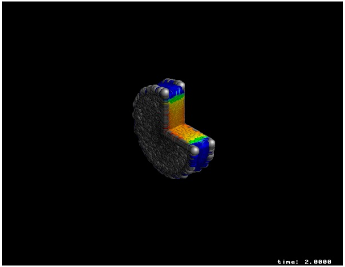
Jyväskylä Summer School 2016

Hydrodynamics in heavy-ion collisions

August 11, 2016

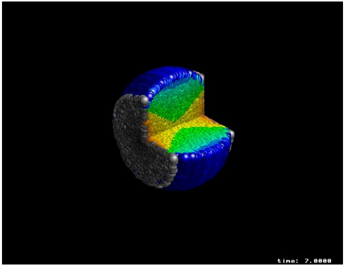
Section 1

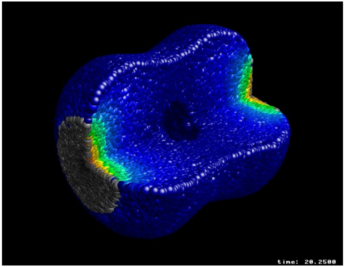
Introduction



Stages of ultrarelativistic heavy-ion collisions

- Initial particle production $\tau \ll 1$ fm/c: Two (Lorentz-contracted) nuclei go through each other, leaving highly exited matter between
- Non-equilibrium evolution of the produced matter (aka thermalization) $\tau \lesssim 1$ fm/c
- (Fluid dynamical) evolution of the QGP $\tau \sim 5$ fm/c





Stages of ultrarelativistic heavy-ion collisions

- Transition back to hadronic matter (QCD phase transition)
- Hadronic evolution (interacting hadron gas)
- Transition to free particles (Freeze-out)

The macroscopic properties of QCD matter (like Equation of State, viscosity) are a direct input to the fluid dynamics. $\longrightarrow$ Relatively straightforward to test different assumptions.

QGP, QCD transition region, Hadronic matter

Units

Natural units:

- $\hbar = c = k_B = 1$. Select GeV or fm as a unit.
- $\hbar c = 1 \approx 0.1973$ GeV fm $\longrightarrow$ Can be used to convert GeV to fm, and vice versa.
- e.g. $t = 1 \text{ fm} = \frac{1 \text{ fm}}{1} = \frac{1 \text{ fm}}{0.1973 \text{ GeV fm}} = 5.07 \text{ GeV}^{-1}$

Conversion to "non-natural" units:

- $t = 1 \text{ fm} = 1 \text{ fm/c} \underbrace{=}_{c \sim 3 \cdot 10^9 \text{ m/s} = 3 \cdot 10^{24} \text{ fm/s}} 3.33 \cdot 10^{-23} \text{ s} \sim$ Typical time scale in HI-collisions
- $T = 100 \text{ MeV} = 100 \text{ MeV}/k_B \underbrace{=}_{k_B = 1.381 \cdot 10^{-23} \text{ J/K} = 8.620 \cdot 10^{-5} \text{ eV/K}} 1.16 \cdot 10^{12} \text{ K}$

Lorentz transformations

Relativistic fluid dynamics is build out of Lorentz 4-vectors that transform according to Lorentz transformations. Basic building blocks: scalas, vectors and tensors: A, A^{μ}, $A^{\mu\nu}$, ...
Particle (or fluid element) position given by 4-vector:

$$x^{\mu} = (t, x, y, z) .$$

Lorentz transformations leave the proper time $d\tau$ invariant

$$d\tau^2 = dt^2 - d\mathbf{x}^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} , \tag{1}$$

where $g_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$ is metric tensor. Note that

$$d\tau^2 = (1 - \frac{d\mathbf{x}}{dt}^2) dt^2 = (1 - \mathbf{v}^2) dt^2 = \gamma^{-2} dt^2 \tag{2}$$

Proper time is a time measured in the frame moving with particle/fluid. $d\tau = d\tau'$ if

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu} ,$$

where $\Lambda^{\mu}_{\ \nu}$ satisfies

$$\Lambda^{\mu}_{\ \alpha} \Lambda^{\nu}_{\ \beta} g_{\mu\nu} = g_{\alpha\beta} \tag{3}$$

Boost by velocity v_z:

$$\begin{aligned} t' &= \gamma\,(t - v_z z) \\ z' &= \gamma\,(z - v_z t) \end{aligned} \tag{4}$$

In general A^μ is a 4-vector if it transforms in Lorentz transformation as

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu \tag{5}$$

and for higher rank tensors

$$A'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\mu{}_\beta A^{\alpha\beta} \tag{6}$$

From each A^μ it is possible to construct a vector

$$A_\mu = g_{\mu\nu} A^\nu \tag{7}$$

that transforms as

$$A'_\mu = \Lambda_\mu{}^\nu A_\nu \tag{8}$$

- A^μ contravariant vector
- A_μ covariant vector
- Metric tensor can be used to lower and rise the indices.

