UNIFORMIZATION OF COFAT DOMAINS ON METRIC TWO-SPHERES

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Abstract. We extend *Schramm's cofat uniformization theorem* to cofat domains on upper Ahlfors 2-regular metric two-spheres X. Specifically, we show that if $\Omega \subset X$ is a cofat domain, then there exists a $\frac{\pi}{2}$-quasiconformal homeomorphism $f : \Omega \to D$ onto a circle domain $D \subset \mathbb{S}^2$. Moreover, f preserves the point-components and non-trivial complementary components. We also construct examples which show that the above conclusions are not true for countably connected ℓ^α-subdomains of $\mathbb{S}^2$.

1. Introduction

The Riemann mapping theorem asserts that every simply connected proper subdomain of the complex plane is conformally equivalent to the unit disk $\mathbb{D}$. For multiply connected domains, Koebe [20] conjectured that every domain in the Riemann sphere $\hat{\mathbb{C}}$ is conformally equivalent to a *circle domain*, i.e., a domain whose boundary components are points and circles.

Koebe himself proved the conjecture for finitely connected domains [21, 22], while He and Schramm [13] established it for countably connected domains. Subsequently, in [47], Schramm introduced a powerful tool for studying the uniformization of multiply connected domains—the transboundary extremal length (or *the transboundary modulus*). He also proved *the cofat uniformization theorem*, which proves the Koebe's conjecture under geometric conditions; see below. Recently, Karafyllia and Ntalampekos [18] gave an affirmative answer to Koebe's conjecture for the class of Gromov hyperbolic domains. The general case for uncountably connected domains remains an open problem. See also [24, 42, 52] for recent works on the Koebe's conjecture.

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In this paper we extend *Schramm's cofat uniformization theorem* to cofat domains on upper Ahlfors 2-regular metric two-spheres.

We recall some necessary notation. A metric space X is called a *metric two-sphere* if X is homeomorphic to $\mathbb{S}^2$ and has finite two-dimensional Hausdorff measure $\mathcal{H}^2$. Given a domain $\Omega \subset X$, we call a connected component p of $X \setminus \Omega$ *non-trivial* and denote $p \in \mathcal{C}_N(\Omega)$ if $\operatorname{diam}(p) > 0$. Otherwise we call p a *point-component* and denote $p \in \mathcal{C}_P(\Omega)$.

Let $\hat{\Omega} := X/\sim$, where $z \sim w$ if either $z = w \in \Omega$ or $z, w \in p$ for some $p \in \mathcal{C}(\Omega) := \mathcal{C}_N(\Omega) \cup \mathcal{C}_P(\Omega)$. We denote the induced quotient map by $\pi_\Omega : X \to \hat{\Omega}$. Every homeomorphism $f : \Omega \to D$ between subdomains of metric two-spheres can be extended to a homeomorphism $\hat{f} : \hat{\Omega} \to \hat{D}$ between their quotient spaces. We make no distinction between $p \in \mathcal{C}(\Omega)$ and $\pi_\Omega(p) \in \hat{\Omega}$.

Let X be a metric two-sphere. We call $A \subset X$ *τ-fat* if for every $x_0 \in A$ and every open ball $B(x_0, r) \subset X$ that does not contain A, we have

$$\mathcal{H}^2(A \cap B(x_0, r)) \geqslant \tau r^2.$$

A domain $\Omega \subset X$ is *τ-cofat* if there exists $\tau > 0$ such that every $p \in \mathcal{C}_N(\Omega)$ is a τ-fat set. A domain $D \subset \mathbb{S}^2$ is called *a circle domain* if each $p \in \mathcal{C}_N(D)$ is a disk. We now state our main result.

Theorem 1.1. *Suppose that X is an upper Ahlfors 2-regular metric two-sphere. If $\Omega \subset X$ is a cofat domain, then there exists a $\frac{\pi}{2}$-quasiconformal homeomorphism $f \colon \Omega \to D$ onto a circle domain $D \subset \mathbb{S}^2$. Moreover,*

$$\hat{f}(\mathcal{C}_N(\Omega)) = \mathcal{C}_N(D) \quad \textit{and} \quad \hat{f}(\mathcal{C}_P(\Omega)) = \mathcal{C}_P(D). \tag{1}$$

The upper 2-regularity and quasiconformality are defined in (2) and Definition 2.2, respectively. Schramm [47] proved Theorem 1.1 for $X = \mathbb{S}^2$ and a conformal f. Compactifications of $\mathbb{R}^2$ equipped with the ℓ^∞-norm show that the constant $\frac{\pi}{2}$ is best possible (see [41, 46]).

If X is Ahlfors 2-regular and *linearly locally connected* (see e.g. [14, Theorem 8.23]), then Theorem 1.1 can be proved by applying Schramm's theorem (see the previous paragraph) and the quasisymmetric uniformization of Bonk and Kleiner [3, Theorem 1.1].

We now give some background. Non-smooth versions of the uniformization theorem have found applications in various areas of mathematics, such as the calculus of variations, metric geometry, complex dynamics, and geometric group theory (see e.g. [1, 2, 4, 5, 11, 19]).

A fundamental result by Bonk and Kleiner [3] shows that an Ahlfors 2-regular metric surface admits a *quasisymmetric* (see e.g. [14, Chapter

10]) homeomorphism onto the standard $\mathbb{S}^2$ if and only if it is linearly locally connected. The more recent series of works by Lytchak and Wenger [25, 26, 27, 28, 29, 30, 31] establishes a connection between Plateau's problem and non-smooth uniformization.

Bonk, Merenkov, and Ntalampekos [2, 5, 35] further developed the theory by studying the quasisymmetric uniformization of Sierpiński carpets. Meanwhile, Merenkov and Wildrick [34] proved a quasisymmetric version of Koebe's uniformization theorem. See also [6, 10, 12, 16, 17, 40, 43, 44, 45, 50] for related results.

Quasisymmetric mappings have globally controlled distortion, and they can only exist on spaces whose geometries are suitably controlled. Towards a more general uniformization theory, Rajala [41] established a necessary and sufficient condition for a metric surface to admit a *quasiconformal* uniformization. Subsequently, Ntalampekos and Romney [38, 39] and Meier and Wenger [32, 33] (for a locally geodesic X) found *weakly quasiconformal parametrizations* for *all metric surfaces*.

The purpose of our work is to initiate the study of quasiconformal uniformization for multiply connected subdomains of metric surfaces. Towards future research, we propose the following problem.

Problem 1. Can we replace the cofatness in Theorem 1.1 by a more general condition?

One approach to Problem 1 is to consider summability conditions on the diameters of the complementary components. As a byproduct of the proof of Theorem 1.1, we obtain the following result which is of independent interest even for $X = \mathbb{S}^2$.

Given $\alpha > 0$, let $\mathcal{F}_\alpha(X)$ be the collection of *ℓ^α-domains* on X, i.e., the *countably connected* subdomains Ω_α of X satisfying

$$\sum_{p \in \mathcal{C}(\Omega_\alpha)} \operatorname{diam}(p)^\alpha < \infty.$$

Theorem 1.2. *Suppose that X is a metric two-sphere. If $\Omega_2 \in \mathcal{F}_2(X)$, then every conformal homeomorphism $f : \Omega_2 \to D$ onto a circle domain $D \subset \mathbb{S}^2$ satisfies $\hat{f}(\mathcal{C}_N(\Omega_2)) \subset \mathcal{C}_N(D)$.*

See Remark 6.3 for how to extract the proof of Theorem 1.2 from the proof of Theorem 1.1. Theorem 1.2 is no longer true if the countable connectedness assumption is removed from the definition of $\mathcal{F}_2(X)$, even when $X = \mathbb{S}^2$, see [9, Theorem 4.1].

Our next result shows that the exponent $\alpha = 2$ in Theorem 1.2 is sharp even when $X = \mathbb{S}^2$.

Theorem 1.3. *For every $\alpha > 2$ there is a domain $U_\alpha \in \mathcal{F}_\alpha(\mathbb{S}^2)$, such that the closed unit disk $\overline{\mathbb{D}} \in \mathcal{C}_N(U_\alpha)$ but $\hat{f}(\overline{\mathbb{D}}) \in \mathcal{C}_P(D)$ for every conformal homeomorphism $f : U_\alpha \to D$ onto a circle domain $D \subset \mathbb{S}^2$.*

Finally, we demonstrate the failure of the second half of Condition (1) in any $\mathcal{F}_\alpha(\mathbb{S}^2)$.

Theorem 1.4. *For every $\alpha > 0$ there is a domain $\Omega_\alpha \in \mathcal{F}_\alpha(\mathbb{S}^2)$ such that $\{0\} \in \mathcal{C}_P(\Omega_\alpha)$ but $\hat{f}(\{0\}) \in \mathcal{C}_N(D)$ for every conformal homeomorphism $f : \Omega_\alpha \to D$ onto a circle domain $D \subset \mathbb{S}^2$.*

Theorem 1.4 generalizes a construction of Ntalampekos [37, Section 6], which covers the case $\alpha = 2$.

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2. Preliminaries

We denote the open ball in a metric space X with center $x_0 \in X$ and radius $r > 0$ by $B(x_0, r)$, and $S(x_0, r) := \{x \in X : d(x_0, x) = r\}$. The metric on X is denoted by d, and we equip the standard $\mathbb{S}^2$ with the spherical metric $d_{\mathbb{S}^2}$. We call X *Ahlfors 2-regular* if there is an $\alpha \geqslant 1$ such that

$$\alpha^{-1} R^2 \leqslant \mathcal{H}^2(B(x, R)) \leqslant \alpha R^2 \tag{2}$$

for all $x \in X$ and $0 < R < \operatorname{diam}(X)$, and *upper Ahlfors 2-regular* if the second inequality holds in (2).

2.1. Conformal Modulus.

We recall the conformal modulus of curve families and give the geometric definition of quasiconformal mappings.

Definition 2.1 (Conformal modulus)**.** Let $\Gamma \subset X$ be a family of curves in a metric two-sphere X. The conformal modulus of Γ is

$$\operatorname{mod}_{\mathrm{C}}(\Gamma) = \inf_{\rho \in Adm_{\mathrm{C}}(\Gamma)} \int_X \rho^2 \, d\mathcal{H}^2,$$

where $Adm_{\mathrm{C}}(\Gamma)$ is the collection of admissible functions for Γ, i.e., all Borel functions $\rho : X \to [0, \infty]$ satisfying

$$\int_\gamma \rho \, ds \geqslant 1 \qquad \text{for every locally rectifiable curve } \gamma \in \Gamma.$$

We now give the *geometric definition of quasiconformality*. We assume that X and Y are metric two-spheres.

