MATS254 Martingale theory — Tutorials 3, 3.2.2014

1. Let $M = (M_n)_{n=0}^{\infty}$ be a martingale with $M_0 := 0$ and $\mathbb{E}M_n^2 < \infty$ for all $n \geq 0$.

Define the *bracket process* $(\langle M \rangle_n)_{n\geq 0}$ by setting $\langle M \rangle_0 := 0$ and

$$\langle M \rangle_n := \sum_{k=1}^{n} \mathbb{E}\big[(M_k - M_{k-1})^2 \bigm| \mathcal{F}_{k-1}\big] \qquad \text{for } n \geq 1.$$

Show that

(a) $(Z_n)_{n\geq 0}$ is a martingale, where $Z_n := M_n^2 - \langle M \rangle_n$.
(b) $\mathbb{E}M_n^2 = \mathbb{E}\langle M \rangle_n$ for all $n \geq 0$.

2. Let τ be a stopping time on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)_{n\geq 0})$.

(a) Show that $\mathcal{F}_\tau = \{A \in \mathcal{F} \, : \, A \cap \{\tau \leq n\} \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}\}$.
(b) Suppose $A \in \mathcal{F}_\tau$. Is it always true that $A \cap \{\tau > n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$?

3. Let ρ and τ be stopping times on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)_{n\geq 0})$.

(a) Show that $\rho + \tau$ is a stopping time.
(b) Show that $\mathcal{F}_{\min\{\rho,\tau\}} = \mathcal{F}_\rho \cap \mathcal{F}_\tau$.

4. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)_{n=0}^{\infty})$ be a filtered probability space.

(a) Assume that $(Z_n)_{n\geq 0}$ is a super-martingale. Prove that there exists a non-decreasing and predictable $(A_n)_{n\geq 0}$ with $A_0 \equiv 0$, such that $M_n := Z_n + A_n$ is a martingale.
(b) Assume $X_1, X_2, \dots$ are independent random variables such that $\mathbb{E}e^{X_k} < 1$. Determine the process A_n in (a) of $(Z_n)_{n\geq 0}$ where $Z_0 := 0$ and $Z_n := \prod_{k=1}^{n} e^{X_k}$.

5. Assume $(Z_n)_{n\geq 0}$ is a positive sub-martingale, that is, $\mathbb{P}(Z_n > 0) = 1$ for all $n \geq 0$. Show that there exists an a.s. unique positive predictable non-increasing process $(B_n)_{n\geq 0}$ with $B_0 \equiv 1$ such that $M_n := Z_n B_n$ is a martingale.

6. Let $X \in L_1(\Omega, \mathcal{F}, \mathbb{P})$ and $Z \in L_1 \in (\Omega, \mathcal{G}, \mathbb{P})$, where $\mathcal{G} \subset \mathcal{F}$, and assume $\mathbb{E}X = \mathbb{E}Z$. Define

$$\mathcal{D} := \left\{ A \in \mathcal{G} \, : \, \int_A Z \mathrm{d}P = \int_A X \mathrm{d}P \right\}.$$

Show that $\mathcal{D}$ is a λ-system:

(a) $\Omega \in \mathcal{D}$.
(b) If $A, B \in \mathcal{D}$ and $B \subset A$, then $A \setminus B \in \mathcal{D}$.
(c) If $A_1, A_2, \dots \in \mathcal{D}$ and $A_1 \subset A_2 \subset \cdots$, then $A := \cup_{n=1}^{\infty} A_n \in \mathcal{D}$.