ON EXPONENTIAL INSTABILITY OF AN INVERSE PROBLEM FOR THE WAVE EQUATION

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Abstract. For a time-independent potential $q \in L^\infty$, consider the source-to-solution operator that maps a source f to the solution $u = u(t,x)$ of $(\Box + q)u = f$ in Euclidean space with an obstacle, where we impose on u vanishing Cauchy data at $t = 0$ and vanishing Dirichlet data at the boundary of the obstacle. We study the inverse problem of recovering the potential q from this source-to-solution map restricted to some measurement domain. By giving an example where measurements take place in some subset and the support of q lies in the 'shadow region' of the obstacle, we show that recovery of q is exponentially unstable.

1. Introduction

Let $T > 0$ be arbitrary and consider the strictly convex, compact obstacle $\mathcal{O} \coloneqq \overline{B_1(0)} \subset \mathbb{R}^n$ with analytic boundary. Here and throughout, for any $r_0 > 0$ and $x_0 \in \mathbb{R}^n$, $B_{r_0}(x_0)$ denotes the open ball of radius r_0 about x_0 in $\mathbb{R}^n$. Define $U \coloneqq B_{T+4}(0) \setminus \mathcal{O}$ (this choice being motivated later on) and introduce $X \coloneqq (-T,T) \times U$ and $X_+ \coloneqq (0,T) \times U$, where the definitions made so far remain fixed throughout this document. The wave operator in X is denoted by $\Box = \partial_t^2 - \Delta_x$.

Let $q \in L^\infty(U)$. We consider the forward problem of finding a solution $u = u(t,x)$ to

$$\Box u + qu = f \quad \text{in} \quad X\,, \qquad u|_{t<0} = 0\,, \quad u|_{(-T,T)\times\partial U} = 0 \tag{1.1}$$

for any $f \in L^2(X_+)$. In fact, as shown in the appendix, Lemma A.1, there is a continuous linear operator

$$S_q \colon L^2(X_+) \to C([-T,T]; H_0^1(U)) \cap C^1([-T,T]; L^2(U))$$

so that $S_q(f)$ is the unique solution to (1.1).

The aim of this note is to show that the inverse problem of determining q from S_q is exponentially unstable in certain partial data settings. This is achieved by appealing to the machinery in [20] and imposing support conditions on potentials q that induce Gevrey smoothing of $S_q - S_0$. Throughout this document $e_1 = (1,0,\dots,0) \in \mathbb{R}^n$ refers to the first standard basis vector, $r \in (0,1)$ is a fixed constant, and for sets A, B we use the notation $A \Subset B$ to mean that the closure of A is a compact subset of B.

Theorem 1.1. *Let $r \in (0,1)$ be fixed as above, $\Sigma \Subset B_r(2e_1), \Xi \Subset B_r(-2e_1)$ be open, $T' \in (0,T)$, and define $\Omega \coloneqq (0,T') \times \Xi$. Let $\mu \in \mathbb{R}, \delta > 0$ be fixed so that $\mu + \delta > n/2$ and define $K \coloneqq \Big\{ q \in H^\mu(U) \colon \operatorname{supp} q \subset \bar{\Sigma}\,, \|q\|_{H^{\mu+\delta}(U)} \le 1 \Big\}$.*

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If ω is a modulus of continuity so that

$$\|q_1 - q_2\|_{H^\mu(U)} \leq \omega\left(\|\mathbb{1}_\Omega(S_{q_1} - S_{q_2}) \circ \mathbb{1}_\Omega\|_{L^2(\Omega)\to L^2(\Omega)}\right), \qquad q_1, q_2 \in K,$$

then $\omega(s) \gtrsim |\log s|^{-\delta\frac{6n+7}{n}}$ *for s small.*

That S_q is well-defined for $q \in K$ follows from the Sobolev embedding and Lemma A.1. For a graphical representation of the sets $\mathcal{O}, B_r(\pm 2e_1)$ and the notion that $B_r(2e_1)$ lies in the 'shadow region' of $\mathcal{O}$ from the perspective of $B_r(-2e_1)$, we point to Fig. 1 below.

The set U was chosen so that for sources $f \in L^2(X_+)$ with $\operatorname{supp} f \subset [0,T] \times \overline{B_3(0)}$, by the finite speed of propagation, $S_q(f)$ vanishes identically near $(-T,T) \times \partial B_{T+4}(0)$, the 'far' boundary of X. Thus, the extension of $S_q(f)$ by 0 outside of U solves

$$(\Box + q)S_q(f) = f \ \text{ in } \ (-T,T) \times \mathbb{R}^n \setminus \mathcal{O}, \quad S_q(f)|_{t<0} = 0, \quad S_q(f)|_{(-T,T)\times\partial\mathcal{O}} = 0,$$

when $\operatorname{supp} f \subset [0,T] \times \overline{B_3(0)}$, where we recall that $T > 0$ was chosen freely. This clarifies that the only significant geometric feature of the space X is the subset of its boundary $(-T,T) \times \partial\mathcal{O}$. Because $B_r(\pm 2e_1) \subset B_3(0)$, considering only sources f satisfying $\operatorname{supp} f \subset [0,T] \times \overline{B_3(0)}$ is no restriction in the context of Theorem 1.1.

