\&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left[\frac{c_1(\eta)}{X_{1,1}(\eta)}-\frac{1}{X_{0,0}(\eta)}\right]\phi^{\mathrm{in}}_x(\eta)d\eta\end{aligned}$$

We decompose the bracket as $E_1+E_2+E_3$:

$$\begin{aligned}E_1(\eta)&:=\frac{c_1(\eta)-1}{X_{1,1}(\eta)},\\E_2(\eta)&:=\frac{1}{X_{1,1}(\eta)}-\frac{1}{X_{0,1}(\eta)},\\E_3(\eta)&:=\frac{1}{X_{0,1}(\eta)}-\frac{1}{X_{0,0}(\eta)}.\end{aligned}$$

We estimate the pairing $\langle E_i,\phi^{\mathrm{in}}_x\rangle$ for $i=1,2,3$, where $\mathrm{supp}(\phi^{\mathrm{in}}_x)\subset\{|\eta|\leq k/2\}$.

Note that $c_1=a_k$ by definition. Therefore,

$$\int_{\mathbb{R}^n}E_1(\eta)\phi^{\mathrm{in}}_x(\eta)d\eta=\int_{\mathbb{R}^n}\frac{(a_k(\eta)-1)\phi^{\mathrm{in}}_x(\eta)}{X_{1,1}(\eta)}d\eta.$$

Since ϕ^{in}_x is smooth and supported in $\{|\eta|\leq k/2\}$, Lemma A.2 implies that there exists $M\in\mathbb{N}$ such that

$$\left|\int_{\mathbb{R}^n}E_1(\eta)\phi^{\mathrm{in}}_x(\eta)d\eta\right|\leq C\|(a_k-1)\phi^{\mathrm{in}}_x\|_{\mathbb{s}_M} \tag{A.29}$$

uniformly in $\varepsilon\in(0,1]$ and $k\geq1$. Combining (A.11) with

$$|a_k(\eta)-1|=\frac{||ke_1+\eta|-k|}{2k}\leq\frac{|\eta|}{2k}\qquad\text{on }\{|\eta|\leq k/2\},$$

we conclude that

$$\|(a_k-1)\phi^{\mathrm{in}}_x\|_{\mathbb{s}_M}\leq\frac{C}{k}\|\phi^{\mathrm{in}}_x\|_{\mathbb{s}_{M+1}}. \tag{A.30}$$

Recall that

$$\phi_x^{\text{in}}(\eta) = e^{ix\cdot\eta}\chi_k(\eta)\widehat{h}(\eta)$$

Since $|\partial_\eta^\nu \chi_k| < C/k^{|\nu|}$ and the polynomials, in x variables, are bounded in K, we conclude that

$$\sup_{x\in K} \|\phi_x^{\text{in}}\|_{\mathbb{s}_{M+1}} \leq C\|\widehat{h}\|_{\mathbb{s}_{M+1}} \tag{A.31}$$

Combining (A.29), (A.30), and (A.31) gives

$$\sup_{x\in K} \left| \int_{\mathbb{R}^n} E_1(\eta)\phi_x^{\text{in}}(\eta)d\eta \right| \leq \frac{C}{k}\|\widehat{h}\|_{\mathbb{s}_{M+1}} \tag{A.32}$$

uniformly in $\varepsilon \in (0,1]$ and $k \geq 1$.

Next, we estimate

$$\langle E_2, \phi_x^{\text{in}}\rangle = \int_{\mathbb{R}^n} E_2(\eta)\phi_x^{\text{in}}(\eta)d\eta = \int_{\mathbb{R}^n} \left[\frac{1}{X_{1,1}(\eta)} - \frac{1}{X_{0,1}(\eta)}\right]\phi_x^{\text{in}}(\eta)d\eta.$$

We use the integral identity

$$\frac{1}{A} - \frac{1}{B} = (B-A)\int_0^1 \frac{d\theta}{[\theta A + (1-\theta)B]^2} \tag{A.33}$$

