RIGID CIRCLE DOMAINS WITH NON-REMOVABLE BOUNDARIES

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Dedicated to Jang-Mei Wu.

Abstract. We give a negative answer to the *rigidity conjecture* of He and Schramm by constructing a rigid circle domain Ω on the Riemann sphere $\hat{\mathbb{C}}$ with conformally non-removable boundary. Here rigidity means that every conformal map from Ω onto another circle domain is a Möbius transformation, and non-removability means that there is a homeomorphism of $\hat{\mathbb{C}}$ which is conformal on $\hat{\mathbb{C}} \setminus \partial\Omega$ but not everywhere.

Our construction is based on a theorem of Wu, which states that the product of any Cantor set E with a sufficiently thick Cantor set F is non-removable. We show that one can choose E and F so that the complement of the union of $E \times F$ and suitably placed disks is rigid.

The proof of rigidity involves a metric characterization of conformal maps, which was recently proved by Ntalampekos. The other direction of the rigidity conjecture, i.e., whether removability of the boundary implies rigidity, remains open.

1. Introduction

A subdomain Ω of the Riemann sphere $\hat{\mathbb{C}}$ is a *circle domain* if every connected component of $\partial\Omega$ is a circle or a point. The long-standing *Koebe conjecture* [Koe08] asserts that every subdomain of $\hat{\mathbb{C}}$ admits a conformal map f onto a circle domain. Koebe proved that every *finitely connected* domain satisfies the conjecture and that f is unique up to postcomposition by a Möbius transformation.

Uniqueness is equivalent to *ridigity*: a circle domain Ω is *(conformally) rigid*, if every conformal map $f : \Omega \to \Omega'$ onto another circle domain is the restriction of a Möbius transformation. Complements of Cantor sets K with positive area are basic examples of non-rigid circle domains; solving the *Beltrami equation* (see e.g. [Ahl66], [AIM09]) with coefficient $\mu = \frac{1}{2}\chi_K$ yields a quasiconformal homeomorphism which is conformal only in $\hat{\mathbb{C}} \setminus K$.

In a breakthrough work [HS93], He and Schramm applied the rigidity of *countably connected circle domains* to verify Koebe's conjecture for all countably connected domains. In [HS94], they moreover proved the rigidity of circle domains whose boundary has *σ-finite length.* Further sufficient conditions for rigidity were established by Ntalampekos and Younsi in [NY20], [You16], and [Nta23] (see also [BKM09], [Bon11], [Mer12]).

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Towards a characterization of rigid circle domains Ω, He and Schramm [HS94] pointed out connections to *conformal removability*, and conjectured that the rigidity of Ω is equivalent to the conformal removability of $\partial\Omega$. Here a compact $K \subset \hat{\mathbb{C}}$ is *conformally removable* (or *CH-removable*), if every homeomorphism $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, which is conformal on $\hat{\mathbb{C}} \setminus K$, is a Möbius transformation. Our main result gives a negative answer.

THEOREM 1.1. *There is a rigid circle domain $\Omega \subset \hat{\mathbb{C}}$ such that $\partial\Omega$ is conformally non-removable.*

The proof below shows that the answer to another version of the rigidity conjecture given in [HS94], which asks if rigidity is equivalent to the conformal removability of every Cantor set contained in $\partial\Omega$, is also negative.

It follows from the definitions that if the boundary of a rigid circle domain *is* a Cantor set K, then K is conformally removable. The other direction of the rigidity conjecture, which asks if circle domains with removable boundaries are rigid, remains open even for domains with Cantor set boundaries.

Conformal removability is an active and challenging research topic, see e.g. [JS00], [You15], [Nta19], [Nta24a], [Nta], and the references therein. A major difficulty is that constructing non-trivial conformal maps f outside exceptional sets becomes considerably harder if one also requires the existence of a *homeomorphic* extension of f to $\hat{\mathbb{C}}$.

A basic example of a non-removable set is $K = E \times [0, 1]$ for any Cantor set E: one can apply an essentially 1-dimensional construction, starting with a continuous measure on E, to produce a non-trivial homeomorphism which is conformal off K. Much more involved constructions of non-removable sets were given by Kaufman [Kau84], Bishop [Bis94], and Wu [Wu98], whose result is an important ingredient of the proof of Theorem 1.1. Here cap is the logarithmic capacity, see e.g. [Pom92, Ch. 9].

THEOREM 1.2 ([Wu98]). *Let E and F be two Cantor sets in $\mathbb{R}$. If*

$$\operatorname{cap}([a, b] \setminus F) < \operatorname{cap}([a, b]) \tag{1.1}$$

for some interval $[a, b]$, then $E \times F$ is conformally non-removable.

Although the proof of Theorem 1.2 is subtle, the rough idea is similar to the case $E \times [0, 1]$ above. Namely, by Ahlfors and Beurling [AB50], Condition (1.1) yields a non-trivial conformal embedding $f : \hat{\mathbb{C}} \setminus (\{0\} \times F) \to \hat{\mathbb{C}}$. Such an f cannot admit a continuous extension to $\hat{\mathbb{C}}$. However, given a continuous probability measure μ on E, one can produce a global homeomorphism that is conformal off $E \times F$, by considering averages of the map f with respect to μ in the real variable. Thus, $E \times F$ is non-removable.

To prove Theorem 1.1 we choose a thick Cantor set F satisfying (1.1), a thin Cantor set E, and closed disks D_j. We let $\Omega = \hat{\mathbb{C}} \setminus ((E \times F) \cup (\cup_j D_j))$, and show that the disks D_j can be placed so that every conformal map $f : \Omega \to \Omega'$ between circle domains must have bounded *eccentric distortion.* Therefore we can, after extending f to a homeomorphism of $\hat{\mathbb{C}}$ using a familiar reflection (Schottky group) method, apply a recent result of Ntalampekos [Nta24b] to prove that f is a Möbius transformation. Thus Ω is rigid. But since $\partial\Omega \supset E \times F$, Theorem 1.2 shows that $\partial\Omega$ is conformally non-removable.

The non-removability of $E \times F$ immediately implies the non-rigidity of $\hat{\mathbb{C}} \setminus E \times F$. Thus, our construction shows that, unlike conformal removability, rigidity is not preserved under taking subsets. More precisely, we obtain the following corollary by choosing Ω as above and $\Omega' = \hat{\mathbb{C}} \setminus E \times F$.

Corollary 1.3. *There are circle domains $\Omega, \Omega' \subset \hat{\mathbb{C}}$ so that Ω is rigid and $\partial\Omega' \subset \partial\Omega$, but Ω' is non-rigid.*

2. Proof of Theorem 1.1: Construction of Ω

To start the proof of Theorem 1.1, we construct the complement of a circle domain Ω by using the basic building block in Section 2.1 followed by an iteration procedure in Section 2.2.

2.1. **Basic building block.** We fix an integer $N \geqslant 2$ and let I_*^n, $n \in \{1, \dots, N\}$, be closed segments with equal length $\ell(I_*^n) =: 2s$ obtained by removing $N-1$ open segments of length a from $[0,1]$. The segments are ordered so that I_*^1 contains 0 and I_*^N contains 1. Notice that $s \leqslant \frac{1}{2N}$.

We remove another open segment of length a from the middle of each I_*^n to obtain segments $J_*^n(\mathrm{Do})$ and $J_*^n(\mathrm{Up})$ of equal length $s - \frac{a}{2}$, containing the left, respectively right, endpoint of I_*^n.