Theorem 1.1 will come as a consequence of the qualitative statement of the propagation of singularities (Proposition 2.4) together with facts from functional analysis to get a quantitative statement. Briefly: we have access to a complete description of the propagation of smooth singularities via [17, Thm. 24.5.3] if $q \in C^\infty$. Furthermore, due to [23] (in particular [22], and see also [14, Thm. 2.1]), together with [15, Thm. 7.3], we have a complete description of the Gevrey-σ singularities for $\sigma \geq 1$ if $q \in C^\omega$. We reduce to the case $q = 0$ by considering $S_q - S_0$ so that these propagation results become applicable. The existence of the obstacle $\mathcal{O}$ prevents (Gevrey-3-)singularities of sources supported in $(0,T) \times \Sigma$ from propagating and giving rise to singularities of the solution in Ω. The machinery of [20] will turn this smoothing behavior into an instability statement.

The significance of Theorem 1.1 lies in the fact that the Boundary Control method, pioneered by M. Belishev in [4] and extended to Riemannian manifolds in [5], see also [19], implies that one can indeed recover q uniquely from the knowledge of S_q (on Ω) for $T < \infty$ sufficiently large: in the setting of a closed manifold and with $q \in C^\infty$, this is shown in [32] (see Remark 1.2 therein), whereas for $q \in L^\infty$, in Euclidean space without the presence of an obstacle see [12, Thm. 1.1]. In the setting without an obstacle, [12, Thm. 1.2] proves log-log stability of the solution operator S_q. For emphasis we point out that in the situation of Theorem 1.1, despite the fact that measurements take place in a bounded domain for a large period of time, recovery is shown to be exponentially unstable.

For a different choice of measurement set Ω, recovery of q from S_q can be proved by reduction to the closely related inverse problem using the Dirichlet-to-Neumann (DN) map as data, which we briefly elaborate on here (see also [18] for a discussion of various equivalent inverse problems). For $W \subset U$ open, $q \in L^\infty(W)$ and $H^1_+((0,T) \times \partial W) := \{\varphi \in H^1((0,T) \times \partial W) \colon \varphi(0,\cdot) = 0\}$ one considers the operator $\Lambda_q \colon H^1_+((0,T)\times\partial W) \to L^2((0,T)\times\partial W)$ defined via $\Lambda_q \colon \varphi \mapsto \partial_\nu v|_{(0,T)\times\partial W}$, where ν is the outward pointing unit normal at ∂W and $v \in C([0,T]; H^1(W)) \cap$

$C^1([0,T];L^2(W))$ is the unique solution of

$$\Box v + qv = 0 \quad \text{in} \quad (0,T)\times W\,, \qquad v|_{t=0} = 0 = \partial_t v|_{t=0}\,, \quad v|_{(0,T)\times\partial W} = \varphi\,.$$

That Λ_q is well-defined is guaranteed, for example, by [6, Lem. 1.2]. One then asks whether the knowledge of the map Λ_q uniquely determines q, which is answered positively in [29] if $T > \operatorname{diam} W$. Here we note that in the case of data on the full boundary as we have here, the Dirichlet-to-Neumann and Neumann-to-Dirichlet data formulations are equivalent.

If we denote by $\eta\colon \{\varphi \in C^\infty((0,T)\times\partial W)\colon \operatorname{supp}\varphi \Subset \{t\geq 0\}\} \to C^\infty((0,T)\times U)$ an extension operator (that is $\eta(\varphi)|_{(0,T)\times\partial W} = \varphi$ for all φ), then a calculation shows that for all $\varphi \in C^\infty((0,T)\times\partial W)$ with $\operatorname{supp}\varphi \Subset \{t\geq 0\}$,

$$\Lambda_q(\varphi) = \partial_\nu(\eta(\varphi) - S_q((\Box + q)\eta(\varphi)))|_{(0,T)\times\partial W}\,.$$

We conclude (using a density argument) that if $W \subset U$ is open, $\operatorname{supp} q \subset W$, and the measurement set Ω contains a neighborhood of the boundary $(-T,T)\times\partial W$, the inverse problem for the source-to-solution map with data on Ω can be reduced to that with the DN map as data, which admits a large body of literature. In particular, for the DN-map formulation with $W = B_R(0)\setminus\mathcal{O}$ with R fixed (so one has measurements on both the outer and inner boundaries of W), [36] shows that recovery of q is Hölder stable if T is sufficiently large, see also [6]. We refer also to [26, 7, 3] and the references therein for further stability results for the case of DN-map data.

Another related case is where one has access to measurements on the full outer boundary $\partial B_R(0)$, but there is a vanishing Dirichlet boundary condition on $\partial\mathcal{O}$. In this case, if T is sufficiently large, the DN-map determines Hölder stably the integrals of q over line segments not touching $\mathcal{O}$ via geometrical optics solutions [35]. From these integrals one can determine q Hölder stably when $n \geq 3$ (e.g. by inverting the X-ray transform on two-dimensional slices), whereas for $n = 2$ one would also need to use broken lines reflecting on $\partial\mathcal{O}$ [10]. Theorem 1.1 corresponds to a setting where one has measurements in a smaller region, thus leading to exponential instability of the inverse problem.

We close the introduction by mentioning that in the setting of the Calderón problem, it follows from [1] and [24] that recovery of a potential for the Schrödinger operator using the Dirichlet-to-Neumann map is logarithmically stable, and that this stability is optimal, see also the discussion in [20, § 1.1]. In the absence of a potential q, the Gel'fand problem of recovering properties of a manifold (such as its metric) from spectral data of the Laplacian is discussed in [2, 8, 9], the first of which establishes an abstract stability result and the latter two show log-log stability.