Take $A = X_{1,1}$ and $B = X_{0,1}$. Note that

$$B - A = i\varepsilon(a_k - 1), \qquad \theta X_{1,1} + (1-\theta)X_{0,1} = X_{\theta,1},$$

for $\theta \in [0,1]$. Therefore,

$$E_2(\eta) = i\varepsilon(a_k(\eta) - 1)\int_0^1 \frac{d\theta}{X_{\theta,1}(\eta)^2}, \tag{A.34}$$

and hence,

$$\int_{\mathbb{R}^n} E_2(\eta)\phi_x^{\text{in}}(\eta)d\eta = \int_0^1 \left(\int_{\mathbb{R}^n} \frac{i\varepsilon(a_k(\eta)-1)\phi_x^{\text{in}}(\eta)}{X_{\theta,1}(\eta)^2}d\eta\right)d\theta.$$

Since ϕ_x^{in} is supported in $\{|\eta| \leq k/2\}$, Lemma A.2 implies that

$$\left|\int_{\mathbb{R}^n} \frac{i\varepsilon(a_k(\eta)-1)\phi_x^{\text{in}}(\eta)}{X_{\theta,1}(\eta)^2}d\eta\right| \leq C\|i\varepsilon(a_k-1)\phi_x^{\text{in}}\|_{\mathbb{s}_M} \leq C\|(a_k-1)\phi_x^{\text{in}}\|_{\mathbb{s}_M}.$$

uniformly in $\varepsilon \in (0,1]$, $\theta \in [0,1]$, and $k \geq 1$. Integrating over $\theta \in [0,1]$ gives

$$\left|\int_{\mathbb{R}^n} E_2(\eta)\phi_x^{\text{in}}(\eta)d\eta\right| \leq C\|(a_k-1)\phi_x^{\text{in}}\|_{\mathbb{s}_M}$$

Then, (A.30) and (A.31) imply that

$$\sup_{x\in K} \left| \int_{\mathbb{R}^n} E_2(\eta)\phi_x^{\text{in}}(\eta)d\eta \right| \leq \frac{C}{k}\|\widehat{h}\|_{\mathbb{s}_{M+1}} \tag{A.35}$$

uniformly in $\varepsilon \in (0,1]$ and $k \geq 1$.

Next, we estimate

$$\langle E_3, \phi_x^{\text{in}}\rangle = \int_{\mathbb{R}^n} E_3(\eta)\phi_x^{\text{in}}(\eta)d\eta = \int_{\mathbb{R}^n} \left[\frac{1}{X_{0,1}(\eta)} - \frac{1}{X_{0,0}(\eta)}\right]\phi_x^{\text{in}}(\eta)d\eta.$$

Take $A = X_{0,1}$, $B = X_{0,0}$ and note that

$$B - A = -\frac{|\eta|^2}{2k}, \qquad \vartheta X_{0,1} + (1-\vartheta)X_{0,0} = X_{0,\vartheta}$$

for $\vartheta \in [0,1]$. Therefore, by (A.33),

$$E_3(\eta) = -\frac{|\eta|^2}{2k}\int_0^1 \frac{d\vartheta}{X_{0,\vartheta}(\eta)^2}.$$

and hence,

$$\int_{\mathbb{R}^n} E_3(\eta)\phi_x^{\mathrm{in}}(\eta)d\eta = -\int_0^1 \left(\int_{\mathbb{R}^n} \frac{\frac{|\eta|^2}{2k}\phi_x^{\mathrm{in}}(\eta)}{X_{0,\vartheta}(\eta)^2}d\eta\right) d\vartheta.$$

Since ϕ_x^{in} is supported in $\{|\eta| \le k/2\}$, Lemma A.2 implies that

$$\left|\int_{\mathbb{R}^n} \frac{\frac{|\eta|^2}{2k}\phi_x^{\mathrm{in}}(\eta)}{X_{0,\vartheta}(\eta)^2}d\eta\right| \le \frac{C}{k}\||\cdot|^2\phi_x^{\mathrm{in}}(\cdot)\|_{\mathbb{S}_M} \le \frac{C}{k}\|\phi_x^{\mathrm{in}}(\cdot)\|_{\mathbb{S}_{M+2}}$$

uniformly in $\varepsilon \in (0,1]$, $\vartheta \in [0,1]$, and $k \ge 1$. Therefore,

$$\left|\int_{\mathbb{R}^n} E_3(\eta)\phi_x^{\mathrm{in}}(\eta)d\eta\right| \le \frac{C}{k}\|\phi_x^{\mathrm{in}}(\cdot)\|_{\mathbb{S}_{M+2}}.$$