Divergence operator is a covariant vector

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = \Lambda_\mu{}^\nu \partial'_\nu \tag{9}$$

When $x'^\nu = \Lambda^\nu{}_\mu x^\mu$ then ∂_μ transforms as

$$\partial'_\mu = \Lambda_\mu{}^\nu \partial_\nu \tag{10}$$

- Note. In general curvilinear coordinates divergence of vector or higher rank tensors is not a tensor! Replace by covariant derivative

For a four-vector A^μ the covariant derivative is

$$A^\mu_{;\alpha} \equiv \partial_\alpha A^\mu + \Gamma^\mu_{\alpha\beta} A^\beta \,, \tag{11}$$

where the Christoffel symbol is

$$\Gamma^\mu_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\nu} \left(\partial_\beta g_{\alpha\nu} + \partial_\alpha g_{\nu\beta} - \partial_\nu g_{\alpha\beta}\right) \tag{12}$$

Similarly, the covariant derivative of covariant vectors is given by

$$A_{\mu;\alpha} \equiv \partial_\alpha A_\mu - \Gamma^\beta_{\mu\alpha} A_\beta \tag{13}$$

For scalars the covariant derivative reduces to the ordinary divergence.
The covariant derivative of second-rank tensors is

$$A^{\mu\nu}_{;\alpha} \equiv \partial_\alpha A^{\mu\nu} + \Gamma^\mu_{\alpha\beta} A^{\beta\nu} + \Gamma^\nu_{\alpha\beta} A^{\mu\beta} \,. \tag{14}$$

4-velocity (note dx^μ/dt is not 4-vector)

$$u^\mu = \frac{d}{d\tau} x^\mu(\tau) = \left(\frac{dt}{d\tau}, \frac{d\mathbf{x}}{d\tau} \right) = \gamma \left(1, \mathbf{v}\right),$$

where τ is the time measure in coordinate system moving with the particle/fluid element. We can write co-moving time derivative for any tensor

$$\frac{d}{d\tau} A^\mu = u^\alpha \partial_\alpha A^\mu. \tag{15}$$

Check that $u^\alpha \partial_\alpha x^\mu = u^\mu$

For a general coordinate transformation:

$$\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} \tag{16}$$

$$\Lambda_\mu{}^\nu = \frac{\partial x^\nu}{\partial x'^\mu} \tag{17}$$

Why tensors?
If two tensors are equal in one frame

$$A^\mu = B^\mu, \tag{18}$$

they remain equal in any other frame. Thus, if we build our theory using only tensors, the equations are automatically frame independent.

Thermodynamics

In statistical physics all thermodynamical quantities are derived from the partition function. A convenient framework for this is the grand canonical ensemble, where the external restrictions are given by the temperature T, volume V and chemical potential μ. The grand canonical partition function $\mathcal{Z}_G$ is defined as

$$\mathcal{Z}_G = \sum_{\{Nr\}} \exp \beta(\mu N - E_{Nr}), \tag{19}$$

where $\beta = 1/T$. The sum is taken over all possible microstates $\{Nr\}$ of the system, where N is the number of particles in the microstate r and E_{Nr} is the energy of the microstate. The partition function gives the probability $p_{\{Nr\}}$ of the microstate when temperature, volume and chemical potentials are fixed,

$$p_{\{Nr\}} = \frac{1}{\mathcal{Z}_G} \exp\left[\beta(\mu N - E_{Nr})\right]. \tag{20}$$