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2. The Proof

Let $p(t,x,\tau,\xi) = -\tau^2 + |\xi|^2$ be the principal symbol of $\Box$, which will remain fixed throughout. We begin with two purely geometric statements that can be summarized informally as: generalized bicharacteristic arcs of p with a point in $T^*((0,T)\times B_r(-2e_1))$ cannot have 'originated' from a neighborhood of $T^*(\{0\}\times B_r(2e_1))$. The following two statements are immediate when one draws a picture (see Fig. 1), but we give the full details nevertheless, first fixing some terminology. If H is a hyperplane in $\mathbb{R}^n$ defined by $H = \{x\colon (x-x^\circ)\cdot x^\circ = 0\}$ where $x^\circ \in \mathbb{R}^n$ is some point, we say that a point $x \in \mathbb{R}^n$ lies above (resp. below) H if $(x-x^\circ)\cdot x^\circ > 0$ (resp. < 0).

Lemma 2.1. *Let $t^\circ > 0$ and $\gamma\colon [0,t^\circ] \to \bar{U}$ be a line segment with $\gamma(0) \in B_r(-2e_1)$, $\gamma(t^\circ) \in \partial\mathcal{O}$, and H the tangential hyperplane of $\partial\mathcal{O}$ at the point $\gamma(t^\circ)$, oriented so that the origin lies below H. The set $B_r(2e_1)$ lies below H.*

Proof. Denoting $x^\circ = \gamma(t^\circ)$, the normal of $\partial\mathcal{O}$ at x° is given by x°, so that H is defined as $H = \{x\colon (x-x^\circ)\cdot x^\circ = 0\}$.

Because any line segment below H intersecting H at x° must pass through $\mathcal{O}$ and γ describes a line segment in $\bar{U}$, all of $\gamma(t)$ and thus $\gamma(0)$ must lie above (or on) H. Let $\gamma(0) = x_0 \in B_r(-2e_1)$, which can be written as $x_0 = -2e_1 + y_0$ for some $|y_0| < r$. Because x_0 lies above (or on) H, we conclude that

$$0 \le (x_0 - x^\circ)\cdot x^\circ = -2x_1^\circ - 1 + y_0\cdot x^\circ \le -2x_1^\circ - 1/4\,,$$

where we used that $|x^\circ| = 1$ and the Cauchy-Schwarz inequality, and x_1° is the first component of x°. We conclude that $x_1^\circ < 0$.

Similarly, any point $x \in B_r(2e_1)$ can be written as $x = 2e_1 + y$ with $|y| < r$ so that

$$(x - x^\circ)\cdot x^\circ < 2x_1^\circ r - 1 < r - 1\,,$$

where we used $x_1^\circ < 0$. This shows that $B_r(2e_1)$ lies below H, completing the proof. ∎

Lemma 2.2. *For every $\nu^* = (t^*, x^*, \tau^*, \xi^*) \in T^*((0,T)\times B_r(-2e_1))\setminus\{0\}$ with $|\tau^*| = 1$ and $\nu^* \in p^{-1}(0)$, the forward generalized bicharacteristic arc γ (according to* [17, Def. 24.3.7] *or* [25]*) on $\mathbb{R}\times U$ with initial condition $\gamma(t^*) = \nu^*$ satisfies $\gamma(t) \in T^*(\{t\}\times(U\setminus B_r(2e_1)))\setminus\{0\}$ for $t\in[0,t^*]$. Here,* forward *means that the projection of γ onto the t component is an increasing function.*

Proof. The following general remark will be used throughout the rest of the proof. Because γ travels with unit speed forward in time ($|\pi_\tau\gamma(t)| = |\tau^*| = 1, t \in [0,t^*]$) and the 'far' boundary $\mathbb{R}\times\partial B_{T+4}(0)$ was chosen far away, and $\pi_x\gamma(t^*) \in B_r(-2e_1)$, we have $\pi_x\gamma(t) \notin \partial B_{T+4}(0)$ for all $t \in [0,t^*]$. We make a case distinction, see also Fig. 1 for a pictorial view of the behavior of the ray $\pi_x\gamma$.

Case 1: the generalized bicharacteristic arc defined by $\gamma(t)$, when projected onto the x-component, does not intersect $\partial\mathcal{O}$ while $t \in [0,t^*]$. This means that $\pi_x\gamma(t), t\in[0,t^*]$ describes a line segment with $\pi_x\gamma(0) \in B_r(-2e_1)$, and if it were to intersect $B_r(2e_1)$, some point $\pi_x\gamma(t)$ must lie on $\partial\mathcal{O}$, a contradiction. We thus find that $\pi_x\gamma(t) \notin B_r(2e_1)$ for all $t\in[0,t^*]$, which completes the consideration of this case.

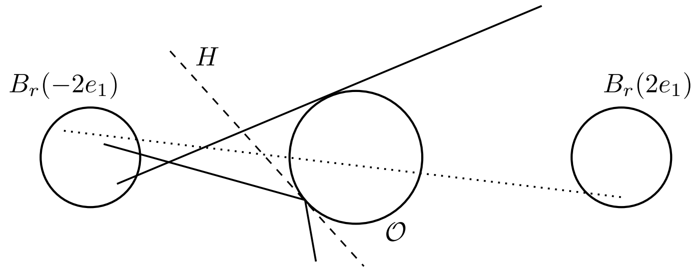

FIGURE 1. A pictorial representation of $B_r(\pm 2e_1), \mathcal{O}$ in U considered the ambient space with $\partial B_{T+4} \subset \partial U$ outside of frame. The following discussion is in the context of the proof of Lemma 2.2. The dotted line represents case 1: a line segment with endpoints in $B_r(\pm 2e_1)$ must pass through the obstacle $\mathcal{O}$ (in particular intersecting $\partial\mathcal{O}$); a ray cannot follow this path. The continuous lines are the two possible cases 2a, 2b: a ray starting in $B_r(-2e_1)$ intersects $\partial\mathcal{O}$ either glancingly and continues as the same line segment, staying away from $B_r(2e_1)$. Or, the ray intersects $\partial\mathcal{O}$ transversally, being reflected according to Snell's law off the tangential plane H at the intersection point, represented here by the dashed line. In the latter case one sees that the ray always stays on one side of H, whereas $B_r(2e_1)$ lies on the other side of H.