Then, (A.31) implies that

$$\sup_{x\in K}\left|\int_{\mathbb{R}^n} E_3(\eta)\phi_x^{\mathrm{in}}(\eta)d\eta\right| \le \frac{C}{k}\|\widehat{h}\|_{\mathbb{S}_{M+3}} \tag{A.36}$$

uniformly in $\varepsilon \in (0,1]$, $\vartheta \in [0,1]$, and $k \ge 1$. Combining (A.32), (A.35), and (A.36) gives

$$\sup_{\varepsilon\in(0,1]}\sup_{x\in K}\left|\widetilde{R}_{k,\varepsilon}h^{\mathrm{in}}(x) - T_{e_1}^{\varepsilon}h^{\mathrm{in}}(x)\right| \le \frac{C}{k}\|\widehat{h}\|_{\mathbb{S}_M} \tag{A.37}$$

for some $M \in \mathbb{N}$, uniformly on $k > 1$.

It remains to pass from the ε-level estimate (A.37) to the estimate at $\varepsilon = 0$. We write

$$\begin{aligned}\sup_{x\in K}\left|\widetilde{R}_k h^{\mathrm{in}}(x) - T_{e_1}h^{\mathrm{in}}(x)\right| \le{}& \left\|\widetilde{R}_k h^{\mathrm{in}} - \widetilde{R}_{k,\varepsilon}h^{\mathrm{in}}\right\|_{C^0(K)} + \left\|T_{e_1}h^{\mathrm{in}} - T_{e_1}^{\varepsilon}h^{\mathrm{in}}\right\|_{C^0(K)} \\ &+ \left\|\widetilde{R}_{k,\varepsilon}h^{\mathrm{in}} - T_{e_1}^{\varepsilon}h^{\mathrm{in}}\right\|_{C^0(K)}.\end{aligned}$$

Then, by (A.37), for $\varepsilon \in (0,1]$, we have

$$\begin{aligned}\sup_{x\in K}\left|\widetilde{R}_k h^{\mathrm{in}}(x) - T_{e_1}h^{\mathrm{in}}(x)\right| \le{}& \left\|\widetilde{R}_k h^{\mathrm{in}} - \widetilde{R}_{k,\varepsilon}h^{\mathrm{in}}\right\|_{C^0(K)} + \left\|T_{e_1}h^{\mathrm{in}} - T_{e_1}^{\varepsilon}h^{\mathrm{in}}\right\|_{C^0(K)} \\ &+ \frac{C}{k}\|\widehat{h}\|_{\mathbb{S}_M}.\end{aligned}$$

For the first term, using (4.2) and Sobolev embedding, we obtain

$$\lim_{\varepsilon\to 0}\left\|\widetilde{R}_k h^{\mathrm{in}} - \widetilde{R}_{k,\varepsilon}h^{\mathrm{in}}\right\|_{C^0(K)} = 0.$$

By Lemma A.1, the second term tends to zero as $\varepsilon \to 0^+$. Hence, taking $\varepsilon \to 0^+$ gives the following.

$$\sup_{x\in K}\left|\widetilde{R}_k h^{\mathrm{in}}(x) - T_{e_1}h^{\mathrm{in}}(x)\right| \le \frac{C}{k}\,\|\widehat{h}\|_{\mathbb{S}_M}. \tag{A.38}$$

Combining this with (A.27) we complete the proof. □

Data availability

No data were used or generated in the research described in this article.

Declaration of competing interests

The authors declare that they have no competing interests.

Declaration of generative AI use

During the preparation of this work, the authors used Claude and ChatGPT to help simplify and verify certain minor calculations and to improve the clarity and readability of the writing. The authors reviewed and verified the AI-assisted content and take full responsibility for the manuscript.

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