Next, we fix $\epsilon > 0$ and define the following subsets of $\mathbb{C}$ for $n \in \{1, \dots, N\}$:

$$\begin{aligned} I^n(\mathrm{Le}) &= -\epsilon + iI_*^n,\ I^n(\mathrm{Ri}) = \epsilon + iI_*^n, \\ J^n(\mathrm{Le}, \mathrm{Do}) &= -\epsilon + iJ_*^n(\mathrm{Do}),\ J^n(\mathrm{Le}, \mathrm{Up}) = -\epsilon + iJ_*^n(\mathrm{Up}), \\ J^n(\mathrm{Ri}, \mathrm{Do}) &= \epsilon + iJ_*^n(\mathrm{Do}),\ J^n(\mathrm{Ri}, \mathrm{Up}) = \epsilon + iJ_*^n(\mathrm{Up}). \end{aligned}$$

We denote by R^n the closed rectangle whose vertical sides are the segments $I^n(\mathrm{Le})$ and $I^n(\mathrm{Ri})$, and by iy^n the center of R^n. Finally, let $D^n(\mathrm{Le})$ and $D^n(\mathrm{Ri})$ be the closed disks with radius s and centers $-2\epsilon - s + iy^n$ and $2\epsilon + s + iy^n$, respectively.

2.2. **Iteration.** Given the sequence of integers N_j defined below, we denote by $\mathcal{N}_k$ the collection of words of length k with letters $n_j \in \{1, \dots, N_j\}$, i.e.,

$$\mathcal{N}_k = \Big\{\tilde{n} = n_1 n_2 \cdots n_k : n_j \in \{1, \dots, N_j\} \text{ for all } j \in \{1, \dots, k\}\Big\}.$$

We also denote $\mathcal{W}_k = \mathcal{M}_k \times \mathcal{N}_k$, where

$$\mathcal{M}_k = \Big\{\tilde{m} = m_1 m_2 \cdots m_k : m_j \in \{\mathrm{Le}, \mathrm{Ri}\} \text{ for all } j \in \{1, \dots, k\}\Big\}.$$

We apply the construction in Section 2.1 with parameters $N = N_1, a = a_1, s = s_1, \epsilon = \epsilon_1$ satisfying

$$N_1 = 2, \quad \epsilon_1 = \frac{a_1}{100} = \frac{s_1}{10^5}. \tag{2.1}$$

We obtain rectangles R^1, R^2, as well as the other sets defined above. We complete the first step of the construction by "duplicating", i.e., if $n_1 \in \{1, 2\}$ we define

$$R^w = \begin{cases} -2 + R^{n_1}, & w = (\mathrm{Le}, n_1) \in \mathcal{W}_1, \\ 2 + R^{n_1}, & w = (\mathrm{Ri}, n_1) \in \mathcal{W}_1, \end{cases}$$

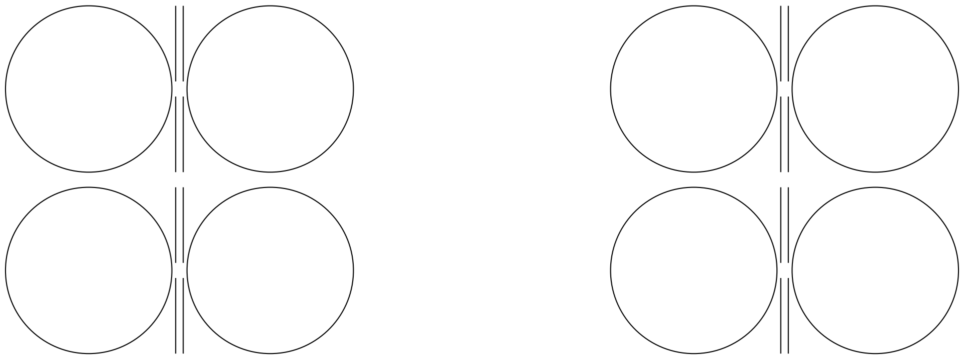

FIGURE 1. First step of the construction.

and use similar notation for the other sets constructed. Altogether, after the first step we have two copies of the sets constructed in Section 2.1; one on the left half-plane and another one on the right half-plane, e.g., $J^{(\mathrm{Le},2)}(\mathrm{Le},\mathrm{Up}) := J^2(\mathrm{Le},\mathrm{Up}) - 2$ and $J^{(\mathrm{Ri},2)}(\mathrm{Le},\mathrm{Up}) := J^2(\mathrm{Le},\mathrm{Up}) + 2$. See Figure 1 for an illustration.

We then assume that rectangles

$$R^w = [x^{\tilde{m}} - t_{k-1}, x^{\tilde{m}} + t_{k-1}] \times [y^{\tilde{n}} - r_{k-1}, y^{\tilde{n}} + r_{k-1}] \tag{2.2}$$

and disks $D^w(\mathrm{Le}), D^w(\mathrm{Ri})$ with radius r_{k-1} and centers

$$x^{\tilde{m}} \pm (2t_{k-1} + r_{k-1}) + iy^{\tilde{n}} \tag{2.3}$$

have been constructed for all $w = (\tilde{m}, \tilde{n}) \in \mathcal{W}_{k-1}$, $k \geqslant 2$. The coordinate $x^{\tilde{m}}$ does not depend on $\tilde{n}$, and the coordinate $y^{\tilde{n}}$ does not depend on $\tilde{m}$. By the first step above we have $r_1 = s_1$, $x^{\mathrm{Le}} = -2$, $x^{\mathrm{Ri}} = 2$, and $t_1 = \epsilon_1$.

Intervals $J^w(\cdot, \mathrm{Do})$ and $J^w(\cdot, \mathrm{Up})$ are obtained by removing an open interval of length δ_{k-1} from $I^w(\cdot)$;

$$\begin{aligned} I^w(\cdot) &= x^{\tilde{m}} \pm t_{k-1} + i[y^{\tilde{n}} - r_{k-1}, y^{\tilde{n}} + r_{k-1}], \\ J^w(\cdot, \mathrm{Do}) &= x^{\tilde{m}} \pm t_{k-1} + i\left[y^{\tilde{n}} - r_{k-1}, y^{\tilde{n}} - \frac{\delta_{k-1}}{2}\right], \\ J^w(\cdot, \mathrm{Up}) &= x^{\tilde{m}} \pm t_{k-1} + i\left[y^{\tilde{n}} + \frac{\delta_{k-1}}{2}, y^{\tilde{n}} + r_{k-1}\right]. \end{aligned} \tag{2.4}$$

We fix such a w. Our goal is to construct the segments, rectangles and disks corresponding to all the "children" $w' = (\tilde{m}m_k, \tilde{n}n_k) \in \mathcal{W}_k$ of w. We denote by ϕ^w_{Do} and ϕ^w_{Up} the homotheties (i.e., maps $z \mapsto \alpha z + \beta$, where $\alpha > 0$ and $\beta \in \mathbb{C}$) for which

$$\phi^w_{\mathrm{Do}}\big(x^{\tilde{m}} + i(y^{\tilde{n}} - r_{k-1})\big) = 0 \quad \text{and} \quad \phi^w_{\mathrm{Do}}\Big(x^{\tilde{m}} + i(y^{\tilde{n}} - \frac{\delta_{k-1}}{2})\Big) = i,$$

$$\phi^w_{\mathrm{Up}}\Big(x^{\tilde{m}} + i(y^{\tilde{n}} + \frac{\delta_{k-1}}{2})\Big) = 0 \quad \text{and} \quad \phi^w_{\mathrm{Up}}\big(x^{\tilde{m}} + i(y^{\tilde{n}} + r_{k-1})\big) = i.$$