Case 2: there is $t^\circ \in (0, t^*)$ so that $\pi_{(t,x)}\gamma(t^\circ) \in (0,T) \times \partial\mathcal{O}$. We remark here that the normal of $\mathbb{R} \times \partial\mathcal{O}$ at any $(t,x) \in (0,T) \times \partial\mathcal{O}$ is given by $(0,x)$.

Case 2a: the generalized bicharacteristic arc γ intersects the boundary $(0,T)\times\partial\mathcal{O}$ tangentially (glancingly) when $t = t^\circ$ (which is to say that $\lim_{t\to^+ t^\circ} \pi_{(\tau,\xi)}\gamma(t) \perp (0, \pi_x\gamma(t^\circ)))$. Due to the fact that $\mathcal{O}$ is strictly convex and the remarks before [17, Def. 24.3.2], any glancing intersection of γ with the boundary $(0,T) \times \partial\mathcal{O}$ must be diffractive. By the definition of a generalized bicharacteristic arc, γ is unperturbed by diffractive intersections and continues as the same line segment. This implies that $\pi_x\gamma(t), t \in [0,t^*]$ intersects ∂U only at $t = t^\circ$, so that $\pi_x\gamma\colon [0,t^*] \to \bar{U}$ describes a line segment, which, as in case 1 implies that $\pi_x\gamma(t) \notin B_r(2e_1)$ for all $t \in [0,t^*]$.

Case 2b: the generalized bicharacterisitc arc γ intersects the boundary $(0,T)\times\partial\mathcal{O}$ transversally at $t = t^\circ$. We are thus in the case of an intersection in the hyperbolic region. Let $\gamma(t^\circ_\pm) := \lim_{t\to^\pm t^\circ} \gamma(t) = ((t^\circ, x^\circ), \sigma_\pm)$ and let $H = \{y \in X\colon (y - (0,x^\circ)) \cdot (0,x^\circ)\}$ be the tangential hypersurface of $(0,T) \times \partial\mathcal{O}$ at $(0,x^\circ)$.

We decompose $\sigma_\pm$ into tangential and normal components with respect to H: $\sigma_\pm = \sigma^t_\pm + \sigma^n_\pm$, where $\sigma^n_\pm \in \mathbb{R}(0,x^\circ)$ and $\sigma^n_+ \neq 0$ by assumption of transversal intersection. In fact, because hyperbolic intersections are isolated (and there cannot have been a glancing intersection) we must have $\sigma^n_+ \cdot (0,x^\circ) > 0$ since $\pi_{(t,x)}\gamma(t) \in X$ for $t \in (t^\circ, t^\circ + \varepsilon)$ for some $\varepsilon > 0$ (ie. γ 'comes from inside the set X'). By the definition of a generalized bicharacteristic arc, we must have $\sigma^+_t = \sigma^-_t$, and because $\gamma(t^\circ_\pm) \in p^{-1}(0)$, we find that $\sigma^n_- = -\sigma^n_+$ so that $\sigma^n_- \cdot (0,x^\circ) < 0$.

We conclude that $\pi_{(t,x)}\gamma(t)$ lies above the hypersurface H for t near but unequal to t°. Further (using that $(0,T) \times \mathcal{O}$ lies below H aside from at $\mathbb{R} \times \{x^\circ\}$), we see that $\pi_x\gamma(t)$ does not intersect ∂U for $t \in [0,t^*] \setminus t^\circ$, so that $\pi_{(t,x)}\gamma$ always lies above the hypersurface H for $t \in [0,t^*] \setminus t^\circ$.

However, $(0,T)\times B_r(2e_1)$ lies below the hypersurface H by Lemma 2.1, completing the proof. □

Recall the definition of Gevrey spaces.

Definition 2.3. For $d\in\mathbb{N}, \sigma\in[1,\infty)$ and $W\subset\mathbb{R}^d$ open, we let $G^\sigma(W)$ be the set of all $f\in C^\infty(W;\mathbb{C})$ so that for every compact $K\subset W$ there is some $C>0$ so that for all multi-indices α one has

$$\max_K |\partial^\alpha f| \le C^{1+|\alpha|}\,|\alpha|^{\sigma|\alpha|}\,.$$

We remark that $G^1(W)$ is the set of analytic functions on W. The definition of the Gevrey-σ wavefront set, an analogue of the smooth wavefront set in Gevrey-σ-regularity, can be found in [16, § 8.4] and [31]; we shall not directly require this notion.

Let $Y \coloneqq (-T,T)\times B_r(-2e_1)\subset X$. We exploit our knowledge of the propagation of singularities together with the geometric statement Lemma 2.2 to give

Proposition 2.4. *For any open $\Sigma\Subset B_r(2e_1)$ and any $g\in L^2(U)$ with $\operatorname{supp} g\subset[0,T]\times\bar{\Sigma}$, if $w\in H^1(X)$ satisfies*

$$\Box w = g \quad in \quad X\,, \qquad w|_{t<0}=0\,, \quad w|_{(-T,T)\times\partial U}=0\,,$$

then $w|_Y\in G^3(Y)$.