The grand canonical potential is defined as

$$\Omega_G(T, V, \mu) = -T \ln \mathcal{Z}_G. \tag{21}$$

All thermodynamic quantities can be calculated once $\Omega_G(T, V, \mu)$ or $\mathcal{Z}_G$ is known. From the thermodynamical identities one obtains

$$\Omega_G(T, V, \mu) = -pV, \tag{22}$$

i.e. if the pressure of the system is known as a function of T, V and μ, the complete thermodynamics of the system is known.
Entropy density s, pressure p and particle density n_i can be obtained by partial differentiation of the partition function:

$$s = \frac{1}{V}\frac{\partial T \ln \mathcal{Z}_G}{\partial T}, \tag{23}$$

$$p = T\frac{\partial \ln \mathcal{Z}_G}{\partial V}, \tag{24}$$

$$n = \frac{T}{V}\frac{\partial \ln \mathcal{Z}_G}{\partial \mu}. \tag{25}$$

For a mixture of noninteracting particles the logarithm of the partition function can be written as a sum of logarithms of the single-particle partition functions:

$$\ln \mathcal{Z}_G = \sum_i \ln \mathcal{Z}_i, \tag{26}$$

where $\mathcal{Z}_i$ is the partition function of particle type i. For noninteracting fermions and bosons the logarithm of the single-particle partition function can be calculated from the definition (19), by replacing the sum with an integral, $\sum_{Nr} \to \sum_N \int \frac{d^3\mathbf{p}}{(2\pi)^3}$. This gives the well-known result

$$\ln \mathcal{Z}_i = \frac{g_i V}{T} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{e^{\beta(E_i - \mu_i)} \pm 1}, \tag{27}$$

where g_i is the degeneracy factor and μ_i the chemical potential of the particle. The energy of the particle is $E_i = \sqrt{\mathbf{p}^2 + m_i^2}$, when the interactions between the particles can be neglected. The plus sign is for fermions and the minus sign for bosons. From the above results we obtain:

$$p(T, \{\mu_i\}) = \sum_i g_i \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{p^2}{3E_i} \frac{1}{e^{\beta(E_i - \mu_i)} \pm 1}, \tag{28}$$

$$n(T, \{\mu_i\}) = \sum_i g_i \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{e^{\beta(E_i - \mu_i)} \pm 1}, \tag{29}$$

$$e(T, \{\mu_i\}) = \sum_i g_i \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{E_i}{e^{\beta(E_i - \mu_i)} \pm 1}, \tag{30}$$

where the sums are over all particle species included in the EoS.

Useful thermodynamical identities

First law of thermodynamics

$$dE = TdS - pdV + \sum_i \mu_i dN_i, \tag{31}$$

Another useful identity is

$$sT = e + p - \sum_i \mu_i n_i, \tag{32}$$

Combining these two gives 1st law of thermodynamics in the form

$$de = Tds + \sum_i \mu_i dn_i, \tag{33}$$

and

$$dp = sdT + \sum_i n_i d\mu_i \tag{34}$$

Section 2

Energy-momentum tensor and conservation laws

Energy-momentum tensor

The basic quantity in fluid dynamics (hydrodynamics): Energy momentum tensor $T^{\mu\nu}$

- T^{00} = energy density
- T^{0i} = momentum density into direction i
- T^{i0} = energy flux into direction i
- T^{ij} = flux of i-momentum into the direction j

For conserved charges/particle numbers, charge/particle 4-current:

- N^0 = particle/charge density
- N^i = particle/charge flux into direction i

In hydrodynamical limit:

- the state of the matter is completely determined by $T^{\mu\nu}$ and N^μ.
- The dynamics (spacetime evolution) can be described in terms of $T^{\mu\nu}$ and N^μ alone (+ properties of the matter: EoS, transport coefficients)

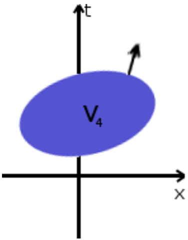
Conservation laws

Let Σ be a 4-dimensional spacetime volume.
Net number of particles flowing through $\mathrm{d}\Sigma$:

$$\mathrm{d}\Sigma_\mu N^\mu$$

Net energy and momentum flowing through $\mathrm{d}\Sigma$:

$$\mathrm{d}\Sigma_\mu T^{\mu\nu}$$



$$\int_{\partial V_4} \mathrm{d}\Sigma_\mu(x) N^\mu(x) = \int_{\partial V_4} \mathrm{d}\Sigma_\mu(x) T^{\mu\nu}(x) = 0$$

Global conservation laws.