Definition 2.2. A homeomorphism $f : \Omega \to D$ between domains $\Omega \subset X$ and $D \subset Y$ is K*-quasiconformal*, or $K - QC$, if

$$K^{-1} \operatorname{mod}_{\mathrm{C}}(\Gamma) \leqslant \operatorname{mod}_{\mathrm{C}}(f\Gamma) \leqslant K \operatorname{mod}_{\mathrm{C}}(\Gamma)$$

for every curve family Γ in Ω. Here $f\Gamma := \{f \circ \gamma : \gamma \in \Gamma\}$.

We can also define quasiconformal mappings between metric spaces via the so-called metric definition (see [15, Section 14]). In the setting of this paper, the geometric definition is more natural due to the uniformization theory which has recently been developed using modulus.

We will apply the following uniformization result by Rajala [41, Theorem 1.4-1.6] to establish the existence of quasiconformal mappings in Theorem 1.1. The sharp constant $\frac{\pi}{2}$ was proved by Romney [46].

Theorem 2.3. *Suppose X is an upper Ahlfors 2-regular metric two-sphere. Then there exists a $\frac{\pi}{2} - QC$ mapping $h : X \to \mathbb{S}^2$.*

We also need the stability of quasiconformal mappings under locally uniform convergence. See [23, Theorem 7.1] for a proof of the following proposition.

Proposition 2.4. *Let $f_j : \Omega \to D_j \subset \mathbb{S}^2$ be $K - QC$ for each $j \geqslant 1$, and suppose that f_j converges to $f : \Omega \to D \subset \mathbb{S}^2$ locally uniformly. If f is a homeomorphism, then f is $K - QC$.*

2.2. **Transboundary Modulus.** We will apply Schramm's transboundary modulus [47]. The decomposition space $\hat{\Omega}$ associated to a domain $\Omega \subset X$ is defined in the introduction before Theorem 1.1.

Definition 2.5. Let X be a metric two-sphere and fix a domain $\Omega \subset X$. The *transboundary modulus* $\operatorname{mod}_{\mathrm{T}}(\Gamma)$ of a family Γ of curves in $\hat{\Omega}$ is

$$\operatorname{mod}_{\mathrm{T}}(\Gamma) = \inf_{\rho \in Adm_{\mathrm{T}}(\Gamma)} \int\limits_{\Omega} \rho^2 \, d\mathcal{H}^2 + \sum_{p \in \mathcal{C}(\Omega)} \rho(p)^2,$$

where $Adm_{\mathrm{T}}(\Gamma)$ is the collection of *admissible functions for* Γ, i.e., Borel measurable functions $\rho\colon \hat{\Omega} \to [0, \infty]$ for which

$$1 \leqslant \int\limits_{\gamma} \rho \, ds + \sum_{p \in \mathcal{C}(\Omega) \cap |\gamma|} \rho(p) \quad \text{for all } \gamma \in \Gamma.$$

Here $|\gamma|$ denotes the trajectory of the curve γ and $\int_\gamma \rho \, ds$ is the curve integral of the restriction of γ to Ω. More precisely, the restriction is a countable union of disjoint curves γ_j, each of which maps onto a component of $|\gamma| \setminus \mathcal{C}(\Omega)$, and we can define

$$\int_\gamma \rho\, ds = \sum_j \int_{\gamma_j} \rho\, ds.$$

We will apply the quasi-invariance of the transboundary modulus under quasiconformal mappings between metric surfaces.

Theorem 2.6. *Let Ω and D be subdomains of metric surfaces X and Y, respectively. If $f : \Omega \to D$ is a $K-QC$ mapping, then for every curve family Γ in $\hat{\Omega}$, we have*

$$K^{-1} \operatorname{mod}_{\mathrm{T}}(\Gamma) \leqslant \operatorname{mod}_{\mathrm{T}}(f\Gamma) \leqslant K \operatorname{mod}_{\mathrm{T}}(\Gamma).$$

The proof of Theorem 2.6 involves notions from the Sobolev theory in metric measure spaces, such as upper gradients. Since we will not apply such notions elsewhere in this paper, we do not give the definitions here. Instead, we refer to [15, 51] for the definitions and basic properties.

Proof. Let $\rho \in Adm_{\mathrm{T}}(\Gamma)$. By Definition 2.2, $f^{-1} : D \to \Omega$ is also $K-QC$. It then follows from Williams' theorem [51, Theorem 1.1] that f^{-1} belongs to the Newtonian Sobolev space $N^{1,2}_{loc}(D, \Omega)$. Moreover, for $\mathcal{H}^2$-almost every $z \in D$, we have

$$g_{f^{-1}}(z)^2 \leqslant K \cdot J_{f^{-1}}(z). \tag{3}$$

Here $g_{f^{-1}}$ and $J_{f^{-1}}$ denote the minimal weak upper gradient and the volume derivative of f^{-1}, respectively.

Then we can define $\tilde{\rho}(z) = \rho(f^{-1}(z)) \cdot g_{f^{-1}}(z)$ if $z \in D$ and $\tilde{\rho}(p) = \rho(\hat{f}^{-1}(p))$ if $p \in \mathcal{C}(D)$. We claim that $\tilde{\rho} : \hat{D} \to [0, \infty]$ is admissible for $\hat{f}\Gamma$. Indeed, for almost every $\widetilde{\gamma} := \hat{f} \circ \gamma \in \hat{f}\Gamma$, we have

$$\begin{aligned} \int_{\widetilde{\gamma}} \tilde{\rho}\, ds + \sum_{p \in \mathcal{C}(D) \cap |\widetilde{\gamma}|} \tilde{\rho}(p) &= \int_{\widetilde{\gamma}} \rho\left(f^{-1}(z)\right) \cdot g_{f^{-1}}(z)\, ds + \sum_{\bar{p} \in \mathcal{C}(\Omega) \cap |\gamma|} \rho(\bar{p}) \\ &\geqslant \int_\gamma \rho(u)\, ds + \sum_{\bar{p} \in \mathcal{C}(\Omega) \cap |\gamma|} \rho(\bar{p}) \\ &\geqslant 1. \end{aligned}$$

Here the first inequality holds since $g_{f^{-1}}$ is a weak upper gradient of f^{-1}. Combining with (3), we can estimate $\operatorname{mod}_{\mathrm{T}}(\hat{f}\Gamma)$:

$$\operatorname{mod}_{\mathrm{T}}(\hat{f}\Gamma) \leqslant \int_D (\tilde{\rho})^2\, d\mathcal{H}^2 + \sum_{p \in \mathcal{C}(D)} \tilde{\rho}(p)^2 \leqslant K \Big[\int_\Omega \rho^2\, d\mathcal{H}^2 + \sum_{p \in \mathcal{C}(\Omega)} \rho(p)^2 \Big]$$

for all $\rho \in Adm_{\mathrm{T}}(\Gamma)$, so we have $\operatorname{mod}_{\mathrm{T}}(\hat{f}\Gamma) \leqslant K \operatorname{mod}_{\mathrm{T}}(\Gamma)$. The reverse inequality $\operatorname{mod}_{\mathrm{T}}(\Gamma) \leqslant K \operatorname{mod}_{\mathrm{T}}(\hat{f}\Gamma)$ is proved in a similar way. $\square$

3. Approximation with finitely connected domains

In this section we start the proof of Theorem 1.1. We use the following approximation scheme.

Fix a cofat domain $\Omega \subset X$ on an upper Ahlfors 2-regular metric two-sphere. By Theorem 2.3 there exists a $\frac{\pi}{2} - QC$ homeomorphism $h : \Omega \to \Omega'$ onto a domain $\Omega' := h(\Omega) \subset \mathbb{S}^2$.

In order to prove Theorem 1.1, we may assume that $\operatorname{card} \mathcal{C}_N(\Omega) = \infty$, since otherwise the existence of a conformal homeomorphism $g : \Omega' \to D$ onto a circle domain $D \subset \mathbb{S}^2$ follows from Koebe's theorem [2, Theorem 9.5], so we get a $\frac{\pi}{2} - QC$ mapping $f := g \circ h : \Omega \to D$; notice that Definition 2.2 yields that the composition of a $K - QC$ mapping and a conformal mapping is $K - QC$.

Since Ω is cofat, each $p \in C_N(\Omega)$ has positive Hausdorff 2-measure. In particular, the collection $C_N(\Omega)$ is finite or countable, and we can enumerate $C_N(\Omega) = \{p_0, p_1, p_2, \ldots\}$. Let $\Omega_j := X \setminus \{p_0, p_1, \ldots, p_j\}$ be a sequence of finitely connected domains, so that $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega$.

In order to prove Theorem 1.1, we may assume that

$$\widetilde{\Omega} := X \setminus \overline{\bigcup_{p \in \mathcal{C}_N(\Omega)} p} = \Omega.$$

Indeed, if we can prove the existence of a homeomorphism $f : \tilde{\Omega} \to D$ as in Theorem 1.1, then also the restriction of f to Ω satisfies the conditions in Theorem 1.1.

By Theorem 2.3 we know that there are $\frac{\pi}{2} - QC$ mappings $h_j : \Omega_j \to \Omega'_j$, such that each $\Omega'_j \subset \mathbb{S}^2$ is a finitely connected domain on $\mathbb{S}^2$. Now by Koebe's theorem, we can find conformal homeomorphisms $g_j : \Omega'_j \to D_j \subset \mathbb{S}^2$ onto circle domains $D_j \subset \mathbb{S}^2$. Moreover,

$$q_{j,\ell} := \hat{g}_j(\hat{h}_j(p_\ell))$$

is a disk for all $\ell = 0, 1, \ldots, j$; here $\hat{h}_j : \hat{\Omega}_j \to \hat{\Omega}'_j$ and $\hat{g}_j : \hat{\Omega}'_j \to \hat{D}_j$ are the unique homeomorphic extensions of h_j and g_j. By postcomposing with Möbius transformations on $\mathbb{S}^2$, we may assume that

$$q_{j,0} = \mathbb{S}^2 \setminus \mathbb{D}(0,1) \quad \text{for all } j = 1, 2, \ldots. \tag{4}$$

We get a sequence of quasiconformal mappings $(f_j)_j$, where $f_j := g_j \circ h_j : \Omega_j \to D_j$. Each f_j maps $\Omega_j \subset X$ onto a circle domain $D_j \subset \mathbb{S}^2$.