Proof. From [17, Thm. 24.1.4], the fact that Σ is away from ∂U, and a compactness argument we know that there is some $\varepsilon>0$ so that $w=0$ in $X\cap((\{0\}\times\partial U)+B_\varepsilon(0))$ (Minkowski sum). Combined with an application of [17, Thm. 23.2.7] together with a compactness argument over $U\setminus((\partial U+B_\varepsilon(0))\cup B_r(2e_1))$ implies that

$$\exists\varepsilon>0\colon w|_{X\cap((\{0\}\times U\setminus B_r(2e_1))+B_\varepsilon(0))}=0\,. \tag{2.1}$$

In particular, we have already shown that for the same $\varepsilon>0$ in (2.1), we have $0=w|_{(-T,\varepsilon)\times B_r(-2e_1)}\in G^3((-T,\varepsilon)\times B_r(-2e_1))=G^3(Y\cap\{t<\varepsilon\})$.

Now assume that $\nu^*=(t^*,x^*,\tau^*,\xi^*)\in \mathrm{WF}(w)\cap T^*(Y\cap\{t>0\})$ where we may assume that $|\tau^*|=1$ by renormalizing and we know from [16, Thm. 8.3.1] (and $\operatorname{supp} g\subset[0,T]\times\bar{\Sigma}$) that $\nu^*\in p^{-1}(0)$. By Lemma 2.2, the forward generalized bicharacteristic arc γ from Lemma 2.2 with $\gamma(t^*)=\nu^*$ satisfies $\gamma(t)\in T^*(\{t\}\times(U\setminus B_r(2e_1)))\setminus\{0\}$ for all $t\in[0,t^*]$.

We note that the support condition we assume on g implies that $g\in\mathcal{N}(\bar{X})$ (defined in [17, Def. 18.3.30]), and according to [17, Cor. 18.3.31], we may assume $w\in\mathcal{N}(\bar{X})$. Note further that the Hamiltonian H_p, is never radial because $\partial_{(\tau,\xi)}p\neq0$ for $(\tau,\xi)\neq0$ (see e.g. [27, Rem. 2.1]).

Therefore, from an application of [17, Thm. 24.5.3] (or combination of [17, Thm. 23.2.9, Thm. 24.2.1, Thm. 24.4.1]), we find that $\gamma(0)\in\mathrm{WF}(w)$ (because $\gamma(t^*)\in\mathrm{WF}(w)$, and $\operatorname{supp} g\subset[0,T]\times\bar{\Sigma}$, $\Sigma\Subset B_r(2e_1)$). However, (2.1) and the fact that $\pi_{(t,x)}\gamma(0)\in\{0\}\times U\setminus B_r(2e_1)$ then leads to the desired contradiction so that $w|_Y\in C^\infty(Y)$.

Further, because [23, Thm. 1.4] states that Gevrey-3 singularities for w propagate precisely the same as smooth singularities near the analytic boundary $(-T,T)\times\partial U\subset\partial X$ (since g vanishes there), replacing the application of [16, Thm. 8.3.1] with [16, Thm. 8.6.1], and using the propagation of Gevrey-3 singularities in the interior ([15, Thm. 7.3]), we have completed the proof. □

The reason we do not hope to achieve better that Gevrey-3 regularity for w above comes from the fact that Gevrey-σ singularities for $\sigma \in [1,3)$ do not behave like the smooth ones at the boundary $\partial\mathcal{O}$; instead, they behave like 'analytic rays' (see [23, 33]). According to [33] and [34, Thm. 0.5], analytic rays can and do bend around the boundary of the obstacle and penetrate into the 'shadow region', in which the non-regular q is supported (see also [30]).

Since we will appeal to [20, Thm. 4.2(b)] in order to prove Theorem 1.1, we will have to replace the domain Y by some closed manifold. We will also have to show mapping properties of S_q into some Banach space rather than G^3. We thus recall the definition of Gevrey spaces on closed manifolds from [20, § 2.6] and their decomposition into a union of Banach spaces.

Definition 2.5. Let $d \in \mathbb{N}, \sigma \in [1,\infty)$. For a closed smooth manifold (M,g) of dimension d and any $\rho > 0$, for $(\varphi_j)_{j\in\mathbb{N}} \subset L^2(M)$ an orthonormal basis of eigenfunctions for $-\Delta_g$, we introduce the subspace $A^{\sigma,\rho}(M)$ of $L^2(M)$ defined as those $u \in L^2(M)$ so that

$$\|u\|_{A^{\sigma,\rho}(M)} = \left(\sum_{j=0}^{\infty} e^{2\rho j^{\frac{1}{d\rho}}} |\langle u, \varphi_j\rangle|^2\right)^{1/2} < \infty\,.$$

Furthermore, if (M,g) is analytic, we let $G^\sigma(M)$ be the set of all $u \in C^\infty(M;\mathbb{C})$ so that for all $k \in \mathbb{N}$ and some $C > 0$,

$$\left\|\nabla^k u\right\|_{L^\infty(M)} \le C^{k+1} k^{\sigma k}\,,$$

with the convention that $0^0 = 1$.

If (M,g) is a closed analytic manifold and $u \in G^\sigma(M)$ for some $\sigma \in [1,\infty)$, then in each coordinate chart, the function u is of class G^σ according to Definition 2.3, see [13].

The importance of the spaces $A^{\sigma,\rho}$ lies in the fact that $\bigcup_{\rho>0} A^{\sigma,\rho}(M) = G^\sigma(M)$ for closed analytic manifolds (M,g), see [20, § 2.6, § B]. There it is also shown that each $A^{\sigma,\rho}(M)$ is a Banach space when (M,g) is a closed smooth manifold.

We shall only need a select amount of properties of the spaces $A^{\sigma,\rho}$, which we state for the readers' convenience.