Then

$$\begin{aligned}\phi^w_{\mathrm{Do}}(J^w(\mathrm{Le},\mathrm{Do})) &= \phi^w_{\mathrm{Up}}(J^w(\mathrm{Le},\mathrm{Up})) = -\hat{\epsilon}_{k-1} + i[0,1] \quad \text{and}\\ \phi^w_{\mathrm{Do}}(J^w(\mathrm{Ri},\mathrm{Do})) &= \phi^w_{\mathrm{Up}}(J^w(\mathrm{Ri},\mathrm{Up})) = \hat{\epsilon}_{k-1} + i[0,1],\\ \text{where} \quad \hat{\epsilon}_{k-1} &= \frac{t_{k-1}}{r_{k-1} - \frac{\delta_{k-1}}{2}}.\end{aligned} \tag{2.5}$$

Let $N = N_k \in 2\mathbb{N}$ and $a = a_k, s = s_k, \epsilon = \epsilon_k > 0$ be the numbers for which

$$\begin{aligned}N_k &= 200\left\lceil \max\left\{\frac{1}{\hat{\epsilon}_{k-1}}, \frac{s_{k-1}}{a_{k-1}}\right\}\right\rceil, \quad \frac{s_k}{a_k} = \exp((2N_k)^{2k}),\\ \epsilon_k &= \frac{\min\{\hat{\epsilon}_{k-1}, a_k\}}{100}.\end{aligned} \tag{2.6}$$

Remark 2.1. The idea behind the construction of Ω is that we can first choose N_k (number of new rectangles $R^{w'}$) to be as large as we wish, then $\frac{a_k}{s_k}$ (relative size of removed segments) as small as we wish, and finally ϵ_k (relative width) as small as we wish. The requirement on N_k and the upper bound on ϵ_k in terms of $\hat{\epsilon}_{k-1}$ in (2.6) imply that the sets defined below are disjoint subsets of R^w; see Lemma 2.2 below.

The requirement on $\frac{s_k}{a_k}$ guarantees that the holes between the vertical segments are thin. Such thinness will lead to a thick Cantor set F (on the imaginary axis) satisfying Condition (1.1), and to the non-removability of $\partial\Omega$. The upper bound on ϵ_k in terms of a_k will lead to a Cantor set E (on the real axis), which "is thinner than F is thick". The precise meaning of such thinness will be given in terms of conformal modulus estimates in Section 3.3, which will be applied to prove the ridigity of Ω.

Applying the construction in Section 2.1 with the parameters $\frac{N_k}{2}, a_k, s_k, \epsilon_k$ defined in (2.6) yields rectangles and disks

$$R_* := R_*^{\tilde{n}n_k}, \quad D_*(\mathrm{Le}) := D_*^{\tilde{n}n_k}(\mathrm{Le}), \quad D_*(\mathrm{Ri}) := D_*^{\tilde{n}n_k}(\mathrm{Ri}) \tag{2.7}$$

for all $n_k \in \{1, \dots, \frac{N_k}{2}\}$. Finally, given $w' = (\tilde{m}m_k, \tilde{n}n_k)$, we define

$$R^{w'} = (\phi^w_\kappa)^{-1}(R_* + \lambda), \quad D^{w'}(\cdot) = (\phi^w_\kappa)^{-1}(D_*(\cdot) + \lambda), \tag{2.8}$$

for $\cdot = \mathrm{Le}$ or Ri, where

$$\lambda = \begin{cases} \frac{-\hat{\epsilon}_{k-1}}{2} & m_k = \mathrm{Le},\\ \frac{\hat{\epsilon}_{k-1}}{2}, & m_k = \mathrm{Ri},\end{cases}$$

and

$$\kappa = \begin{cases} \mathrm{Do}, & n_k \in \{1, \dots, \frac{N_k}{2}\},\\ \mathrm{Up}, & n_k \in \{\frac{N_k}{2} + 1, \dots, N_k\}.\end{cases}$$

The segments $I^{w'}(\cdot), J^{w'}(\cdot,\cdot)$ are defined in a similar manner. See Figure 2 for an illustration.

These sets can be represented as in (2.2), (2.3), (2.4) above, replacing w by w' and the lengths $t_{k-1}, r_{k-1}, \delta_{k-1}$ by t_k, r_k, δ_k. We record some basic properties for future reference.

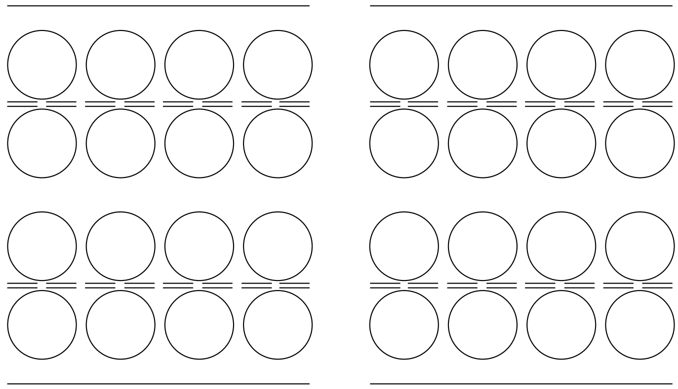

FIGURE 2. The next generation inside a rectangle R^w. The figure is rotated by 90 degrees to improve the layout, and the parameters in the actual construction are different.

Lemma 2.2. *The sets* $R^{w'}$, $D^{w'}(\mathrm{Le})$, *and* $D^{w'}(\mathrm{Ri})$ *are pairwise disjoint subsets of* R^w. *Moreover, if* $k \geqslant 1$ *then*

$$t_k \leqslant \frac{\delta_k}{100} \leqslant \frac{1}{100} \quad \text{and} \quad \delta_k \leqslant \exp(-(2N_k)^{2k}) r_k \leqslant \exp(-(2N_k)^{2k}), \tag{2.9}$$

and if $k \geqslant 2$ *then*

$$N_k \geqslant 20N_{k-1} \geqslant 20N_1 = 40, \quad r_k \leqslant \frac{\delta_{k-1}}{100} \leqslant 1, \quad \text{and} \quad t_k \leqslant \frac{t_{k-1}}{100}. \tag{2.10}$$

Proof. Recall that the rectangles and disks in Section 2.1 are pairwise disjoint. Thus, in view of (2.5) and the definition (2.8) of the sets $R^{w'}$, $D^{w'}(\mathrm{Le})$, and $D^{w'}(\mathrm{Ri})$, the claims concerning them follow if we can show that the sets in (2.7) satisfy

$$R_* \cup D_*(\mathrm{Le}) \cup D_*(\mathrm{Ri}) \subset \left\{ z = x + iy : -\frac{\hat{\epsilon}_{k-1}}{4} < x < \frac{\hat{\epsilon}_{k-1}}{4} \right\} \tag{2.11}$$

for every m_k and n_k. Since the rectangle R_* lies in the "middle" of the two disks, replacing the union on the left side of (2.11) by $D_*(\mathrm{Le}) \cup D_*(\mathrm{Ri})$ results in an equivalent claim. Moreover, since $D_*(\mathrm{Ri})$ lies in the right half-plane and the two disks can be mapped to each other by a reflection across the imaginary axis, it suffices to show that

$$x < \frac{\hat{\epsilon}_{k-1}}{4} \quad \text{for every } x \in D_*(\mathrm{Ri}). \tag{2.12}$$

By the construction of $D_*(\mathrm{Ri})$, we have

$$x < 2(\epsilon_k + s_k) \quad \text{for every } x \in D_*(\mathrm{Ri}).$$

By the last part of (2.6), we have $\epsilon_k \leqslant \hat{\epsilon}_{k-1}/100$. Recall also that we have applied the construction in Section 2.1 with parameter $N = N_k/2$, so that $s_k \leqslant N_k^{-1}$ (see Section 2.1). In particular, by the first part of (2.6) we have $s_k \leqslant \hat{\epsilon}_{k-1}/100$. Combining the estimates yields (2.12).