For every $\ell \in \mathbb{N}$, any subsequence of $(q_{j,\ell})_j$ has a further subsequence Hausdorff converging to a limit disk or a point. Therefore we can use a diagonal argument to find a subsequence $(f_{j_k})_k$, converging locally uniformly in Ω, so that $q_{j_k,\ell} \to q_\ell$ for each ℓ. We continue to denote the subsequence by $(f_j)_j$. By Condition (4) and Proposition 2.4, the limit mapping f is non-constant and therefore a $\frac{\pi}{2}-QC$ mapping from Ω onto a domain $D := f(\Omega) \subset \mathbb{S}^2$. Each q_ℓ, $\ell \in \mathbb{N}$, is a disk or a point, and $q_0 = \mathbb{S}^2 \setminus \mathbb{D}(0,1)$.

Theorem 1.1 now follows once we establish the following.

Theorem 3.1. *The $\frac{\pi}{2}-QC$ mapping $f : \Omega \to D$ has the following properties:*

$$\operatorname{diam}(\hat{f}(p)) = 0 \quad \text{for all } p \in \mathcal{C}_P(\Omega), \tag{5}$$

$$q_\ell = \hat{f}(p_\ell) \quad \text{and} \quad \operatorname{diam}(q_\ell) > 0 \quad \text{for all } \ell = 0,1,2,\ldots. \tag{6}$$

Here $\hat{f} : \hat{\Omega} \to \hat{D}$ denotes the homeomorphism induced by f.

4. Modulus estimates

The aim of the following is to develop the estimates required for the proof of Theorem 3.1.

In this section we assume that X is a metric two-sphere that satisfies the upper Ahlfors 2-regularity condition (2) with constant $\alpha \geqslant 1$. Recall that if $\Omega \subset X$ is a domain, then we denote the quotient map by $\pi_\Omega : X \to \hat{\Omega}$. We also denote by $A(x_0, r) \subset X$ the annulus $B(x_0, 2r) \setminus \overline{B}(x_0, r)$.

Given a domain $\Omega \subset X$, $\bar{p} \in \hat{\Omega}$, $x_0 \in \bar{p}$, and $r > 0$, we denote the family of curves joining $\pi_\Omega(S(x_0,r))$ and $\pi_\Omega(S(x_0,2r))$ in $\pi_\Omega(\overline{A}(x_0,r))$ by $\Gamma_{x_0,r}$. Recall that $\bar{p}$ may or may not be a point of Ω, and that if it is not then we apply the notation for $\bar{p}$ also for $\pi_\Omega^{-1}(\bar{p}) \in \mathcal{C}(\Omega)$. The following estimate is the main technical result towards Theorem 1.1.

Proposition 4.1. *Let $\Omega \subset X$ be a finitely connected τ-cofat domain. There is an $M > 0$ depending only on α and τ so that $\operatorname{mod}_{\mathrm{T}} \Gamma_{x_0,r} \leqslant M$.*

We postpone the proof of Proposition 4.1 until Section 5. We now fix a τ-cofat domain as in Theorem 1.1. Moreover, given $j = 1,2,\ldots$, we let Ω_j be the finitely connected domain constructed in Section 3. We fix $\bar{p} \in \hat{\Omega}$ and $x_0 \in \bar{p}$ as above, and choose radii $R, r > 0$ such that $100r < R < \frac{\operatorname{diam} X}{2}$. We denote by $\Gamma^j_{R,r}$ the family of curves joining $\pi_{\Omega_j}(S(x_0,r))$ and $\pi_{\Omega_j}(S(x_0,R))$ in $\hat{\Omega}_j \setminus \{\bar{p}\}$.

Proposition 4.2. *The following estimate holds:*

$$\limsup_{j\to\infty} \operatorname{mod}_{\mathrm{T}} \Gamma^j_{R,r} \to 0 \qquad \text{as} \ \ r \to 0.$$

Proof. We choose a sequence of radii R_n decreasing to zero as follows: let $R_1 = \frac{R}{20}$. Then, assuming $R_1, \ldots, R_{n-1}$ are defined let

$$R_n = \frac{R_n'}{20},$$

where $R_n' \leqslant R_{n-1}/2$ is the smallest radius for which some $p \in \mathcal{C}_N(\Omega)$ other than $\bar{p}$ intersects both $S(x_0, R_{n-1}/2)$ and $S(x_0, R_n')$. If no $p \in \mathcal{C}_N(\Omega)$ other than $\bar{p}$ intersects $S(x_0, R_{n-1}/2)$, we set $R_n' = R_{n-1}/2$. Then R_n does not depend on j and $R_n \to 0$ as $n \to \infty$.

The annuli $A_n := B(x_0, 2R_n) \setminus \overline{B}(x_0, R_n)$ and $\pi_{\Omega_j}(A_n)$ are pairwise disjoint for a fixed $j = 0, 1, 2....$ We denote by Γ_n^j the family of all curves joining $\pi_{\Omega_j}(S(x_0, R_n))$ and $\pi_{\Omega_j}(S(x_0, 2R_n))$ in $\hat{\Omega}_j$.

By Proposition 4.1, we know that $\operatorname{mod}_{\mathrm{T}} \Gamma_n^j \leqslant M$, where M does not depend on j or n. Thus, if we fix $N \in \mathbb{N}$, then for every $1 \leqslant n \leqslant N$, there is an admissible function ρ_n for the curve family Γ_n^j such that

$$\int\limits_{\Omega_j} \rho_n^2 \, d\mathcal{H}^2 + \sum_{p \in \mathcal{C}(\Omega_j)} \rho_n(p)^2 \leqslant 2M. \tag{7}$$

We claim that $\widetilde{\rho_N} := \frac{\rho_1 + \cdots + \rho_N}{N}$ is admissible for $\Gamma^j_{R, R_{N+1}}$. Indeed, each $\gamma \in \Gamma^j_{R, R_{N+1}}$ intersects $\pi_{\Omega_j}(S(x_0, R_i))$ for $i = 1, 2, \ldots, N$ but avoids $\bar{p}$, so we have

$$\begin{aligned}
\int\limits_{\gamma} \widetilde{\rho_N} \, ds + \sum_{p \in \mathcal{C}(\Omega_j) \cap |\gamma|} \widetilde{\rho_N}(p) &\geqslant \sum_{i=1}^{N} \int\limits_{\gamma \cap A_i} \widetilde{\rho_N} \, ds + \sum_{i=1}^{N} \sum_{p \in C(\Omega_j) \cap |\gamma| \cap A_i} \widetilde{\rho_N}(p) \\
&\geqslant \sum_{i=1}^{N} \frac{1}{N} = 1.
\end{aligned}$$

Finally, we can estimate the limiting modulus as follows:

$$\begin{aligned}
\operatorname{mod}_{\mathrm{T}} \Gamma^j_{R, R_{N+1}} &\leqslant \int\limits_{\Omega_j} \widetilde{\rho_N}^2 \, d\mathcal{H}^2 + \sum_{P \in \mathcal{C}(\Omega_j)} \rho_N(p)^2 \\
&\leqslant \frac{1}{N^2} \sum_{i=1}^{N} \left[\int\limits_{\Omega_j} \rho_i^2 \, d\mathcal{H}^2 + \sum_{p \in \mathcal{C}(\Omega_j)} \rho_i(p)^2 \right] \\
&\leqslant \frac{1}{N^2} \cdot N \cdot 2M = \frac{2M}{N} \to 0 \text{ as } N \to \infty.
\end{aligned}$$

Recall that the last inequality holds by (7). Since R_N decreases to zero as $N \to \infty$ and M, N do not depend on j, the claim follows. □

We now describe the setting where Proposition 4.2 will be applied. The proof of the following lemma is an exercise.

Lemma 4.3. *Let X be a metric space homeomorphic to $\mathbb{S}^2$. For every $\kappa > 0$ there is an $\epsilon > 0$ such that the following holds: if $a, b \in X$ and $d(a, b) < \epsilon$, then a and b can be joined by a curve $I = I_{a,b}$ with* $\operatorname{diam}(I) < \kappa$.

Fix $\widetilde{p} \in \mathcal{C}(\Omega)$ and any Jordan curve $J \subset \Omega$ (recall that a *Jordan curve* in Ω is a subset of Ω which is homeomorphic to $\mathbb{S}^1$). Fix $0 < \kappa < \operatorname{dist}(\widetilde{p}, J)$, and let $0 < \epsilon \leqslant \kappa$ be as in Lemma 4.3. Given $j = 1, 2, \ldots$ let $f_j : \Omega_j \to D_j$ be the quasiconformal mapping in Section 3.

Next, fix $b \in \Omega \cap N_\epsilon(\widetilde{p})$, where $N_\epsilon(\widetilde{p})$ is the ϵ-neighborhood of $\widetilde{p}$ in X. Choose $a \in \partial\widetilde{p}$ such that no other point of $\partial\widetilde{p}$ is closer to b. By Lemma 4.3, we may choose a curve $I = I_{a,b}$ with diameter less than κ joining a and b. Notice that I does not intersect J.

For every $j \geqslant 1$, denote

$$\Lambda_j := \{\text{curves in } \hat{\Omega}_j \setminus \{\pi_{\Omega_j}(\widetilde{p})\} \text{ that join } \pi_{\Omega_j}(I) \text{ and } \pi_{\Omega_j}(J)\}.$$

We will apply the following circle domain estimate. Recall that $d_{\mathbb{S}^2}$ is the spherical metric on the standard $\mathbb{S}^2$.

Proposition 4.4. *There exists a homeomorphism $\psi_a : [0,\infty) \to [0,\infty)$ such that*

$$\limsup_{j\to\infty} \operatorname{mod}_{\mathrm{T}} \hat{f}_j(\Lambda_j) \geqslant \limsup_{j\to\infty} \psi_a(d_{\mathbb{S}^2}(f_j(b), \hat{f}_j(\widetilde{p}))).$$

Proof. The proof given in [8, Proposition 3.2 (1)] can be adapted to our setting. □

5. Proof of Proposition 4.1

Recall that X is a metric two-sphere which is upper Ahlfors 2-regular with a constant α. We fix a finitely connected τ-cofat domain $\Omega \subset X$, $\bar{p} \in \hat{\Omega}$, $x_0 \in \bar{p}$, and $r > 0$. We denote by $\Gamma_{x_0,r}$ the family of all curves joining $\pi_\Omega(S(x_0,r))$ and $\pi_\Omega(S(x_0,2r))$ in $\pi_\Omega(\overline{A}(x_0,r)) \subset \hat{\Omega}$.

Proposition (Proposition 4.1)**.** There is an $M > 0$ depending only on α and τ such that $\operatorname{mod}_{\mathrm{T}} \Gamma_{x_0,r} \leqslant M$.