Lemma 2.6 ([20, § 2.6, Lem. B.1])**.** *Let $\sigma \in [1,\infty)$ and (M,g) be a closed analytic manifold. If $u \in G^\sigma(M)$, there is some $\rho > 0$ so that $u \in A^{\sigma,\rho}(M)$. Furthermore, if for some $C, R > 0$, $u \in L^2(M)$ satisfies*

$$\left\|(-\Delta_g)^t u\right\|_{L^2(M)} \le C R^{2t} (2t)^{2t\sigma} \tag{2.2}$$

for all $t \in \mathbb{N}$, then for some $\rho_0 > 0$ and any $\rho \le \rho_0$, we have $u \in A^{\sigma,\rho}(M)$.

Proof. The first statement is a consequence of [20, Lem. B.1]. For the second, let $u \in L^2(M)$ satisfy (2.2). Once we have shown that u satisfies (2.2) for all $t \in [0,\infty)$, not just the integers, the proof is completed by [20, Lem. B.1]. The proof of the sufficiency statement in [20, Lem. B.1(a)] only requires (2.2) for integers t, which we use to conclude that $u \in G^\sigma(M)$, so that the necessity statement of [20, Lem. B.1(a)] gives (2.2) for possibly different $C, R > 0$ and all $t \in [0,\infty)$, which completes the proof. □

Recall that we defined $Y = (-T,T) \times B_r(-2e_1)$. We set up a functional analytic result that will allow us to replace Y by some closed manifold $N \supset Y$. It will also later be used to show that $S_q - S_0$ maps into $A^{3,\rho}$ for some $\rho > 0$. We will rely on [21], where for any $\sigma > 1$, our notation G^σ corresponds to $\mathcal{E}^{\{M_p\}}$ in that paper, where M_p is the sequence $M_0 = 1, M_p = p!^\sigma, p \in \mathbb{N}$.

Proposition 2.7. *Let $\sigma > 1$ and N be a closed analytic $(n+1)$-dimensional manifold with $N \Supset Y$. Let $F\colon L^2(X_+) \to G^\sigma(Y)$ be linear with $F\colon L^2(X_+) \to L^2(Y)$ continuous.*

For every $\chi \in G^\sigma(N) \cap C_c^\infty(Y)$ there is some $\rho > 0$ so that $\chi F\colon L^2(X_+) \to A^{\sigma,\rho}(N)$ is continuous.

Proof. Because $F\colon L^2(X_+) \to L^2(Y)$ is linear and continuous, it has closed graph. Because $F\colon L^2(X_+) \to G^\sigma(Y) \subset H^{(n+1)/2+1}(Y)$ and the following inclusion map $\iota\colon H^{(n+1)/2+1}(Y) \to L^2(Y)$ is continuous, the closed graph theorem implies that $F\colon L^2(X_+) \to H^{(n+1)/2+1}(Y)$ is continuous and the dual map

$$F^*\colon H^{-(n+1)/2-1}(Y) \to L^2(X_+)',$$

is well-defined. In particular, we may define

$$H\colon Y \to L^2(X_+)', \quad x \mapsto \chi(x)F^*(\delta_x),$$

and for all $u \in L^2(X_+)$, applying Lemma 2.6 to χ,

$$\langle H(x), u\rangle = \chi(x)(Fu)(x) \in G^\sigma(Y) \quad \text{as a function of } x.$$

Thus, [21, Thm. 3.10] implies that in fact $H \in G^\sigma(Y, L^2(X_+)')$, where the meaning of this space is explained in [21, Def. 3.9], which in our case reduces to: for every compact $K \subset Y$ there are constants $C, R > 0$ so that for all multi-indices α,

$$\sup_{x\in K} \|\partial_x^\alpha H(x)\|_{L^2(X_+)'} \le CR^{|\alpha|}\alpha!^\sigma.$$

(We point also to the remarks near [37, Thm. 27.1,§ 40] for an explanation as to what differentiation of a topological-vector-space-valued function means.) In particular, since $\operatorname{supp}\chi =: K \subset Y \subset N$ is compact, we find that

$$\begin{aligned}\sup_{\|u\|_{L^2(X_+)}=1} \|\partial^\alpha(\chi F u)\|_{L^2(N)} &= \sup_{\|u\|_{L^2(X_+)}=1} \|\langle \partial^\alpha H(\cdot), u\rangle\|_{L^2(N)} \\ &\le C' \sup_{x\in K} \|\partial_x^\alpha H(x)\|_{L^2(X_+)'} \le C'CR^{|\alpha|}\alpha!^\sigma,\end{aligned}$$

where $C' > 0$ is the volume of N. With an application of Lemma 2.6 the proof is complete. □

The above result has immediate consequences for $S_q - S_0$. In the following proofs, for any set W, $\mathbb{1}_W$ will both denote the multiplication by the characteristic function of W in L^2 as well as the restriction to $L^2(W)$ of some function defined on a larger set.