It remains to prove (2.9) and (2.10). First, notice that $t_k = \frac{\epsilon_k}{a_k}\delta_k$ for every $k \geqslant 1$ by the above construction. Thus the first part of (2.9) follows from the second part of (2.1) and the last part of (2.6).

Next, notice that $\delta_k = \frac{a_k}{s_k} r_k$ and $r_k \leqslant 1$ for every $k \geqslant 1$, again by construction. Thus the second part of (2.9) follows from the second parts of (2.1) and (2.6).

The first part of (2.10) follows by recalling that $N_1 = 2$ and noticing that the first two parts of (2.6) give

$$N_k \geqslant \frac{200 s_{k-1}}{a_{k-1}} = 200\exp((2N_{k-1})^{2(k-1)}) \geqslant 20 N_{k-1}.$$

The second inequality in the second part of (2.10) follows from the construction. For the first inequality, notice that $r_k N_k \leqslant r_{k-1}$ and $\frac{r_{k-1}}{\delta_{k-1}} = \frac{s_{k-1}}{a_{k-1}}$. Combining with the first part of (2.6) yields

$$r_k = \frac{r_k N_k}{N_k} \leqslant \frac{r_{k-1}}{N_k} \leqslant \delta_{k-1}\frac{s_{k-1}}{a_{k-1}N_k} \leqslant \frac{\delta_{k-1}}{100}.$$

The last part of (2.10) follows from the last part of (2.6) by noticing that $\frac{t_k}{t_{k-1}} = \frac{\epsilon_k}{\hat{\epsilon}_{k-1}}$. The proof is complete. □

2.3. **Definition of Ω.** We carry out the above construction for every $w \in \mathcal{W} := \cup_k \mathcal{W}_k$. Lemma 2.2 guarantees that all the disks $D^w(\cdot)$ are pairwise disjoint. We now define the circle domain Ω as follows:

$$\begin{aligned} \hat{\mathbb{C}} \setminus \Omega &= \Big(\bigcup_{w \in \mathcal{W}} (D^w(\text{Le}) \cup D^w(\text{Ri}))\Big) \cup \bigcap_{k=1}^{\infty} \Big(\bigcup_{w \in \mathcal{W}_k} R^w \Big) \\ &= \Big(\bigcup_{w \in \mathcal{W}} (D^w(\text{Le}) \cup D^w(\text{Ri}))\Big) \cup \Big(E \times F \Big). \end{aligned}$$

By Lemma 2.2, $E \times F$ is the product of Cantor sets $E \subset [-3, 3]$ and $F \subset [0, 1]$ which is disjoint from the union of the disks $D^w(\cdot)$.

Lemma 2.3. *The set*

$$E \times F = \bigcap_{k=1}^{\infty} \big(\bigcup_{w \in \mathcal{W}_k} R^w \big)$$

has Lebesgue measure zero.

Proof. Given $k \in \mathbb{N}$, the Cantor set E is covered by 2^k intervals of length $2t_k$. By (2.9) we have $t_1 \leqslant \frac{1}{100}$, and by (2.10) we have $t_{k+1} \leqslant \frac{t_k}{100}$ for all $k \in \mathbb{N}$. Since $2^{k+1}100^{-k} \to 0$ as $k \to \infty$, E has zero length. The claim follows by Fubini's theorem. □

2.4. **Non-removability of ∂Ω.** In this section we apply Theorem 1.2 to show that ∂Ω is conformally non-removable.

THEOREM 2.4. *The boundary of Ω is conformally non-removable.*

Proof. We will prove Condition (1.1) for F and $[a,b]=[0,1]$, i.e.,

$$\operatorname{cap}([0,1]\setminus F)<\operatorname{cap}([0,1]).$$

Theorem 1.2 then gives the desired conclusion. The construction of Ω can in fact be carried out so that $\operatorname{cap}([0,1]\setminus F)$ is smaller than any predetermined $\epsilon>0$, so crude estimates are sufficient below.

Recall the following properties of the logarithmic capacity:

(i) For intervals $[c,d]\subset\mathbb{R}$ we have

$$\operatorname{cap}([c,d])=\frac{d-c}{4} \qquad \text{(see [Pom92, p. 207]).} \tag{2.13}$$

(ii) If E_ℓ are Borel sets and if $E=\cup_{\ell=1}^{\infty}E_\ell$ satisfies $\operatorname{diam}E<\delta$, then

$$\frac{1}{\log\frac{\delta}{\operatorname{cap}(E)}}\leqslant\sum_{\ell=1}^{\infty}\frac{1}{\log\frac{\delta}{\operatorname{cap}(E_\ell)}} \qquad \text{(see [Pom92, Cor. 9.13]).} \tag{2.14}$$

The set $[0,1]\setminus F$ is the union of the removed intervals:

$$[0,1]\setminus F=\Big(\cup_{\ell=1}^{3}\Delta(\ell)\Big)\cup\Big(\bigcup_{k=2}^{\infty}\Big(\bigcup_{\tilde n\in\mathcal{N}_k}\bigcup_{\ell=1}^{2N_k-2}\Delta(\tilde n,\ell)\Big)\Big).$$

Here $\Delta(\ell)$ is the projection to the imaginary axis of one of the three intervals removed from $-2-\epsilon_1+i[0,1]$ in the first step of the construction, and $\Delta(\tilde n,\ell)$ is the projection to the imaginary axis of one of the $2N_k-2$ intervals removed from $I^w(\mathrm{Le})$ in the k:th step, for any $w=(\tilde m,\tilde n)$. The combined cardinality of the segments $\Delta(\tilde n,\ell)$, $\tilde n\in\mathcal{N}_k$, is

$$\#_k=2(N_k-1)\Pi_{j=1}^{k-1}N_j.$$

Since $N_j\leqslant N_k$ for all $j\leqslant k$ by (2.10), it follows that $\#_k\leqslant 2N_k^k$. By (2.9), the length δ_k of such segments is bounded from above by $\exp(-(2N_k)^{2k})$.

We recall that $\operatorname{diam}([0,1]\setminus F)=1$. Since $N_k\geqslant 20N_{k-1}$ and $N_1=2$ by (2.10), we can apply the above estimates together with (2.13) and (2.14) to conclude that

$$\frac{1}{\log\frac{2}{\operatorname{cap}([0,1]\setminus F)}}\leqslant\sum_{k=1}^{\infty}\frac{2N_k^k}{\log(8\exp((2N_k)^{2k})}\leqslant\frac{1}{4}. \tag{2.15}$$

Combining (2.13) and (2.15) yields

$$\operatorname{cap}([0,1]\setminus F)\leqslant 2\exp(-4)<\frac{1}{4}=\operatorname{cap}([0,1]),$$

as desired. The proof is complete. □

3. Proof of Theorem 1.1: Rigidity of Ω

In this section we prove the rigidity of Ω.