Proof of Proposition 4.1. We define $\rho : \hat{\Omega} \to [0,\infty]$ such that

1) $\rho(x) = \frac{1}{r}$, if $x \in \Omega \cap A(x_0,r)$;
2) If $p \in \mathcal{C}(\Omega)$ and $p \cap A(x_0,r) \neq \emptyset$,
then $\rho(p) = \begin{cases} 1, \text{ if } \operatorname{diam}(p) \geqslant \frac{r}{10}; \\ \frac{\operatorname{diam}(p)}{r}, \text{ if } \operatorname{diam}(p) < \frac{r}{10}; \end{cases}$
3) $\rho(x) = 0$ elsewhere.

We denote

$$\mathcal{C}(x_0,r) := \{p \in \mathcal{C}(\Omega) : p \cap A(x_0,r) \neq \emptyset\},$$

and

$$\mathcal{G} := \left\{p \in \mathcal{C}(x_0,r) : \operatorname{diam}(p) \geqslant \frac{r}{10}\right\}.$$

Lemma 5.1. *ρ is admissible for $\Gamma_{x_0,r}$.*

Proof. We fix a curve $\gamma \in \Gamma_{x_0,r}$. If $\gamma(t) = \bar{p} \in \mathcal{G}$ for some t, then

$$\int_\gamma \rho\, ds + \sum_{p\in C(\Omega)\cap|\gamma|} \rho(p) \geqslant \rho(\bar{p}) = 1.$$

If $|\gamma| \cap \mathcal{G} = \emptyset$, then the curve γ can only "pass through" the complementary components with small diameters, see Figure 1. We denote $\gamma = \gamma_1 * \gamma_{p^1} * \gamma_2 * \gamma_{p^2} * \cdots * \gamma_n$, where γ_i is the curve in Ω and γ_{p^i} is the constant curve with value $\gamma_{p^i} = p^i \in \mathcal{C}(x_0, r) \setminus \mathcal{G}$. Then

$$\begin{aligned}\int_\gamma \rho\, ds + \sum_{p\in C(\Omega)\cap|\gamma|} \rho(p) &= \sum_l \int_\gamma \rho\, ds + \sum_i \rho\left(p^l\right)\\ &\geqslant \frac{1}{r}\sum_l \ell(\gamma_l) + \frac{1}{r}\sum_l \operatorname{diam}(p^l)\\ &\geqslant \frac{1}{r}\cdot r = 1,\end{aligned}$$

where we applied the triangle inequality. □

We denote the cardinality of a set A by $\#A$.

Lemma 5.2. *$\#\mathcal{G} \leqslant N$, where $N > 0$ depends only on α and τ.*

Proof. Given $p \in \mathcal{G}$, choose $a \in p \cap A(x_0, r)$. Since $\operatorname{diam}(p) \geqslant \frac{r}{10}$, we have $p \not\subset B(a, r/40)$. By the τ-fatness of p and the upper Ahlfors 2-regularity of X, we have

$$\tau \cdot (r/40)^2 \leqslant \mathcal{H}^2(p \cap B(a, r/40)) \leqslant \mathcal{H}^2(B(x_0, 3r)) \leqslant \alpha \cdot 9r^2. \tag{8}$$

Summing (8) over for all $p \in \mathcal{G}$ yields

$$\frac{\tau r^2}{1600} \cdot (\#\mathcal{G}) \leqslant \sum_{p\in\#\mathcal{G}} \mathcal{H}^2(p \cap B(a, r/40)) \leqslant \alpha \cdot 9r^2,$$

so we have the uniformly upper bound of $\#\mathcal{G}$:

$$\#\mathcal{G} \leqslant \frac{14400\alpha}{\tau} =: N(\alpha, \tau).$$

□

Finally, we prove the uniform boundedness of $\operatorname{mod}_{\mathrm{T}} \Gamma_{x_0,r}$. Since ρ is an admissible function for $\Gamma_{x_0,r}$ by Lemma 5.1, we have

$$\operatorname{mod}_{\mathrm{T}} \Gamma_{x_0,r} \leqslant \int_\Omega \rho^2 d\mathcal{H}^2 + \sum_{p\in\mathcal{C}(\Omega)} \rho(p)^2. \tag{9}$$

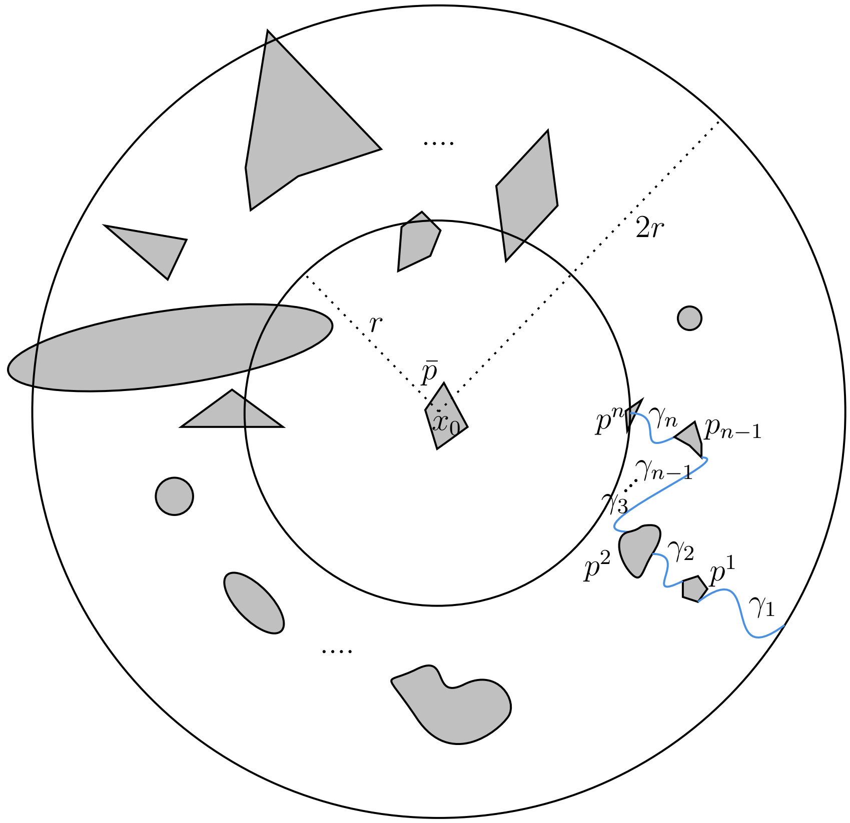

FIGURE 1. Part of a sample curve $\gamma = \gamma_1 * p^1 * \cdots * p^n$ that satisfies $\gamma(t) \notin \mathcal{G}$, each $\operatorname{diam}(p^i) < \frac{r}{10}$.

We divide the right-hand side of (9) into two terms and estimate them separately. First, we have

$$\begin{aligned} \int_\Omega \rho^2 d\mathcal{H}^2 &= \int_{\Omega\cap A(x_0,r)} \frac{1}{r^2} d\mathcal{H}^2 = \frac{1}{r^2} \cdot \mathcal{H}^2(\Omega \cap A(x_0,r)) \\ &\leqslant \frac{1}{r^2}\mathcal{H}^2(B\,(x_0,3r)) \leqslant 9\alpha, \end{aligned} \tag{10}$$

by Lemma 5.2 and the upper 2-regularity of X. Secondly, we can estimate the sum in (9) as follows:

$$
\begin{aligned}
\sum_{p\in\mathcal{C}(\Omega)} \rho(p)^2 &= \sum_{p\in\mathcal{G}} \rho(p)^2 + \sum_{p\notin\mathcal{G}} \rho(p)^2 = \#\mathcal{G} + \sum_{p\notin\mathcal{G}} \frac{\operatorname{diam}(p)^2}{r^2} \\
&\leqslant N + \sum_{p\notin\mathcal{G}} \frac{\mathcal{H}^2(p)}{\tau\cdot r^2} \leqslant N + \frac{\mathcal{H}^2(B(x_0,3r))}{\tau\cdot r^2} \qquad (11)\\
&\leqslant N + \frac{9\alpha}{\tau}.
\end{aligned}
$$

By (9), (10), and (11), we have $\operatorname{mod}_{\mathrm{T}} \Gamma_{x_0,r} \leqslant 9\alpha + N + \frac{9\alpha}{\tau}$. □

6. Proof of Theorem 3.1

In this section, we prove that the $\frac{\pi}{2}-QC$ mapping f constructed in Section 3 preserves the classes of complementary components and the image is a circle domain on $\mathbb{S}^2$.

We first recall that $f:\Omega\to D$ is the locally uniform limit of a sequence of $\frac{\pi}{2}-QC$ mappings $(f_j)_j$ given in Section 3. Moreover, if $\ell=1,2,\dots$ then $p_\ell\in\mathcal{C}_N(\Omega)$ and

$$\hat{f}_j(p_\ell)\to q_\ell \quad \text{as} \quad j\to\infty \tag{12}$$

in the Hausdorff sense. We prove Theorem 3.1 in three steps:

Step 1: From point to point. To prove (5), we assume $\widetilde{p}=\{a\}\in\mathcal{C}_P(\Omega)$ and suppose towards contradiction that $\hat{f}(\widetilde{p})\in\mathcal{C}_N(D)$. Then we can find a $c>0$ and a sequence $(b_k)_k$ of points in Ω converging to a, such that for every $k\in\mathbb{N}$, we have

$$\limsup_{j\to\infty} d_{\mathbb{S}^2}(f_j(b_k), f_j(a)) \geqslant c > 0.$$

Recall that Λ_j denotes the family of all curves joining $\pi_{\Omega_j}(I)$ and $\pi_{\Omega_j}(J)$ in $\hat{\Omega}_j\setminus\{\pi_{\Omega_j}(a)\}$; here $J\subset\Omega$ is a fixed Jordan curve,

$$\kappa_k < R := d(a,J) \quad \text{for every } k, \text{ and } \kappa_k\to 0 \quad \text{as } k\to\infty.$$

Moreover, $I\subset B(a,\kappa_k)$ is a curve in X joining a and b_k. By Proposition 4.4, we have

$$\limsup_{j\to\infty} \operatorname{mod}_T \hat{f}_j(\Lambda_j) > 0. \tag{13}$$

On the other hand, every curve $\gamma\in\Lambda_j$ intersects both $\pi_{\Omega_j}(S(a,\kappa_k))$ and $\pi_{\Omega_j}(S(a,R))$. Therefore, if we denote the family of curves joining $\pi_{\Omega_j}(S(a,\kappa_k))$ and $\pi_{\Omega_j}(S(a,R))$ in $\hat{\Omega}_j$ by $\Gamma^j_{R,\epsilon}$, we have

$$\operatorname{mod}_{\mathrm{T}}(\Lambda_j) \leqslant \operatorname{mod}_{\mathrm{T}}(\Gamma^j_{R,\kappa_k}). \tag{14}$$

By (14), Proposition 4.2, and the assumption $\widetilde{p} \in \mathcal{C}_P(\Omega)$, we have

$$\limsup_{j\to\infty} \operatorname{mod}_{\mathrm{T}}(\Lambda_j) = 0.$$

This contradicts (13) and Theorem 2.6 (quasi-invariance of the trans-boundary modulus), so $\hat{f}(\widetilde{p})$ must be a point-component.