Gauss theorem states that surface integrals over closed surface can be converted into volume integrals:

$$0 = \int_{\partial V_4} \mathrm{d}\Sigma_\mu(x) N^\mu(x) = \int_{V_4} \mathrm{d}^4x\, \partial_\mu N^\mu(x)$$
$$0 = \int_{\partial V_4} \mathrm{d}\Sigma_\mu(x) T^{\mu\nu}(x) = \int_{V_4} \mathrm{d}^4x\, \partial_\mu T^{\mu\nu}(x)$$

These must hold for an abritrary spacetime volume $V_4 \longrightarrow$ Local conservation of energy, momentum and charge:

$$\partial_\mu T^{\mu\nu} = 0 \qquad \partial_\mu N^\mu = 0 \tag{35}$$

These are the basic equations of fluid dynamics

- In general $T^{\mu\nu}$ and N^μ contain 14 independent components, but the conservation laws provide only 5 equations.

Kinetic definition of $T^{\mu\nu}$

Consider a system where the state of the matter is characterized by a single particle momentum distribution function $f(x, p)$. This distribution is a scalar function that gives a probability density of observing particle with momentum p at position x.

The expectation or average values of the components of $T^{\mu\nu}$ and N^{μ} can be calculated as the moments of $f(x, p)$

$$T^{00} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} p^0 f(x,p) \tag{36}$$

$$T^{0i} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} p^i f(x,p) \tag{37}$$

$$T^{i0} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{p^i}{p^0} p^0 f(x,p) \tag{38}$$

$$T^{ij} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{p^i}{p^0} p^j f(x,p) \tag{39}$$

and

$$N^{0} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} f(x,p) \tag{40}$$

$$N^{i} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} p^i f(x,p) \tag{41}$$

These definitions can be written in a tensor form:

$$T^{\mu\nu} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 p^0} p^\mu p^\nu f(x,p) \tag{42}$$

$$N^{\mu} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 p^0} p^\mu f(x,p) \tag{43}$$

$$\left.\begin{array}{lcl} \frac{d^3\mathbf{p}}{p^0} & = & \text{Lorentz scalar} \\ f(x,p) & = & \text{Lorentz scalar} \\ p^\mu & = & \text{Lorentz vector} \end{array}\right\} \longrightarrow \begin{array}{lcl} T^{\mu\nu} & = & \text{Second rank tensor} \\ N^\mu & = & \text{4-vector} \end{array} \tag{44}$$

Section 3

Ideal fluid dynamics

Local thermal equilibrium

What greatly simplifies the system is an assumption of thermal equilibrium. In this case the state of the matter is determined by two parameters, temperature T and chemical potential μ, and the single particle distribution is given by the usual Fermi-Dirac or Bose-Einstein form:

$$f_{\mathrm{eq}}(p; T, \mu) = \frac{1}{\exp\left((E-\mu)/T\right) \pm 1}, \quad (45)$$

If the system is in local thermal equilibrium, the distribution is still given the form above, but T, and μ can depend on x, i.e. $(T, \mu) \longrightarrow (T(x), \mu(x))$.
Note that the distribution above is not Lorentz-invariant as such, but rather written in frame where matter is at rest. The Lorentz invariance can be made explicitly by replacing

$$E \longrightarrow E_{\mathrm{LR}} = u^{\mu} p_{\mu} = u \cdot p \quad (46)$$

Because $u \cdot p = E$ in the local rest frame where $u^{\mu} = (1, 0, 0, 0)$, and because $u \cdot p$ is scalar it is the same regardless of the frame where it is calculated. The distribution is now:

$$f_{\mathrm{eq}}\left(T(x), \mu(x), u(x)\right) = \frac{1}{\exp\left((u \cdot p - \mu)/T\right) \pm 1} \quad (47)$$

Decomposition of $T^{\mu\nu}$ in local thermal equilibrium

$$T^{\mu\nu} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 p^0} p^\mu p^\nu \frac{1}{\exp\left((u \cdot p - \mu)/T\right) \pm 1} \tag{48}$$

Can depend only on vector/tensor quantities u^μ $g^{\mu\nu}$

$$T^{\mu\nu} = A' u^\mu u^\nu + B' g^{\mu\nu} \tag{49}$$

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \tag{50}$$

$$T^{\mu\nu} = A u^\mu u^\nu + B \Delta^{\mu\nu} \tag{51}$$

$$A = u_\mu u_\nu T^{\mu\nu} \tag{52}$$

$$B = -\frac{1}{3} \Delta_{\mu\nu} T^{\mu\nu} \tag{53}$$

(54)