THEOREM 3.1. *Domain Ω is rigid; every conformal map $f:\Omega\to\Omega'$ onto another circle domain Ω' is the restriction of a Möbius transformation.*

Theorem 1.1 follows from Theorems 2.4 and 3.1. The key property towards Theorem 3.1 is that we can surround every $z \in E \times F$ with unions of nearby disks and families of paths in Ω of large conformal modulus, see Section 3.3. It follows (see Section 3.4) that there is a sequence of disks $D^w(z)$ whose relative distances to z are small both in the domain and after mapping with any conformal map from Ω onto another circle domain. Combining with the extension procedure in Section 3.5 and Ntalampekos' metric characterization of conformal maps (see Section 3.6) shows that f must be the restriction of a Möbius transformation.

3.1. **Conformal modulus.** The *conformal modulus* $\mathrm{mod}(\Gamma)$ of a family of paths Γ in $\mathbb{C}$ is

$$\mathrm{mod}(\Gamma) = \inf \int_{\mathbb{C}} \rho^2 \, dA,$$

where the infimum is over all *admissible functions*, i.e., Borel functions $\rho : \mathbb{C} \to [0, \infty]$ satisfying $\int_\gamma \rho \, ds \geqslant 1$ for all locally rectifiable $\gamma \in \Gamma$. We will apply the following basic properties, see e.g. [Ahl66, Ch. I]:

Proposition 3.2. (i) *A sense-preserving homemorphism $f : U \to V$ between subdomains of $\hat{\mathbb{C}}$ is conformal if and only if every path family Γ on $U \cap \mathbb{C}$ satisfies*

$$\mathrm{mod}(f\Gamma) = \mathrm{mod}(\Gamma).$$

(ii) *If Γ_1 and Γ_2 are path families so that for every $\gamma_1 \in \Gamma_1$ there is a $\gamma_2 \in \Gamma_2$ which is a restriction of γ_1, then*

$$\mathrm{mod}(\Gamma_2) \geqslant \mathrm{mod}(\Gamma_1).$$

(iii) *If Γ is the family of horizontal (vertical) segments connecting the vertical (horizontal) edges of rectangle $(\zeta, \zeta + t) \times (\xi, \xi + s)$, then*

$$\mathrm{mod}(\Gamma) = \frac{s}{t} \qquad \Big(\mathrm{mod}(\Gamma) = \frac{t}{s}\Big).$$

(iv) *If Γ is the family of circles $S(z_0, s)$, $s_1 < s < s_2$, then*

$$\mathrm{mod}(\Gamma) = \frac{\log \frac{s_2}{s_1}}{2\pi}.$$

(v) *If Λ is the family of paths joining $S(z_0, s_1)$ and $S(z_0, s_2)$, $s_1 < s_2$, then*

$$\mathrm{mod}(\Lambda) = \frac{2\pi}{\log \frac{s_2}{s_1}}.$$

3.2. **Neighboring disks.** We say that $w = (\tilde{m}, \tilde{n} n_k) \in \mathcal{W}_k$ is a *bottom* if $n_k = 1$ or $n_k = \frac{N_k}{2} + 1$, and a *top* if $n_k = \frac{N_k}{2}$ or $n_k = N_k$. Given $l = \pm 1$, we denote

$$w + l = (\tilde{m}, \tilde{n}(n_k + l)).$$

We fix $z \in E \times F$ and $k \in \mathbb{N}$, and let $w = (\tilde{m}, \tilde{n} n_k)$ be the element of $\mathcal{W}_k$ for which $z \in R^w$. The *ordered* collection of the *k:th level neighbors* of z is

$$\mathfrak{N}_k(z) = \{D^{w-1}(\mathrm{Ri}), D^w(\mathrm{Ri}), D^{w+1}(\mathrm{Ri}), D^{w+1}(\mathrm{Le}), D^w(\mathrm{Le}), D^{w-1}(\mathrm{Le})\} \tag{3.1}$$

if w is not a top or a bottom,

$$\mathfrak{N}_k(z) = \{D^w(\text{Ri}), D^{w+1}(\text{Ri}), D^{w+1}(\text{Le}), D^w(\text{Le})\} \tag{3.2}$$

if w is a bottom, and

$$\mathfrak{N}_k(z) = \{D^{w-1}(\text{Ri}), D^w(\text{Ri}), D^w(\text{Le}), D^{w-1}(\text{Le})\}$$

if w is a top. We determine a cyclic ordering: if $\mathfrak{N}_k(z) = \{D_1, \ldots, D_\ell\}$, then

$$D_j < D_{j+1} \text{ for } j \in \{1, \ldots, \ell-1\}, \text{ and } D_\ell < D_1. \tag{3.3}$$

We use the notation $\tau D := \overline{\mathbb{D}}(z_0, \tau r)$ for a disk $D = \overline{\mathbb{D}}(z_0, r)$ and $\tau > 0$.

Lemma 3.3. *If $D \in \mathfrak{N}_k(z)$, then $z \in 4D$.*

Proof. Recall that the height of R^w and radius of each $D \in \mathfrak{N}_k(z)$ are $2r_k$ and r_k, respectively. We assume that w is not a bottom and $D = D^{w-1}(\text{Le})$; the other cases are proved similarly. The width of R^{w-1} (and R^w) and distance between D and R^{w-1} are $2t_k$ and t_k, respectively. Thus, if the center of D is z_0, then the distance between z_0 and the center p_0 of the right vertical edge $I^{w-1}(\text{Ri})$ of R^{w-1} is $r_k + 3t_k$. The distance between p_0 and the top right corner q_0 of R^w is $3r_k + \frac{\delta_k}{2}$. Since $\max\{t_k, \delta_k\} \leqslant \frac{r_k}{100}$ by (2.9), we conclude from the Pythagorean theorem that

$$|z_0 - z| \leqslant |z_0 - q_0| \leqslant \Big((r_k + 3t_k)^2 + (3r_k + \frac{\delta_k}{2})^2\Big)^{1/2} < 4r_k.$$

The proof is complete. □

3.3. **Surrounding path families on Ω.** If $A_1, A_2 \subset \mathbb{C}$ and if $U \subset \mathbb{C}$ is a domain, we say that a path $\gamma : [\alpha, \beta] \to \overline{U}$ *connects A_1 and A_2 in U*, if $\gamma(\alpha) \in A_1$, $\gamma(\beta) \in A_2$, and $\gamma(t) \in U$ for all $\alpha < t < \beta$. In the following, $|\gamma|$ refers to the image of γ. We apply the notation of Section 3.2 for the collection $\mathfrak{N}_k(z)$ of k:th level neighbors.