Step 2: The image is a circle domain. To prove the first claim of (6), i.e., that $\hat{f}(p_\ell) = q_\ell$ for all the non-trivial components in (12), we use the idea in the proof of (5).

From a quasiconformal version of Carathéodory's kernel convergence theorem [36, Lemma 2.14], we know that $q_\ell \subset \hat{f}(p_\ell)$. We argue by contradiction and suppose that $q_\ell \subsetneq \hat{f}(p_\ell)$. Then there are a $c > 0$ and a sequence $(b_k)_k$ of points in Ω such that $d(b_k, p_\ell) \to 0$ as $k \to \infty$ and

$$\limsup_{j\to\infty} d_{\mathbb{S}^2}(f_j(b_k), \hat{f}_j(p_\ell)) \geqslant c > 0. \tag{15}$$

Combining (15) with Proposition 4.4, we have

$$\limsup_{j\to\infty} \operatorname{mod}_{\mathrm{T}} \hat{f}_j(\Lambda_j) > 0, \tag{16}$$

where Λ_j is the curve family defined before Proposition 4.4. On the other hand, since (14) remains true in the current setting, recalling that $d(b_k, p_\ell) \to 0$ as $k \to \infty$ and applying Proposition 4.2 yields

$$\limsup_{j\to\infty} \operatorname{mod}_{\mathrm{T}}(\Lambda_j) = 0.$$

This contradicts (16) and Theorem 2.6 (quasi-invariance of the trans-boundary modulus). It follows that $\hat{f}(p_\ell)$ is the disk q_ℓ. Combining Steps 1 and 2, we conclude that D is a circle domain.

Step 3: From non-trivial to non-trivial. Finally, we need to show the second claim of (6) i.e., $\hat{f}(\mathcal{C}_N(\Omega)) \subset \mathcal{C}_N(D)$.

Fix $p_\ell \in \mathcal{C}_N(\Omega)$ and a Jordan curve $J \subset \Omega$. Then $p_\ell \in \mathcal{C}_N(\Omega_j)$ and $J \subset \Omega_j$ for every $j \geqslant \ell$. Denote by Λ'_j the family of all curves joining $\pi_{\Omega_j}(p_\ell)$ and $\pi_{\Omega_j}(J)$ in $\hat{\Omega}_j$. Towards a contradiction, we assume that $\hat{f}(p_\ell) = q_\ell$ is a point. We claim that

$$\limsup_{j\to\infty} \operatorname{mod}_{\mathrm{T}} \hat{f}_j(\Lambda'_j) = 0. \tag{17}$$

We need the following lemma to prove (17).

Lemma 6.1. *For every $R > 0$, there exist an index $j_R \in \mathbb{N}$ and a radius $0 < r_R < R$, such that for every $j > j_R$ and every $q \in \mathcal{C}(D_j)$ with $q \cap S(q_\ell, R) \neq \emptyset$, we have $q \cap S(q_\ell, r_R) = \emptyset$.*

Proof. We argue by contradiction. Suppose that there exist $R_0 > 0$, a subsequence $(f_{j_i})_i$ of $(f_j)_j$, and a sequence of complementary components $p^i \in \mathcal{C}_N(\Omega_{j_i})$, such that for each $q^i := \hat{f}_{j_i}(p^i) \in \mathcal{C}(D_{j_i})$ satisfying $q^i \cap S(q_\ell, R_0) \neq \emptyset$, we have

$$\lim_{i\to\infty} d_{\mathbb{S}^2}(q_\ell, q^i) = 0. \tag{18}$$

After taking a subsequence if necessary, we may assume that $(p^i)_i$ converges to some $\bar{p} \in \mathcal{C}(\Omega)$ in the Hausdorff sense. Let $\bar{q} := \hat{f}(\bar{p})$. Then, by Carathéodory's kernel convergence theorem, for every $\epsilon > 0$, there exist a $\delta > 0$ and an $i_\epsilon \in \mathbb{N}$ such that

$$\hat{f}_{j_i}\big(N_\delta(\bar{p})\big) \subset N_\epsilon(\bar{q}) \quad \text{for all } i > i_\epsilon,$$

where $N_\delta(\bar{p})$ denotes the δ-neighborhood of $\bar{p}$ in X, and $N_\epsilon(\bar{q})$ denotes the ϵ-neighborhood of $\bar{q}$ in $\mathbb{S}^2$.

On the other hand, by the Hausdorff convergence of $(p^i)_i$, there exists an $i_\delta \in \mathbb{N}$ such that $p^i \in N_\delta(\bar{p})$ for every $i > i_\delta$. It follows that

$$d_{\mathbb{S}^2}(\bar{q}, q^i) \to 0 \quad \text{as } i \to \infty.$$

Combining this with (18), we conclude that $\bar{q} = q_\ell$. However, this contradicts the assumption that $q^i \cap S(q_\ell, R_0) \neq \emptyset$ for every $i \in \mathbb{N}$. The proof is complete. □

We continue to prove (17). We construct a sequence of radii $(r_n)_n$ as in the proof of Proposition 4.2: Let

$$r_0 := d_{\mathbb{S}^2}(f(J), q_\ell) > 0.$$

Since $f_j \to f$ locally uniformly in Ω, we may assume that

$$d_{\mathbb{S}^2}(f_j(J), q_\ell) > r_0/2$$

for all sufficiently large indices j. We first define $r_1 := r_0/20$. Assuming that $r_1 > r_2 > \ldots > r_{n-1} > 0$ have been chosen, we apply Lemma 6.1 with $R = r_{n-1}/2$ to obtain an index $j_n \in \mathbb{N}$ and a radius $0 < \widetilde{r}_n < r_{n-1}/2$ such that for every $j > j_n$, and every $q \in \mathcal{C}(D_j)$ with $q \cap S(q_\ell, r_{n-1}/2) \neq \emptyset$, we have $q \cap S(q_\ell, \widetilde{r}_n) = \emptyset$. We then define

$$r_n := \frac{\widetilde{r}_n}{20}.$$

Note that the sequence $(r_n)_n$ is independent of j, and satisfies $r_n \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, define the annulus

$$\widetilde{A}_n := B(q_\ell, 2r_n) \setminus \overline{B}(q_\ell, r_n) \subset \mathbb{S}^2.$$

Then for any fixed $j \in \mathbb{N}$, the sets $\pi_{D_j}(\widetilde{A}_n)$ are pairwise disjoint. We can now follow the argument in the proof of Proposition 4.2 to conclude that (17) holds. The following lemma and the quasi-invariance of the transboundary modulus will lead us to a contradiction with (17).

Lemma 6.2. *We have* $\mathrm{mod}_{\mathrm{T}}\, \Lambda'_j \geqslant c > 0$*, where* c *is independent of* j*.*

Proof. Let γ be a curve in X joining J and ∂p_ℓ, and let $m(x) := \mathrm{dist}(x, |\gamma|)$. Then m is 1-Lipschitz.

Fix a point $\widetilde{b} \in X$ that is not in the same component of $X \setminus J$ as p_ℓ. Then there exists a $\delta' := \delta'(\widetilde{b}) > 0$ such that $E_t := m^{-1}(t)$ separates $|\gamma|$ and $\{\widetilde{b}\}$ when $0 < t < \delta'$. Since X is homeomorphic to $\mathbb{S}^2$, there exists a continuum $\eta_t \subset E_t$ which also separates $|\gamma|$ and $\{\widetilde{b}\}$ in X; see [49, Ch. 2, Lemma 5.20].

Since $\mathrm{diam}(p_\ell) > 0$, there exists a $0 < \delta < \delta'$ such that η_t intersects both ∂p_ℓ and J for every $0 < t < \delta$. Applying the Coarea inequality ([7]) to the 1-Lipschitz function m and $g = 1$, we have $\mathcal{H}^1(\eta_t) < \infty$ for almost every $0 < t < \delta$. Thus, by ([48, Proposition 15.1]) there exists an injective 1-Lipschitz curve γ_t such that $|\gamma_t| \subset \eta_t$ and γ_t connects ∂p_ℓ and J in X, so we have $\hat{\gamma}_t := \pi_{\Omega_j}(\gamma_t) \in \Lambda'_j$.

Now we can use the curve $\hat{\gamma}_t$ to find a lower bound for $\mathrm{mod}_{\mathrm{T}}\, \Lambda'_j$. Let ρ be admissible for Λ'_j. Since $\hat{\gamma}_t \in \Lambda'_j$ for almost every $0 < t < \delta$ and the restriction of $\hat{\gamma}_t$ to Ω_j is injective, applying the Coarea inequality yields

$$\begin{aligned}
\delta &\leqslant \int_0^\delta \left[\int_{\hat{\gamma}_t} \rho \, ds + \sum_{p \in \mathcal{C}(\Omega_j) \cap |\hat{\gamma}_t|} \rho(p) \right] dt \\
&\leqslant \int_0^\delta \int_{E_t \cap \Omega_j} \rho \, d\mathcal{H}^1 dt + \sum_{p \in \mathcal{C}(\Omega_j)} \rho(p) \cdot \mathrm{diam}(p) \\
&\leqslant \frac{4}{\pi} \left[\int_{\Omega_j} \rho \, d\mathcal{H}^2 + \sum_{p \in \mathcal{C}(\Omega_j)} \rho(p) \cdot \mathrm{diam}(p) \right] =: \frac{4I}{\pi}.
\end{aligned}$$

Moreover, applying Hölder's inequality yields

$$I \leqslant \left[\mathcal{H}^2(\Omega_j) + \sum_{p \in \mathcal{C}(\Omega_j)} \mathrm{diam}(p)^2 \right]^{\frac{1}{2}} \left[\int_{\Omega_j} \rho^2 \, d\mathcal{H}^2 + \sum_{p \in \mathcal{C}(\Omega_j)} \rho(p)^2 \right]^{\frac{1}{2}}. \quad (19)$$

Since $\mathcal{H}^2(X) < \infty$ and $\operatorname{diam}(p)^2 \leqslant \tau^{-1}\mathcal{H}^2(p)$ for every $p \in \mathcal{C}(\Omega_j)$ by cofatness, it follows that

$$\mathcal{H}^2(\Omega_j) + \sum_{p\in\mathcal{C}(\Omega_j)} \operatorname{diam}(p)^2 \leqslant M < \infty, \tag{20}$$

where M does not depend on j. Since the estimates hold for all admissible functions ρ, the claim follows by combining (19) and (20). □

Lemmas 6.2 and 2.6 (quasi-invariance of transboundary modulus) contradict (17). We conclude that $\hat{f}$ maps non-trivial components to non-trivial components. The proof of Theorem 1.1 is complete.