Lemma 2.8. *Let $\Sigma \Subset B_r(2e_1), \Xi \Subset B_r(-2e_1)$ be open, $T' \in (0,T)$ and $\Omega = (0,T') \times \Xi$. There is a closed $(n+1)$-dimensional torus $N \supset \Omega$ (thus an analytic closed manifold), a function $\chi \in G^3(N) \cap C_c^\infty(Y)$, $\rho > 0$, and a continuous $b'\colon [0,\infty) \to [0,\infty)$ so that the following is true.*

For every $q \in L^\infty(U)$ with $\operatorname{supp} q \subset \bar{\Sigma}$, the map

$$F_q := \mathbb{1}_{X_+}\chi(S_q - S_0) \circ \chi\mathbb{1}_{X_+}, \qquad u \mapsto \mathbb{1}_{X_+}\chi(S_q - S_0)(\chi\mathbb{1}_{X_+}u)$$

satisfies

$$F_q\colon L^2(N) \to A^{3,\rho}(N) \quad \textit{with} \quad \|F_q\|_{L^2(N)\to A^{3,\rho}(N)} \le b'\left(\|q\|_{L^\infty(U)}\right), \tag{2.3}$$

and

$$\mathbb{1}_\Omega(S_q - S_0)\circ \mathbb{1}_\Omega = \mathbb{1}_\Omega F_q \circ \mathbb{1}_\Omega\,. \tag{2.4}$$

Before we move on to the proof, we remark that in the definition of F_q, we post-compose with $\mathbb{1}_{X_+}$ at the end only so that the adjoint operator of F_q (which can be determined using Lemma 2.9 below) will be well-defined.

Proof. Consider the torus $N \subset \mathbb{R}^{1+n}$ defined by identifying opposite sides of the cube $[-R,R]^{1+n} \subset \mathbb{R}^{1+n}$, where $R>0$ is chosen sufficiently large so that $Y = (-T,T)\times B_r(-2e_1) \Subset N$ (see also [28, Lem. 3.1.8]). With the euclidean metric, N is a closed analytic manifold. We take $\chi \in G^3(\mathbb{R}^{1+n})$ to be identically 1 in an open neighborhood of Ω and supported in a compact subset of $Y \subset N$ (that Gevery cut-offs exist follows from [31, § 1.4] or [16, § 1.4]).

For any $g\in L^2(X_+)$ and $\tilde q \in L^\infty(U)$, $(S_{\tilde q} - S_0)(g)$ is the unique solution of (1.1) for $f = -\tilde q S_{\tilde q}(g)$ and $q=0$, whereas $S_0(-\tilde q S_{\tilde q}(g))$ is also a solution of (1.1) for the same f and q, so that Lemma A.1 guarantees that $S_0(-\tilde q S_{\tilde q}(g)) = (S_{\tilde q} - S_0)(g)$. In particular, for any $q\in L^\infty(U)$, we have

$$F_q = \mathbb{1}_{X_+}\chi(S_q - S_0)\circ\chi\mathbb{1}_{X_+} = \mathbb{1}_{X_+}\chi S_0\circ(-qS_q)\circ\chi\mathbb{1}_{X_+}\,, \tag{2.5}$$

and because $\mathbb{1}_{X_+}\chi = 1$ on Ω, this proves (2.4).

From Lemma A.1 and Proposition 2.4, we know that $S_0\circ\mathbb{1}_{(0,T)\times\Sigma}\colon L^2(X_+)\to H^1(Y)$ is continuous and maps $S_0\circ\mathbb{1}_{(0,T)\times\Sigma}\colon L^2(X_+)\to G^3(Y)$, so that Proposition 2.7 implies that there is $\rho>0$ so that the map

$$\chi S_0\circ\mathbb{1}_{(0,T)\times\Sigma}\colon L^2(X_+)\to A^{3,\rho}(N)\,,\quad f\mapsto \chi S_0(\mathbb{1}_{(0,T)\times\Sigma}f)$$

is continuous, and its operator norm is independent of q. Thus, using (2.5) and noting that $\mathbb{1}_{X_+}\chi S_0 = \chi S_0$ and $\operatorname{supp} q\subset\bar\Sigma$, we have

$$\begin{aligned}\|F_q\|_{L^2(N)\to A^{3,\rho}(N)} &\le \left\|\chi S_0\circ\mathbb{1}_{(0,T)\times\Sigma}\right\|_{L^2(X_+)\to A^{3,\rho}(N)}\|-qS_q\circ\chi\|_{L^2(X_+)\to L^2(X)}\\ &\le C'\|q\|_{L^\infty}\, b\left(\|q\|_{L^\infty}\right)\,,\end{aligned}$$

for b from (A.1) and some $C'>0$ independent of q, which gives (2.3). ☐

As it will be required later, we introduce an adjoint operator, see also [12, Lem. 3.1].

Lemma 2.9. *Let $q\in L^\infty(U)$ and define*

$$S_q^*\colon L^2(X_+)\to H^1((0,2T)\times U)\,,\quad S_q^* = R\circ S_q\circ R\,,\quad \textit{where}\quad Ru(t,x) := u(T-t,x)\,.$$

The $L^2(X)$-adjoint of $\mathbb{1}_{X_+}S_q\circ\mathbb{1}_{X_+}$ is $\mathbb{1}_{X_+}S_q^*\circ\mathbb{1}_{X_+}$.

Proof. Let us first point out that by construction, for any $g\in L^2(X_+)$, we have $S_q^*(g)\in H^1((0,2T)\times U)$, which satisfies

$$(\Box + q)S_q^*(g) = g \quad \text{in} \quad (0,2T)\times U\,,\quad S_q^*(g)|_{t>T} = 0\,,\quad S_q^*(g)|_{(0,2T)\times\partial U} = 0\,. \tag{2.6}$$

Let $f,g\in L^2(X)$. Using (2.6), we have

$$\int_X S_q(\mathbb{1}_{X_+}f)\mathbb{1}_{X_+}g\,\mathrm{d}(t,x) = \int_{X_+} S_q(\mathbb{1}_{X_+}f)(\Box+q)S_q^*(\mathbb{1}_{X_+}g)\,\mathrm{d}(t,x)\,,$$

where the RHS can be understood as the pairing between an H^1 and an H^{-1} function: by construction, $S_q(f)$ and $S_q^*(g)$ vanish in $t<0$ and $t>T$ respectively (and $S_q(f)$ vanishes in $(0,T)\times\partial U$), and thus this pairing makes sense.