Proposition 3.4. *For every $z \in E \times F$ and $k \in \mathbb{N}$ there are families Γ_j, $j \in \{1, \ldots, \ell\}$, of paths connecting D_j and $D_{j+1} \in \mathfrak{N}_k(z)$ in $\Omega \setminus \{\infty\}$ so that*

(i) *$\text{mod}(\Gamma_j) \geqslant \frac{1}{10}$ for every $j \in \{1, \ldots, \ell\}$, and*
(ii) *if $\gamma_j \in \Gamma_j$ then $\cup_{j=1}^{\ell}(|\gamma_j| \cup D_j)$ separates z from ∞.*

Proof. Fix $z \in R^w$, where $w = (\tilde{m}, \tilde{n}) \in \mathcal{W}_k$. Suppose first that w is not a top or a bottom, and recall the cyclic order (3.3) of elements in $\mathfrak{N}_k(z)$ (defined in (3.1)). See Figure 3 for an illustration of the path families defined below. By (2.9), the distance δ_k between $D^{w-1}(\text{Ri})$ and $D^w(\text{Ri})$ is less than $\frac{r_k}{100}$, where r_k is their radius. By (2.3), the centers of these disks have the same x-coordinate $x^{\tilde{m}} + 2t_k + r_k$, while the y-coordinates are $y^{\tilde{n}} - 2r_k - \delta_k$ and $y^{\tilde{n}}$, respectively. We thus conclude that every vertical segment connecting the horizontal sides of the square

$$\Big(x^{\tilde{m}} + 2t_k + \frac{9r_k}{10}, x^{\tilde{m}} + 2t_k + \frac{11r_k}{10}\Big) \times \Big(y^{\tilde{n}} - \frac{11r_k + 5\delta_k}{10}, y^{\tilde{n}} - \frac{9r_k + 5\delta_k}{10}\Big)$$

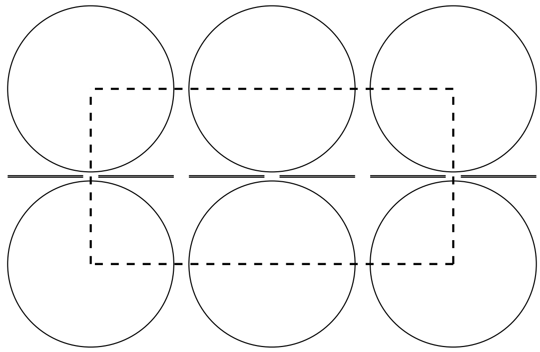

FIGURE 3. The neighboring disks when w is not a bottom or a top. The figure is rotated by 90 degrees to improve the layout, and the surrounding "chain" consisting of the neighboring disks and segments $\gamma_j \in \Gamma_j$ is illustrated by a dashed rectangle.

contains a subsegment joining $D^{w-1}(\mathrm{Ri})$ and $D^{w}(\mathrm{Ri})$ in $\Omega \setminus \{\infty\}$. For the family Γ_1 of such subsegments, Proposition 3.2 (ii) and (iii) yield

$$\mathrm{mod}(\Gamma_1) \geqslant 1 \geqslant \frac{1}{10}. \tag{3.4}$$

We define families Γ_2, Γ_4 and Γ_5 of vertical segments in a similar manner, and apply the argument above to show that (3.4) also holds for such families.

We next define family Γ_3 of horizontal segments connecting $D^{w+1}(\mathrm{Ri})$ and $D^{w+1}(\mathrm{Le})$. We recall from (2.3) that the points minimizing the distance of these disks are

$$x^{\tilde{m}} \pm 2t_k + i(y^{\tilde{n}} + 2r_k + \delta_k) =: x^{\tilde{m}} \pm 2t_k + iy^{+}.$$

In particular, by (2.9) we have that

$$\mathrm{dist}(D^{w+1}(\mathrm{Ri}), D^{w+1}(\mathrm{Le})) = 4t_k \leqslant \frac{\delta_k}{25}.$$

Recalling that the set

$$T^{\tilde{n}+1} := \left\{x + iy : y^{+} - \frac{\delta_k}{2} < y < y^{+} + \frac{\delta_k}{2}\right\}$$

does not intersect $E \times F$, we conclude that every horizontal segment connecting the vertical sides of the square

$$(x^{\tilde{m}} - 10t_k, x^{\tilde{m}} + 10t_k) \times (y^{+} - 10t_k, y^{+} + 10t_k) \subset T^{\tilde{n}+1}$$

contains a subsegment connecting $D^{w+1}(\mathrm{Ri})$ and $D^{w+1}(\mathrm{Le})$ in $\Omega \setminus \{\infty\}$. Applying Proposition 3.2 (ii) and (iii), we see that (3.4) holds for the family Γ_3 of all such subsegments. The same argument applies to Γ_6. We conclude that the desired modulus bounds (i) hold. Condition (ii) follows directly from the definitions of families Γ_j. We have established the proposition when w is not a top or a bottom.

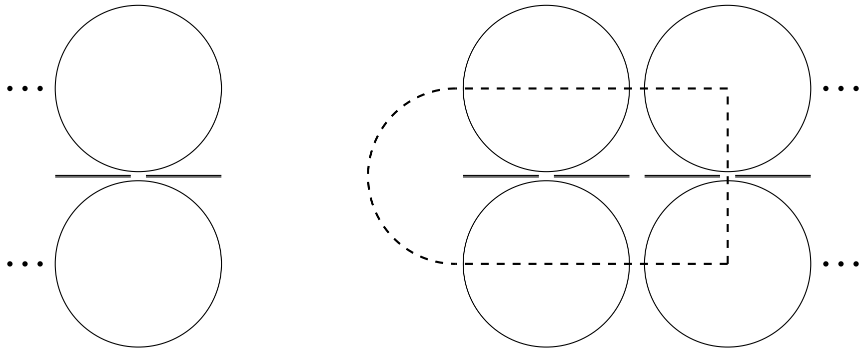

FIGURE 4. The neighboring disks when w is a bottom. The figure is rotated by 90 degrees to improve the layout, and the surrounding "chain" consisting of the neighboring disks and paths $\gamma_j \in \Gamma_j$ is illustrated by a dashed loop.

We now assume that w is a bottom, and recall the cyclic order (3.3) of elements in $\mathfrak{N}_k(z)$ (defined in (3.2)). We can define families Γ_1, Γ_2 of vertical segments and Γ_3 of horizontal segments as above so that (3.4) holds. See Figure 4 for an illustration.

We define the final family Γ_4 of circular arcs as follows: Since $100t_k \leqslant r_k$ by (2.9), for every circle S_r, with radius $\frac{3r_k}{4} < r < \frac{3r_k}{2}$ centered at $x^{\tilde{m}} + i(y^{\tilde{n}} - r_k)$, there is a connected component of $S_r \setminus (D^w(\mathrm{Le}) \cup D^w(\mathrm{Ri}))$ whose closure $\eta(r)$ contains the lower semicircle of S_r and connects $D^w(\mathrm{Le})$ and $D^w(\mathrm{Ri})$. Moreover, since w is a bottom and $100r_k \leqslant \delta_{k-1}$ by (2.10), $\eta(r)$ does not intersect any other complementary components of Ω.