Remark 6.3. We demonstrate how the above arguments can be applied to prove Theorem 1.2, leaving the details to the reader. Suppose that $\Omega_2 \in \mathcal{F}_2(X)$, and let $f : \Omega_2 \to D$ be a conformal mapping onto a circle domain $D \subset \mathbb{S}^2$. Fix $p \in \mathcal{C}_N(\Omega_2)$ and a Jordan curve $J \subset \Omega_2$, and let Γ be the family of curves joining $\pi_{\Omega_2}(p)$ and $\pi_{\Omega_2}(J)$ in $\hat{\Omega}_2$. The proof of Lemma 6.2 can be adapted for ℓ^2-domains to show that $\operatorname{mod}_{\mathrm{T}} \Gamma > 0$.

On the other hand, since the circle domain D is cofat, the proof of Proposition 4.2 can be modified to show that if $\hat{f}(p) \in \mathcal{C}_P(D)$, then $\operatorname{mod}_{\mathrm{T}} \hat{f}(\Gamma) = 0$; here we also need the countable connectedness of D. We arrive at a contradiction with the conformal invariance of the transboundary modulus (Theorem 2.6). Theorem 1.2 follows.

7. Proof of Theorem 1.3

In this section we demonstrate the sharpness of Theorem 1.2 and Remark 6.3. Recall that $\mathcal{F}_\alpha(X)$ is the collection of ℓ^α-domains on X, i.e., the countably connected subdomains Ω_α of X satisfying

$$\sum_{p\in\mathcal{C}(\Omega_\alpha)} \operatorname{diam}(p)^\alpha < \infty.$$

Theorem (Theorem 1.3)**.** For every $\alpha > 2$ there is a domain $U_\alpha \in \mathcal{F}_\alpha(\mathbb{S}^2)$, such that $\overline{\mathbb{D}} \in \mathcal{C}_N(U_\alpha)$ but $\hat{f}(\overline{\mathbb{D}}) \in \mathcal{C}_P(D)$ for every conformal homeomorphism $f : U_\alpha \to D$ onto a circle domain $D \subset \mathbb{S}^2$.

Note that such a conformal homeomorphism f always exists by the He-Schramm theorem [13, Theorem 0.1].

Proof of Theorem 1.3. Fix $\alpha > 2$. We construct a countably connected domain $U_\alpha \subset \mathbb{S}^2$ whose complementary components include the closed unit disk $\overline{\mathbb{D}}$ as well as countably many subarcs of concentric circles "winding around" $\overline{\mathbb{D}}$, such that $\hat{f}(\overline{\mathbb{D}}) \in \mathcal{C}_P(D)$ for every conformal mapping $f : U_\alpha \to D$ onto a circle domain D.

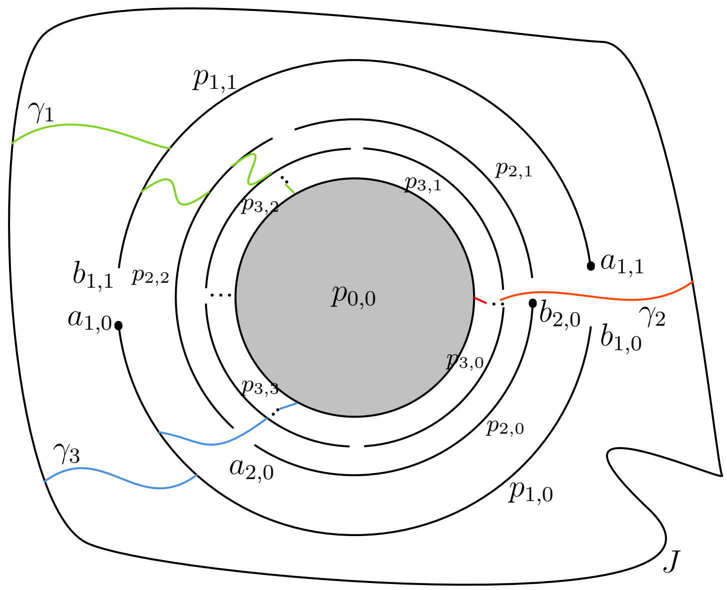

FIGURE 2. Some complementary components of U_α and the Jordan curve $J \subset U_\alpha$.

We now describe the elements of $\mathcal{C}(U_\alpha)$. First, denote

$$p_{0,0} := \overline{\mathbb{D}}.$$

Then, given $k \in \mathbb{N}_+$ and $0 \leqslant j \leqslant k$, let

$$p_{k,j} := \{r_k e^{i\theta_j} : r_k = 1 + \frac{1}{2^{k-1}},\ \theta_j \in [\alpha_{k,j}, \beta_{k,j}]\},$$

where

$$\begin{cases} \alpha_{k,j} := \frac{2(j-1)\pi}{k+1} + \theta_k, \\ \beta_{k,j} := \frac{2j\pi}{k+1} - \theta_k, \end{cases}$$

and

$$\theta_k := \frac{2\pi}{e^{2^k} \cdot 2025^k}. \tag{21}$$

We denote the endpoints of each $p_{k,j}$ by

$$a_{k,j} := r_k e^{i\alpha_{k,j}}, \quad b_{k,j} := r_k e^{i\beta_{k,j}}.$$

Finally, let U_α be the domain for which

$$\mathcal{C}(U_\alpha) := \{p_{k,j} : k \in \mathbb{N},\ j = 0, 1, \ldots, k\}.$$

Figure 2 shows the complementary components $p_{0,0}, p_{1,j}, p_{2,j}$ and $p_{3,j}$.

By the above construction, we have

$$\begin{aligned}\sum_{p\in C(U_\alpha)} \operatorname{diam}(p)^\alpha &\leqslant 2^\alpha + \sum_{k=1}^{\infty} \operatorname{diam}\left(p_{k,j}\right)^\alpha \cdot (k+1) \\ &< 2^\alpha + \sum_{k=1}^{\infty} \left(\frac{2\pi\left(1+\frac{1}{2^{k-1}}\right)}{k+1}\right)^\alpha \cdot (k+1) \\ &< 2^\alpha + (2\pi)^\alpha \cdot 2^\alpha \left(\sum_{k=1}^{\infty} \frac{1}{(k+1)^{\alpha-1}}\right) < \infty.\end{aligned}$$

The last inequality holds since $\alpha > 2$. Thus U_α is an ℓ^α-domain.

Since U_α is a countably connected domain, by the He-Schramm theorem there exists a conformal homeomorphism $f : U_\alpha \to D$ onto a circle domain D. Moreover, f is unique up to postcomposition by a Möbius transformation. Without loss of generality, we assume $\hat{f}(\infty) = \infty$. To show that $\hat{f}(\overline{\mathbb{D}}) \in \mathcal{C}_P(D)$, we fix a Jordan curve $J \subset U_\alpha$ such that if V is the bounded component of $\hat{\mathbb{C}} \setminus J$, then $\hat{\mathbb{C}} \setminus U_\alpha \subset V$. We denote

$$\Gamma_\alpha := \{\text{curves in } \hat{U}_\alpha \text{ that join } \pi_{U_\alpha}(\overline{\mathbb{D}}) \text{ and } \pi_{U_\alpha}(J)\}.$$

Towards a contradiction, we assume $\hat{f}(\overline{\mathbb{D}}) \in \mathcal{C}_N(D)$. The proof can be divided into three steps. First, we define an admissible function ρ for Γ_α. Next, we employ ρ to estimate $\operatorname{mod}_{\mathrm{T}}(\Gamma_\alpha)$. Finally, with the help of the proof of Theorem 3.1 and Remark 6.3, we derive that such an estimate is impossible, thereby $\hat{f}(\overline{\mathbb{D}}) \in \mathcal{C}_P(D)$.

We let $\rho : \hat{U}_\alpha \to [0, \infty]$,

$$\rho(x) = \begin{cases} \frac{1}{(k+1)\log(k+1)}, & x \in p_{k,j}\ (k \geqslant 1), \\ \frac{1}{|x-c_{k,j}|\log\frac{R_k}{g_k}}, & x \in U_\alpha \cap A_{k,j}\ (k \geqslant 1), \\ 0, & \text{otherwise.} \end{cases}$$

Here

$$A_{k,j} := B(c_{k,j}, R_k) \setminus \overline{B}(c_{k,j}, g_k), \quad k \in \mathbb{N}_+ \text{ and } 0 \leqslant j \leqslant k,$$

and the center and radii are defined as follows:

$$c_{k,j} := \begin{cases} \frac{b_{k,j}+a_{k,j+1}}{2}, \text{ when } j = 0, \dots, k-1, \\ \frac{b_{k,k}+a_{k,0}}{2}, \text{ when } j = k, \end{cases}$$

$$g_k := \left|\frac{a_{k,j+1} - b_{k,j}}{2}\right| < \theta_k \cdot r_k, \quad \text{and} \quad R_k := r_k \cdot \theta_k e^{2^k} > e^{2^k} g_k. \tag{22}$$

Recall that since $r_k = 1+\frac{1}{2^{k-1}}$, the first inequality of (22) follows from the fact that the shortest path between two points is a line segment.

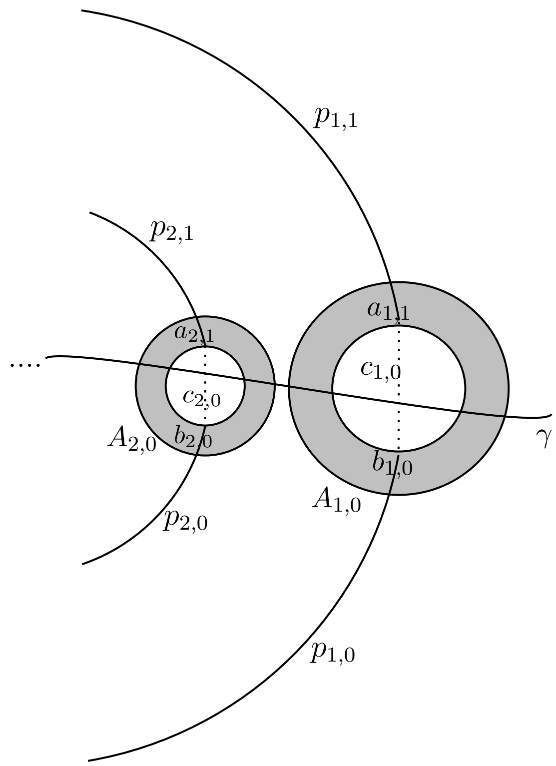

FIGURE 3. The annuli $A_{1,0}$ and $A_{2,0}$ in the definition of ρ.