Decomposition of $T^{\mu\nu}$ in local thermal equilibrium

$$u_\mu u_\nu T^{\mu\nu} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 p^0} (u_\mu p^\mu)^2 f_{\rm eq}(p) = e_0 \tag{55}$$

$$-\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu} = \frac{1}{3}\int \frac{d^3\mathbf{p}}{(2\pi)^3 p^0} \left((u_\mu p^\mu)^2 - g_{\mu\nu} p^\mu p^\nu\right) f_{\rm eq}(p) \tag{56}$$

$$= \frac{1}{3}\int \frac{d^3\mathbf{p}}{(2\pi)^3 p^0} \mathbf{p}^2 f_{\rm eq}(p) = p_0 \tag{57}$$

e_0 energy density in LRF, p_0 = kinetic pressure

$$T^{\mu\nu} = e_0 u^\mu u^\nu - p_0 \Delta^{\mu\nu} \tag{58}$$

5 independent variables (e_0, p_0, u^μ), but in equilibrium pressure given by the equation of state (EoS): $p_0 = p_0(e_0, n_0)$
$\longrightarrow$ 4 independent variables (e_0, u^μ) $\longrightarrow$ The conservation laws $\partial_\mu T^{\mu\nu} = 0$ and $\partial_\mu N^\mu = 0$ can be solved!
This is the ideal/perfect hydrodynamics. Follows from the strong assumption of Local thermal equilibrium!

Equations of motion

$$\partial_\mu N^\mu = 0 \quad \& \quad N^\mu = n_0 u^\mu \tag{59}$$

$$\partial_\mu N^\mu = \partial_\mu(n_0 u^\mu) = u^\mu \partial_\mu n_0 + n_0 \partial_\mu u^\mu \tag{60}$$

$$u^\mu \partial_\mu A = \dot{A} \quad \partial_\mu u^\mu = \theta \tag{61}$$

$$\dot{n}_0 + n_0 \theta = 0 \tag{62}$$

$$\partial_\mu T^{\mu\nu} = 0 \quad \& \quad T^{\mu\nu} = e_0 u^\mu u^\nu - p_0 \Delta^{\mu\nu} \tag{63}$$

$$\partial_\mu T^{\mu\nu} = \partial_\mu(e_0 u^\mu u^\nu - p_0 \Delta^{\mu\nu}) \tag{64}$$

$$= (\dot{e}_0 + \dot{p}_0)\, u^\nu + (e_0 + p_0)\, \theta u^\nu + (e_0 + p_0)\, \dot{u}^\nu - \partial^\nu p_0 = 0 \tag{65}$$

Contractions

$$u_\nu \partial_\mu T^{\mu\nu} = (\dot{e}_0 + \dot{p}_0) + (e_0 + p_0)\, \theta + (e_0 + p_0)\, \underbrace{u_\nu \dot{u}^\nu}_{=0} - \underbrace{u_\nu \partial^\nu p_0}_{=\dot{p}_0} = 0 \tag{66}$$

$$\longrightarrow \dot{e}_0 = -(e_0 + p_0)\theta \tag{67}$$

$$\Delta^\alpha_\nu \partial_\mu T^{\mu\nu} = (e_0 + p_0)\, \underbrace{\Delta^\alpha_\nu \dot{u}^\nu}_{\dot{u}^\alpha} - \underbrace{\Delta^\alpha_\nu \partial^\nu p_0}_{\equiv \nabla^\alpha p_0} = 0 \tag{68}$$

$$\longrightarrow (e_0 + p_0)\dot{u}^\alpha = \nabla^\alpha p_0 \tag{69}$$

What is $\theta = \partial_\mu u^\mu$?
Start from the particle number conservation:

$$\partial_\mu (n_0 u^\mu) = 0 \Longrightarrow \partial_\mu \left(\frac{N}{V} u^\mu\right) = 0 \tag{70}$$

$$\partial_\mu \left(\frac{N}{V} u^\mu\right) = u^\mu \partial_\mu \left(\frac{N}{V}\right) + \frac{N}{V} \partial_\mu u^\mu \tag{71}$$