We conclude that each $\eta(r)$ connects $D^w(\mathrm{Le})$ and $D^w(\mathrm{Ri})$ in $\Omega \setminus \{\infty\}$. Applying Proposition 3.2 (ii) and (iv) shows that the family Γ_4 of such arcs satisfies

$$\mathrm{mod}(\Gamma_4) \geqslant \frac{\log 2}{2\pi} \geqslant \frac{1}{10}.$$

We have proved the desired modulus bounds (i) for bottoms. Tops are treated similarly. Condition (ii) follows again from the definitions of families Γ_j. The proof is complete. □

3.4. **Distortion estimate.** Given a domain $G \subset \hat{\mathbb{C}}$, we denote by $\mathcal{C}(G)$ the collection of connected components of $\hat{\mathbb{C}} \setminus G$ and by $\hat{G}$ the quotient space (equipped with the quotient topology) $\hat{\mathbb{C}}/\sim$, where

$$x \sim y \text{ if either } x = y \in G \text{ or } x, y \in p \text{ for some } p \in \mathcal{C}(G).$$

The corresponding quotient map is $\pi_G : \hat{\mathbb{C}} \to \hat{G}$. Identifying each $x \in G$ and $p \in \mathcal{C}(G)$ with $\pi_G(x)$ and $\pi_G(p)$, respectively, we have $\hat{G} = G \cup \mathcal{C}(G)$. A homeomorphism $f : G \to G'$ has a homeomorphic extension $\hat{f} : \hat{G} \to \hat{G}'$; see [NY20, Section 3] for a detailed discussion.

Let $f : \Omega \to \Omega'$ be a conformal map onto a circle domain Ω'. We eventually want to conclude that f is a Möbius transformation. Post-composing f with another Möbius transformation does not affect the conclusion, so we may assume that $f(\infty) = \infty$. Recall the notation $\tau D := \mathbb{D}(z_0, \tau r)$ for a disk $D = \mathbb{D}(z_0, r)$ and $\tau > 0$. We denote the radius of a disk D by $r(D)$.

We continue to apply the notation of Section 3.2 for the collection $\mathfrak{N}_k(z)$ of k:th level neighbors. In the following we abuse the notation and denote the disk $\pi_{\Omega'}^{-1}(\hat{f}(\pi_\Omega(D)))$, $D \in \mathcal{C}(\Omega)$, by $\hat{f}(D)$.

Proposition 3.5. *For every $z \in E \times F$ and $k \in \mathbb{N}$ there is*

$$D_z^k \in \mathfrak{N}_k(z) \quad \textit{so that} \quad z \in 4D_z^k \quad \textit{and} \quad \hat{f}(\{z\}) \subset 10^{30}\hat{f}(D_z^k). \tag{3.5}$$

Proof. Suppose that $z \in R^w$, where $w = (\tilde{m}, \tilde{n}) \in \mathcal{W}_k$. We recall that the first inclusion in (3.5) holds for all $D_j \in \mathfrak{N}_k(z)$ by Lemma 3.3.

Given $j \in \{1, \ldots, \ell\}$, we denote $D_j' = \hat{f}(D_j)$ and $\Gamma_j' = f(\Gamma_j)$. Here Γ_j is the path family in Proposition 3.4, which together with the conformal invariance of modulus (Proposition 3.2 (i)) yields

$$\operatorname{mod}(\Gamma_j') \geqslant \frac{1}{10} \quad \text{for every } j \in \{1, \ldots, \ell\}. \tag{3.6}$$

We now claim that for every $j \in \{1, \ldots, \ell\}$ there is a $\gamma_j' \in \Gamma_j'$ for which

$$\operatorname{diam}(|\gamma_j'|) \leqslant 2 \cdot (\exp(20\pi) + 1) r(D_j') \leqslant 2 \cdot (\exp(20\pi) + 1) r', \tag{3.7}$$

where $r' = \max\Big\{ r(D_j') \,:\, j \in \{1, \ldots, \ell\} \Big\}$. The second inequality is trivial. To prove the first inequality, recall that every $\gamma \in \Gamma_j'$ intersects D_j' by the definition of Γ_j'. If the first inequality in (3.7) fails then every $\gamma \in \Gamma_j'$ also intersects the boundary of $(\exp(20\pi) + 1)D_j'$. Combining (3.6) with Proposition 3.2 (ii) and (v), we see that

$$\frac{1}{10} \leqslant \operatorname{mod}(\Gamma_j') \leqslant \frac{2\pi}{\log(\exp(20\pi) + 1)},$$

which is a contradiction. We have proved (3.7).

Let γ_j', $j \in \{1, \ldots, \ell\}$, be the paths in (3.7). Part (ii) of Proposition 3.4 shows that

$$T := \cup_{j=1}^{\ell} (D_j' \cup |\gamma_j'|) \quad \text{separates } \hat{f}(\{z\}) \text{ from infinity.}$$

In particular, the distance between D_j' and any point $z' \in \hat{f}(\{z\})$ is bounded from above by the diameter of T (we will soon show that z' is unique). Applying (3.7) and the triangle inequality, we conclude that if $D \in \mathfrak{N}_k(z)$ is one of the neighbors satisfying $r(D') = r'$, where $D' = \hat{f}(D)$, then

$$\begin{aligned} \operatorname{dist}(D', z') &\leqslant \operatorname{diam}(T) \leqslant \sum_{j=1}^{\ell} (2r(D_j') + \operatorname{diam}(|\gamma_j'|)) \\ &\leqslant 12 \cdot (2 + \exp(20\pi)) r' \leqslant 10^{29} r'; \end{aligned}$$

recall that the cardinality ℓ of $\mathfrak{N}_k(z)$ is at most 6. We conclude that also the second inclusion in (3.5) holds if $D_z^k = D'$. The proof is complete. □

3.5. **Global extension of f.** We continue to investigate a conformal map $f : \Omega \to \Omega'$ onto a circle domain Ω' satisfying $f(\infty) = \infty$. We first prove that f extends to a homeomorphism between the closures of Ω and Ω'.

Lemma 3.6. *The map f extends to a homeomorphism $\tilde{f} : \overline{\Omega} \to \overline{\Omega'}$.*

Proof. Since the disk components D of $\hat{\mathbb{C}} \setminus \Omega$ are isolated in $\pi_\Omega(\mathcal{C}(\Omega))$, Carathéodory's theorem shows that f has a homeomorphic extension $\partial D \to \partial \hat{f}(D)$. Hence it suffices to show that $\operatorname{diam}(\hat{f}(\{z\})) = 0$ for every $z \in E \times F$.

Notice that the preimage of $\hat{f}(\mathcal{C}(\Omega))$ under $\pi_{\Omega'}^{-1}$ is bounded in $\mathbb{C}$ since $\infty \in \Omega'$. Consequently, the sequence of disks D_z^k in Proposition 3.5 satisfies $r(\hat{f}(D_z^k)) \to 0$ as $z \to \infty$. Combining with (3.5), we have

$$\operatorname{diam}(\hat{f}(\{z\})) \leqslant 2 \cdot 10^{30} \lim_{k \to \infty} r(\hat{f}(D_z^k)) = 0.$$

The proof is complete. □

We next apply Lemma 3.6 and repeated reflections across the boundary circles of Ω and Ω' to extend f to all of $\hat{\mathbb{C}}$. The method has been applied to prove rigidity of circle domains and Schottky sets e.g. in [HS94], [BKM09], [You16], [NY20], [NR23]. We refer to [NY20, Section 7.1] for the details of the following construction and the proof of Proposition 3.7 below.