Moreover, since θ_k is given by (21), we have

$$R_k = \frac{2\pi r_k}{2025^k} < \frac{1}{100 \cdot 2^{k-1}},$$

which implies that the annuli $A_{k,j}$ are pairwise disjoint. Figure 3 shows the annuli $A_{1,0}$ and $A_{2,0}$.

Lemma 7.1. *ρ is admissible for Γ_α and*

$$\int_\gamma \rho \, ds + \sum_{p \in \mathcal{C}(U_\alpha) \cap |\gamma|} \rho(p) = \infty \quad \text{for all } \gamma \in \Gamma_\alpha.$$

Proof. Fix a curve $\gamma \in \Gamma_\alpha$. We consider cases based on the number of times γ passes through complementary components or "gaps" between them; see Figure 2 for illustration.

Case 1: Suppose that $|\gamma| \cap \overline{B}(c_{k,j}, g_k) \neq \emptyset$ only for finitely many $k \in \mathbb{N}_+$. Then there exists a $K_1 > 0$ such that for every $k \geqslant K_1$, we have $|\gamma| \cap p_{k,j} \neq \emptyset$ for some $j(k) \in \{0, \dots, k\}$. This implies that

$$\int\limits_{\gamma} \rho\, ds + \sum_{p \in \mathcal{C}(U_\alpha) \cap |\gamma|} \rho(p) \geqslant \sum_{k=K_1}^{\infty} \frac{1}{(k+1)\log(k+1)} = \infty.$$

Case 2: Suppose that there is an infinite set $K \subset \mathbb{N}_+$ such that for every $k \in K$ there is an index $j = j(k)$ for which $|\gamma| \cap \overline{B}(c_{k,j}, g_k) \neq \emptyset$. Then

$$\int\limits_{\gamma} \rho\, ds + \sum_{p \in \mathcal{C}(U_\alpha) \cap |\gamma|} \rho(p) \geqslant \sum_{k \in K} \int\limits_{|\gamma| \cap A_{k,j}} \rho\, ds \geqslant \sum_{k \in K} \int\limits_{g_k}^{R_k} \frac{dr}{r \log \frac{R_k}{g_k}} = \infty.$$

□

It follows from Lemma 7.1 that for every $\epsilon > 0$ the function $\widetilde{\rho} := \epsilon\rho$ is also admissible for Γ_α. Therefore,

$$\begin{aligned}
\operatorname{mod}_{\mathrm{T}}(\Gamma_\alpha) &\leqslant \int\limits_{U_\alpha} \epsilon^2 \rho^2\, d\mathcal{H}^2 + \sum_{p \in \mathcal{C}(U_\alpha)} \epsilon^2 \rho(p)^2 \\
&= \sum_{k=1}^{\infty} \int\limits_{A_{k,j}} \frac{\epsilon^2\, d\mathcal{H}^2}{|x - c_{k,j}|^2 \log^2 \frac{R_k}{g_k}} + \sum_{k=1}^{\infty} \frac{\epsilon^2 (1+k)}{(1+k)^2 \log^2(1+k)} \\
&\leqslant \epsilon^2 \left(\sum_{k=1}^{\infty} \int\limits_{g_k}^{R_k} \frac{2\pi}{r \log^2 \frac{R_k}{g_k}} dr + \sum_{k=1}^{\infty} \frac{1}{(1+k)\log^2(1+k)} \right) \\
&\leqslant \epsilon^2 \left(2\pi \sum_{k=1}^{\infty} \frac{1}{2^k} + \sum_{k=1}^{\infty} \frac{1}{(1+k)\log^2(1+k)} \right) \\
&\leqslant \epsilon^2 (4\pi + M).
\end{aligned}$$

Here $M < \infty$ denotes the sum of the second series, and the third inequality follows from (22). Since the above estimate holds for all $\epsilon > 0$, by letting $\epsilon \to 0$ we deduce that

$$\operatorname{mod}_{\mathrm{T}}(\Gamma_\alpha) = 0. \tag{23}$$

On the other hand, if $\hat{f}(\overline{\mathbb{D}}) \in \mathcal{C}_N(D)$, then using the same argument as in Lemma 6.2 and Remark 6.3, we have

$$\operatorname{mod}_{\mathrm{T}}(\hat{f}(\Gamma_\alpha)) > 0.$$

This contradicts (23) and the conformal invariance of the transboundary modulus (Theorem 2.6), so we have $\hat{f}(\overline{\mathbb{D}}) \in \mathcal{C}_P(D)$ for every conformal homeomorphism $f : U_\alpha \to D$ onto a circle domain D. □

8. Proof of Theorem 1.4

In this section we show that no ℓ^α-assumption on the complementary components is sufficient to guarantee the last conclusion of Theorem 1.1, i.e., that point-components are mapped to point-components. Recall that $\mathcal{F}_\alpha(X)$ is the collection of countably connected subdomains Ω_α of X satisfying $\sum_{p\in\mathcal{C}(\Omega_\alpha)} \operatorname{diam}(p)^\alpha < \infty$.

Theorem (Theorem 1.4)**.** For every $\alpha > 0$ there is a domain $\Omega_\alpha \in \mathcal{F}_\alpha(\mathbb{S}^2)$ such that $\{0\} \in \mathcal{C}_P(\Omega_\alpha)$ but $\hat{f}(\{0\}) \in \mathcal{C}_N(D)$ for every conformal homeomorphism $f : \Omega_\alpha \to D$ onto a circle domain $D \subset \mathbb{S}^2$.

Recall that such a conformal homeomorphism f always exists by the He-Schramm theorem.

Proof of Theorem 1.4. Fix $\alpha > 0$. We construct a domain Ω_α by describing each element of $\mathcal{C}(\Omega_\alpha)$. The origin $\{0\}$ is the only element of $\mathcal{C}_P(\Omega_\alpha)$. We parametrize the non-trivial complementary components as follows: Given $j \in \mathbb{N}$, we denote by W_j the collection of finite words $w = w_1 \cdots w_j$, where $w_k \in \{0, 1\}$ for every $1 \leqslant k \leqslant j$. Moreover, denote $W_0 := \{\emptyset\}$ and $W := \bigcup_{j=0}^{\infty} W_j$. Then

$$\mathcal{C}_N(\Omega_\alpha) = \{p_w : w \in W\}.$$

We now define the elements p_w. For each $n = 0, 1, 2, \ldots$, consider the sequences

$$\ell_n := \left(\frac{1}{2^{n+n_0}\cdot(n+n_0)^2}\right)^{\frac{1}{\alpha}} > 0 \quad \text{and} \quad \epsilon_n := \frac{\ell_{n+1}}{2^{n+n_0+1}} > 0, \tag{24}$$

where $n_0 = n_0(\alpha) \in \mathbb{N}_+$ is large enough such that

$$\epsilon_{n-1} \leqslant \epsilon_{n+1} + \ell_{n+1} \quad \text{for every } n = 1, 2, \ldots; \tag{25}$$

an elementary computation shows that such an n_0 exists. We apply an inductive definition. First, let

$$p_\emptyset = [\epsilon_0, \epsilon_0 + \ell_0]$$

be a segment on the positive x-axis. Next, fix $j = 1, 2, \ldots$ and suppose that $w = \bar{w}w_j$, where $\bar{w} \in W_{j-1}$ and $w_j \in \{0, 1\}$. Moreover, assume that

$$p_{\bar{w}} := e^{i\theta_{\bar{w}}} \cdot [\epsilon_{j-1}, \epsilon_{j-1} + \ell_{j-1}] \tag{26}$$

is defined in polar coordinates. We set

$$p_w := e^{i\theta_w} \cdot [\epsilon_j, \epsilon_j + \ell_j], \tag{27}$$

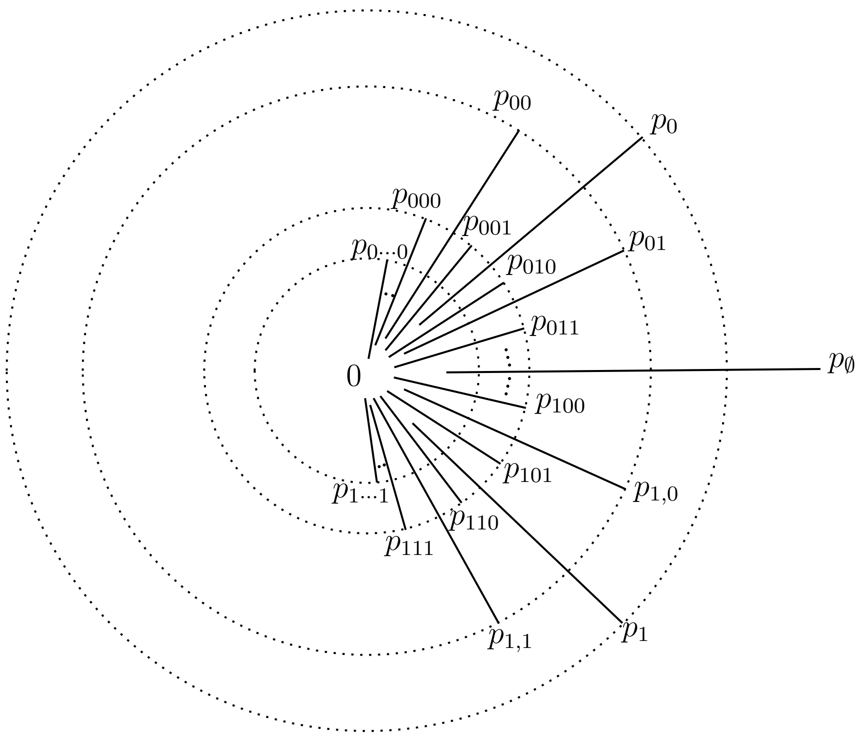

FIGURE 4. Some complementary components of Ω_α.

where the angle θ_w is given by

$$\theta_w := \begin{cases} \theta_{\bar{w}} + \alpha_j, & \text{if } w_j = 0, \\ \theta_{\bar{w}} - \alpha_j, & \text{if } w_j = 1, \end{cases} \qquad \alpha_j := 4^{-10j} \log\left(1 + \frac{1}{2^{j+n_0+3}}\right). \quad (28)$$

Notice that the sequence $(\alpha_j)_j$ decreases rapidly towards 0 as $j \to \infty$. Figure 4 depicts the first few generations of the complementary components p_w. We will apply the following properties.