Taking a sequence of smooth functions $v_j \in C^\infty(X)$ vanishing on $(0,2T)\times\partial U$ and in $t>T$, with $v_j \to S_q^*(\mathbb{1}_{X_+}g)$ in $H^1(X)$, the definition of S_q as the solution operator gives (essentially by partial integration, or rather the definition of a weak solution, see [19, Eq. (2.67)]),

$$\begin{aligned}\int_{X_+} S_q(\mathbb{1}_{X_+}f)(\Box+q)S_q^*(\mathbb{1}_{X_+}g)\mathrm{d}(t,x) &= \lim_{j\to\infty}\int_{X_+} S_q(\mathbb{1}_{X_+}f)(\Box+q)v_j\mathrm{d}(t,x)\\ = \lim_{j\to\infty}\int_{X_+}\mathbb{1}_{X_+}fv_j\mathrm{d}(t,x) &= \int_X \mathbb{1}_{X_+}fS_q^*(\mathbb{1}_{X_+}g)\mathrm{d}(t,x)\,,\end{aligned}$$

which completes the proof. □

Finally, we shall introduce one additional piece of notation from [20, Thm. 3.16]. For $\sigma\in[1,\infty),\rho>0$ and $s\in\mathbb{R}$, and some closed smooth manifold M we let

$$W^{\sigma,\rho}(H^s,H^{-s}) \coloneqq \{T\in B(H^s,H^{-s})\colon T(H^s)\subset A^{\sigma,\rho}(M)\,,\quad T^*(H^s)\subset A^{\sigma,\rho}(M)\}\,,$$

where we wrote H^s for $H^s(M)$ and $B(H^s,H^{-s})$ is the set of bounded linear operators between $H^s(M)$ and $H^{-s}(M)$, and for any $T\in B(H^s,H^{-s})$, $T^*\in B(H^s,H^{-s})$ denotes the formal adjoint of T. It is shown in [20, Thm. 3.16] that $W^{\sigma,\rho}(H^s,H^{-s})$ is a Banach space with the norm

$$\|T\|_{W^{\sigma,\rho}(H^s,H^{-s})} \coloneqq \max\left\{\|T\|_{H^s\to A^{\sigma,\rho}}\,,\|T^*\|_{H^s\to A^{\sigma,\rho}}\right\}\,.$$

We now have all tools in hand to provide the

Proof of Theorem 1.1. We set up notation to apply [20, Thm. 4.2(b)]. For N, F_q from Lemma 2.8, define the operator

$$F\colon K\to B(L^2(N),L^2(N))\,,\quad q\mapsto F_q\,,$$

where $B(L^2(N),L^2(N))$ denotes the set of bounded linear operators between $L^2(N)$ and $L^2(N)$.

As a consequence of Lemma 2.8 (in particular (2.3)) and Lemma 2.9, F maps K into a bounded set in $W^{3,\rho}(H^0,H^0)$ for some $\rho>0$. By an application of [20, Thm. 4.2(b)] we conclude that if ω is a modulus of continuity so that

$$\|q_1-q_2\|_{H^\mu(U)} \le \omega\left(\|F(q_1)-F(q_2)\|_{L^2(N)\to L^2(N)}\right)\,,\qquad q_1,q_2\in K\,, \tag{2.7}$$

then we must have $\omega(s)\gtrsim|\log s|^{-\delta\frac{6n+7}{n}}$ for s small.

Due to (2.4), $\|\mathbb{1}_\Omega(S_{q_1}-S_{q_2})\circ\mathbb{1}_\Omega\|_{L^2(\Omega)\to L^2(\Omega)} \le \|F(q_1)-F(q_2)\|_{L^2(N)\to L^2(N)}$, so that (2.7) completes the proof. □

A. **Existence of Solutions.**

Lemma A.1. *There is a continuous function $b\colon[0,\infty)\to[0,\infty)$ so that for every $q\in L^\infty(U)$ there is a continuous linear operator*

$$S_q\colon L^2(X_+)\to C([-T,T];H_0^1(U))\cap C^1([-T,T];L^2(U))\subset H^1(X)\,,$$

with

$$\|S_q\|_{L^2(X_+)\to H^1(X)} \le b\left(\|q\|_{L^\infty(U)}\right)\,, \tag{A.1}$$

so that in $C([-T,T];H_0^1(U))\cap C^1([-T,T];L^2(U))$ *the unique solution to (1.1) is* $u=S_q(f)$.

Proof. Translating the condition $u|_{t<0}=0$ in (1.1) into vanishing Cauchy data at $t=0$, the uniqueness and existence of the solution $u=u_q^f$ as well as the continuity of S_q are guaranteed by [19, Thm. 2.30] (via extension by 0 to $t<0$ of the solution constructed there).

We turn to finding the explicit norm bound on S_q, assuming at first that $q\in C^1(\bar U)$. One can directly verify that the solution u constructed above is also a weak solution according to the definition given in [11, § 7.2], and by inspection of the proof of [11, § 7.2, Thm. 2] (see also [11, § 7.2, Thm. 5(i)]), one finds the bound in (A.1) for $q\in C^1(\bar U)$. On the other hand, if $q\in L^\infty(U)$ and we let $(q_k)_{k\in\mathbb{N}}\subset C^1(\bar U)$ converge to q in L^∞, by direct verification of the definition of a weak solution (see [19, Eq. (2.66)]), we will see that $S_{q_k}\to S_q$ as operators $L^2(X_+)\to H^1(X)$, completing the proof. □

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