As before, we assume that $f(\infty) = \infty$. We denote the collection of non-degenerate elements (i.e., disks) of $\mathcal{C}(\Omega)$ by $\mathcal{D}$. Given $D \in \mathcal{D}$, let R_D be the reflection across the circle ∂D. The *Schottky group* $\mathcal{S}(\Omega)$ is the free discrete group generated by $\{R_D : D \in \mathcal{D}\}$. Every non-identity element g of $\mathcal{S}(\Omega)$ can be uniquely written as

$$g = R_{D_1} \circ \cdots \circ R_{D_\ell}, \quad \text{where } D_{j+1} \neq D_j \text{ for all } 1 \leqslant j \leqslant \ell - 1. \tag{3.8}$$

Denoting $D' = \hat{f}(D)$, the map f admits a conformal extension

$$f_* : \Omega \cup \bigcup_{D \in \mathcal{D}} (R_D(\Omega) \cup \partial D) \to \Omega' \cup \bigcup_{D \in \mathcal{D}} (R_{D'}(\Omega') \cup \partial D') :$$

we set $f_*(z) = (R_{D'} \circ f \circ R_D)(z)$ for $z \in R_D(\Omega)$ and apply Lemma 3.6 and the Schwarz reflection principle to extend f_* across the boundary circle ∂D. Continuing inductively, we see that f_* can be further extended to the union Ω_* of sets $g(\Omega) \cup g(\partial D_\ell)$, $g \in \mathcal{S}(\Omega)$. Here D_ℓ is the disk in (3.8).

Thus, we have a conformal homeomorphism $f_* : \Omega_* \to \Omega'_*$, where Ω'_* is defined as Ω_* but using elements $g' = \mathcal{S}(\Omega')$ instead of elements $g = \mathcal{S}(\Omega)$. The boundary satisfies $\partial \Omega_* = \hat{\mathbb{C}} \setminus \Omega_* = X \cup Y$, where

$$X = \bigcup_{g \in \mathcal{S}(\Omega)} g(E \times F), \tag{3.9}$$

and for every $z \in Y$ there are disks $D_j \in \mathcal{D}$ so that if $B_0 = D_0$ then

$$\{z\} = \cap_{j=0}^{\infty} B_j, \quad \text{where} \quad B_{j+1} = R_{D_1} \circ \cdots \circ R_{D_j}(D_{j+1}) \subset B_j. \tag{3.10}$$

Given $z \in Y$, the intersection of the disks B_j' bounded by circles $f_*(\partial B_j)$ is a point $z' \in \partial\Omega_*'$. Applying Lemma 3.6 to X and setting $f_*(z) = z'$ for $z \in Y$ shows that f_* has a homeomorphic extension to $\hat{\mathbb{C}}$, see [NY20, Lemma 7.5]. The following proposition summarizes our discussion.

Proposition 3.7. *The map $f_* : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a homeomorphism, and conformal on $\Omega_* = \hat{\mathbb{C}} \setminus (X \cup Y)$. Here X and Y satisfy* (3.9) *and* (3.10)*, respectively.*

3.6. **Eccentric distortion and conformality.** The final step in the proof of the rigidity of Ω is showing that the homeomorphic extension $f_* : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of f (see Proposition 3.7) is conformal on $\hat{\mathbb{C}}$, and therefore a Möbius transformation. We now show how conformality follows from Proposition 3.5 and a recent characterization of conformal maps by Ntalampekos [Nta24b].

Recall that a sense-preserving homemorphism $h : G \to G'$ between subdomains of $\hat{\mathbb{C}}$ is *K-quasiconformal*, $K \geqslant 1$, if the conformal modulus of every path family Γ in $G \cap \mathbb{C}$ satisfies

$$\frac{1}{K}\operatorname{mod}(\Gamma) \leqslant \operatorname{mod}(h\Gamma) \leqslant K \operatorname{mod}(\Gamma). \tag{3.11}$$

By Proposition 3.2 (i), 1-quasiconformality is equivalent with conformality.

The classical *metric definition of quasiconformality* (see e.g. [Väi71, Ch. 4]) is given in terms of the *metric distortion*, which at a point z_0 measures the distortion of images under h of small disks centered at z_0. We apply a more flexible notion of metric distortion which was recently introduced by Ntalampekos: We say that the *eccentricity* of a bounded open set $A \subset \mathbb{C}$ is

$$E(A) = \inf\{M \geqslant 1 : \text{there exists an open disk } B \text{ such that } B \subset A \subset MB\}.$$

The *eccentric distortion* of a topological embedding $h : G \to \hat{\mathbb{C}}$ of an open $G \subset \hat{\mathbb{C}}$ at $z_0 \in G \setminus (\{\infty, h^{-1}(\infty)\})$ is

$$\begin{aligned} E_h(z_0) \quad = \quad & \inf\{M \geqslant 1 : \text{there exists a sequence of open sets } A_k \subset G, \\ & k \in \mathbb{N}, \text{ with } z_0 \in A_k,\ k \in \mathbb{N}, \text{ and } \operatorname{diam}(A_k) \to 0 \text{ as } k \to \infty \\ & \text{such that } E(A_k) \leqslant M \text{ and } E(h(A_k)) \leqslant M \text{ for all } k \in \mathbb{N}\}. \end{aligned}$$

The definition can be extended to $\{\infty, h^{-1}(\infty)\}$ by composing h with Möbius transformations.

THEOREM 3.8. *Let $G \subset \hat{\mathbb{C}}$ be open and $h : G \to \hat{\mathbb{C}}$ a sense-preserving topological embedding. If there is $H \geqslant 1$ so that*

$$E_h(z_0) \leqslant H \quad \textit{for all } z_0 \in G, \tag{3.12}$$

then h is quasiconformal on G. If in addition to (3.12) *also $E_h(z_0) = 1$ for almost every $z_0 \in G$, then h is conformal on G.*

Proof. The first claim is [Nta24b, Theorem 1.2], and the second claim is [Nta23, Lemma 2.5]. □

We can now finish the proof of Theorem 3.1. We need to prove that the map f_* in Proposition 3.7 is conformal on $\hat{\mathbb{C}}$. Conformality of f_* in Ω_* shows that $E_{f_*}(z_0) = 1$ for every $z_0 \in \Omega_*$. We also have $E_{f_*}(z_0) = 1$ for every $z_0 \in Y$, since we can apply the interiors of the disks B_j in (3.10) to test the definition of the eccentric distortion.

Finally, we claim that

$$E_{f_*}(z_0) \leqslant 10^{30} \quad \text{for every } z_0 \in X. \tag{3.13}$$

If $z_0 \in E \times F$, we can test the definition of eccentric distortion with arbitrarily small open neighborhoods of the unions of z_0 and the neighbors D_z^k in Proposition 3.5 to prove (3.13).

If $g \in \mathcal{S}(\Omega)$ is a non-identity element, then every $z_0 \in g(E \times F)$ has an open neighborhood U so that

$$f_*(z) = (g' \circ f_* \circ g^{-1})(z) \quad \text{for all } z \in U,$$

where $g' \in \mathcal{S}(\Omega')$. The eccentric distortion is not affected by compositions with g' and g^{-1}, because they are (anti)conformal. Since $E_{f_*}(g^{-1}(z_0)) \leqslant 10^{30}$ by the previous paragraph, we conclude that (3.13) holds for all $z_0 \in X$.

We have proved that the eccentric distortion of f_* is bounded everywhere and equal to one on $\hat{\mathbb{C}} \setminus X$. The Lebesgue measure of X is zero by Lemma 2.3. Thus, we can apply Theorem 3.8 to conclude that f_* is conformal. The proofs of Theorem 3.1 and Theorem 1.1 are complete.

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