Lemma 8.1. *Let $w = \bar{w}w_j$ be as above. The following properties hold:*

(1) *The elements p_w are pairwise disjoint subsets of the right half-plane*
(2) *For every $\epsilon > 0$ there is a $j_\epsilon \geqslant 1$ such that if $j \geqslant j_\epsilon$, then $p_w \subset B(0, \epsilon)$.*
(3) *There is a family Γ_w of curves joining p_w and $p_{\bar{w}}$ in Ω_α, such that $\mathrm{mod}_{\mathbb{C}}(\Gamma_w) > 4^j$.*
(4) *We have $\sum_{p \in \mathcal{C}(\Omega_\alpha)} \mathrm{diam}(p)^\alpha < \infty$; thus $\Omega_\alpha \in \mathcal{F}_\alpha(\mathbb{S}^2)$.*

Proof. Since $p_\emptyset$ lies on the positive real axis, Condition (1) follows from the construction (28) of the angles θ_w and the choice of the sequence

$(\alpha_j)_j$. To prove Condition (2), we notice that by (24), both ℓ_j and ϵ_j converge to 0 as $j \to \infty$. Consequently, we can choose a $j_\epsilon \geqslant 1$ such that $\ell_j + \epsilon_j < \epsilon$ for every $j \geqslant j_\epsilon$, which implies that $p_w \subset B(0,\epsilon)$ for all $w \in W_j$, $j \geqslant j_\epsilon$.

We now prove Condition (3). We claim that for every

$$\epsilon_{j+1} + \ell_{j+1} < t < \epsilon_j + \ell_j \tag{29}$$

there is a subarc J_t of $S(0,t)$ so that

(i) the length of J_t is $t\alpha_j$,
(ii) one endpoint of J_t lies in $p_{\bar{w}}$ and the other endpoint lies in p_w,
(iii) all other points of J_t lie in Ω_α.

Indeed, if $\epsilon_{j-1} < t < \epsilon_j + \ell_j$, then $S(0,t)$ intersects both $p_{\bar{w}}$ and p_w by (26) and (27). We let J_t be the shorter of the two subarcs of $S(0,t)$ joining $p_{\bar{w}}$ and p_w. Then J_t satisfies Properties (i) and (ii). By (25), such properties remain true for all t that satisfy (29).

To prove Property (iii), we fix a $w' \in W_{j'}$ other than $\bar{w}$ or w. The choice of the sequence $(\alpha_j)_j$ in (28) then guarantees that if t satisfies (29) then $p_{w'} \cap J_t = \emptyset$ if $j' \leqslant j$. Finally, the lower bound in (29) guarantees that $p_{w'} \cap J_t = \emptyset$ also when $j' > j$. We conclude that Property (iii) holds.

We set $\Gamma_w = \{J_t : \epsilon_{j+1} + \ell_{j+1} < t < \epsilon_j + \ell_j\}$, and recall from the definitions of ℓ_j and ϵ_j in (24) that

$$\frac{\epsilon_j + \ell_j}{\epsilon_{j+1} + \ell_{j+1}} \geqslant \frac{\epsilon_j + \ell_{j+1}}{(1 + \frac{1}{2^{j+n_0+2}})\ell_{j+1}} = 1 + \frac{1}{2^{j+n_0+2}+1}. \tag{30}$$

Let ρ be admissible for Γ_w. Then $1 \leqslant \int_{J_t} \rho\, ds$ for every $J_t \in \Gamma_w$. We divide both sides by t and integrate over t to get

$$\log\Big(\frac{\epsilon_j + \ell_j}{\epsilon_{j+1} + \ell_{j+1}}\Big) \leqslant \int\limits_{\epsilon_{j+1}+\ell_{j+1}}^{\epsilon_j+\ell_j} \frac{dt}{t} \leqslant \int\limits_{A_w} \frac{\rho(x)}{|x|}\, d\mathcal{H}^2(x) =: I, \tag{31}$$

where $A_w = \{x \in \Omega_\alpha : x \in J_t \text{ for some } J_t \in \Gamma_w\}$. We apply Hölder's inequality and polar coordinates to see that

$$\begin{aligned} I^2 &\leqslant \Big(\int\limits_{A_w} |x|^{-2}\, d\mathcal{H}^2(x)\Big)\Big(\int\limits_{\Omega_\alpha} \rho(x)^2\, d\mathcal{H}^2(x)\Big) \\ &= \alpha_j \log\Big(\frac{\epsilon_j + \ell_j}{\epsilon_{j+1} + \ell_{j+1}}\Big)\Big(\int\limits_{\Omega_\alpha} \rho(x)^2\, d\mathcal{H}^2(x)\Big). \end{aligned} \tag{32}$$

Combining (30), (31), and (32) with (28), and minimizing over all admissible functions ρ, then yields the desired bound $\operatorname{mod}_C \Gamma_w > 4^j$.

Finally, since the set W_j has 2^j elements for every $j = 0, 1, 2, \ldots$,

$$\sum_{p\in\mathcal{C}(\Omega_\alpha)} \operatorname{diam}(p)^\alpha = \sum_{j=0}^{\infty} 2^j \ell_j^\alpha \leqslant \sum_{j=0}^{\infty} (j+n_0)^{-2} < \infty$$

by the choice (24) of the sequence $(\ell_j)_j$. Condition (4) follows. □

Since Ω_α is a countably connected domain, by the He-Schramm theorem there exists a conformal homeomorphism $f : \Omega_\alpha \to D$ onto a circle domain D. Furthermore, f is unique up to postcomposition by a Möbius transformation. To show that $\hat{f}(\{0\}) \in \mathcal{C}_N(D)$, we assume towards a contradiction that $\hat{f}(\{0\})$ is a point-component.

We denote by Γ the family of curves in $\hat{\Omega}_\alpha$ joining $p_\emptyset$ and $\{0\}$. As in Remark 6.3, we can adapt the proof of Proposition 4.2 to show that $\operatorname{mod}_{\mathrm{T}}(\hat{f}(\Gamma)) = 0$. By the conformal invariance of the transboundary modulus (Theorem 2.6), we have $\operatorname{mod}_{\mathrm{T}}(\Gamma) = 0$. The desired contradiction follows if we can prove that

$$\operatorname{mod}_{\mathrm{T}}(\Gamma) > 0. \tag{33}$$

We denote by W_∞ the collection of all the infinite words $w_1w_2\cdots$, where $w_k \in \{0,1\}$. We equip the space W_∞ with the unique probability measure ν satisfying $\nu(U_w) = 2^{-j}$ for all $j \geqslant 1$ and $w \in W_j$, where

$$U_w := \{w_\infty \in W_\infty : w_\infty = ww_{j+1}w_{j+2}\cdots\}.$$

Let $\rho : \hat{\Omega}_\alpha \to [0,\infty]$ be an admissible function for Γ that satisfies

$$\int_{\Omega_\alpha} \rho^2 \, d\mathcal{H}^2 + \sum_{w\in W} \rho(w)^2 = 1. \tag{34}$$

We claim that there exists a $u_\infty = u_1u_2\cdots \in W_\infty$ so that

$$\sum_{j=1}^{\infty} \rho(p_{\widetilde{u}_j}) \leqslant 1. \tag{35}$$

Here $\widetilde{u}_j := u_1u_2\cdots u_j$. Indeed, by Fubini's theorem we have

$$\int_{W_\infty} \sum_{j=1}^{\infty} \rho(p_{\widetilde{u}_j}) \, d\nu(u_\infty) = \sum_{j=1}^{\infty} \sum_{w\in W_j} \nu(U_w)\rho(p_w) = \sum_{j=1}^{\infty} 2^{-j} \sum_{w\in W_j} \rho(p_w) =: I_\infty.$$

By the Cauchy-Schwarz inequality and since $\#W_j = 2^j$,

$$I_\infty \leqslant \sum_{j=1}^{\infty} 2^{-j/2} \Big(\sum_{w\in W_j} \rho(p_w)^2\Big)^{1/2} \leqslant \Big(\sum_{j=1}^{\infty} 2^{-j}\Big)^{1/2} \Big(\sum_{w\in W} \rho(p_w)^2\Big)^{1/2} \leqslant 1,$$

where the last inequality follows from (34). Combining the estimates shows that there exists a $u_\infty = u_1 u_2 \cdots \in W_\infty$ satisfying (35).

By Lemma 8.1, for every $\widetilde{u}_j = u_1 u_2 \cdots u_j$, $j = 1, 2, \ldots$, there is a curve family $\Gamma_{\widetilde{u}_j}$ joining $p_{\widetilde{u}_{j-1}}$ and $p_{\widetilde{u}_j}$ in Ω_α such that $\mathrm{mod}_{\mathrm{C}}(\Gamma_{\widetilde{u}_j}) > 4^j$. We next claim that for every such j there is a $\gamma_j \in \Gamma_{\widetilde{u}_j}$ such that

$$\int_{\gamma_j} \rho \, ds \leqslant 2^{-j}. \tag{36}$$

Assume towards a contradiction that there is a $j_0 \geqslant 1$ such that $\int_{\gamma_{j_0}} \rho \, ds > 2^{-j_0}$ for every $\gamma_{j_0} \in \Gamma_{\widetilde{u}_{j_0}}$. Then $2^{j_0}\rho$ is an admissible function for $\Gamma_{\widetilde{u}_{j_0}}$, and so

$$\int_{\Omega_\alpha} 4^{j_0} \rho^2 \, d\mathcal{H}^2 \geqslant \mathrm{mod}_{\mathrm{C}}(\Gamma_{\widetilde{u}_{j_0}}) > 4^{j_0}.$$

Thus $\int_{\Omega_\alpha} \rho^2 \, d\mathcal{H}^2 > 1$, which contradicts (34). We have proved (36).

We sequentially concatenate the curves $\pi_{\Omega_\alpha} \circ \gamma_j \in \hat{\Omega}_\alpha$, $j = 1, 2, \ldots$, to obtain a curve $\gamma \in \Gamma$ such that the elements of $|\gamma| \cap \mathcal{C}(\Omega_\alpha)$ are $p_\emptyset$, $p_{\widetilde{u}_j} (j = 1, 2, \ldots)$ and $\{0\}$. Combining (35) and (36), we arrive at

$$\int_\gamma \rho \, ds + \sum_{j=1}^{\infty} \rho(p_{\widetilde{u}_j}) \leqslant 2. \tag{37}$$

Since ρ is any admissible function for Γ that satisfies (34), the definition of the transboundary modulus and (37) show that (33) holds. This gives a contradiction, so we have $\hat{f}(\{0\}) \in \mathcal{C}_N(D)$ for every conformal homeomorphism $f : \Omega_\alpha \to D$ onto a circle domain $D \subset \mathbb{S}^2$. $\square$

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