Define volume element so that N =const. within the volume $V \Longrightarrow$ particle density changes due to the volume changes $\Longrightarrow$

$$u^\mu \partial_\mu \left(\frac{N}{V}\right) + \frac{N}{V} \partial_\mu u^\mu = 0 \tag{72}$$

$$\underbrace{u^\mu \partial_\mu \left(\frac{1}{V}\right)}_{=-\frac{\dot{V}}{V^2}} + \frac{1}{V} \partial_\mu u^\mu = 0 \tag{73}$$

$$\Longrightarrow \frac{\dot{V}}{V} = \partial_\mu u^\mu = \theta \quad = \text{volume expansion rate in LRF} \tag{74}$$

How about energy density:

$$\dot{e}_0 = \frac{de}{d\tau} = -(e_0 + p_0)\theta \Longrightarrow de = -e_0 \underbrace{\theta d\tau}_{=\frac{dV}{V}} - p_0 \underbrace{\theta d\tau}_{=\frac{dV}{V}} \tag{75}$$

$$\Longrightarrow Vde_0 = -e_0 dV - p_0 dV \tag{76}$$

$$\Longrightarrow d(Ve_0) = -p_0 dV \tag{77}$$

$$\Longrightarrow d(E) = -p_0 dV \tag{78}$$

The $p_0\theta$ term represents the work done by the pressure as the system expands.
The final equation $\dot{u}^\mu = \frac{\nabla^\alpha p_0}{e_0 + p_0}$ gives the accelaration of the fluid element due to the pressure gradient.

The final equations of motion now read:

$$\dot{n}_0 = -n_0\theta \quad (79)$$

$$\dot{e}_0 = -(e_0 + p_0)\theta \quad (80)$$

$$\dot{u}^\mu = \frac{\nabla^\alpha p_0}{e_0 + p_0} \quad (81)$$

Contain 6 unknowns and 5 equations. Equation of state in the form $p_0 = p_0(e_0, n_0)$ sufficient to close the system!

Ideal fluid dynamics

For given initial conditions $n_0(t_0, \mathbf{x}), e_0(t_0, \mathbf{x}), u^\mu(t_0, \mathbf{x})$, the equations of motion can be solved $\Longrightarrow$ space-time evolution of n_0, e_0, u^μ.

Some results for ideal fluids: entropy conservation

In ideal fluids the entropy is conserved. This can be shown as:

$$\partial_\mu S_0^\mu = \partial_\mu \left(s_0 u^\mu \right) = \dot{s}_0 + s_0 \theta \tag{82}$$

Laws of thermodynamics say that

$$s_0 = \frac{e_0 + p_0 - \mu n_0}{T}, \tag{83}$$

$$\Longrightarrow \partial_\mu S_0^\mu = \left(\frac{e_0 + p_0 - \mu n_0}{T} \right) \theta + \dot{s}_0 \tag{84}$$

Furthermore thermodynamics relates change of entropy to change in energy and number density, i.e.

$$T ds_0 = de_0 - \mu dn_0 \tag{85}$$

It then follows that

$$\dot{s}_0 = \frac{1}{T} \left(\dot{e}_0 - \mu \dot{n}_0 \right) = -\frac{1}{T} \left(e_0 + p_0 - \mu n_0 \right) \theta \tag{86}$$

The last step follows from the conservation laws $\dot{e}_0 = -(e_0 + p_0)\theta$ and $\dot{n}_0 = -n_0\theta$. It then follows that

$$\partial_\mu S_0^\mu = \left(\frac{e_0 + p_0 - \mu n_0}{T} \right) \theta - \frac{1}{T} \left(e_0 + p_0 - \mu n_0 \right) \theta = 0 \tag{87}$$

Some further results for ideal fluids: speed of sound

Consider small perturbations around hydrostatic equilibrium state with constant e_0, n_0 at rest $u^\mu = (1,0,0,0)$:

$$n = n_0 + \delta n \qquad e = e_0 + \delta e \qquad u^\mu = u_0^\mu + \delta u^\mu = \gamma\,(1, \delta v_x, 0, 0) \tag{88}$$

Substitute these into eom's, and neglect powers $O(\delta^2)$ and higher.

$$\dot{n} = \dot{n_0} + \dot{\delta n} = -(n_0 + \delta n)\partial_\mu u^\mu = -n_0 \partial_\mu \delta u^\mu - \delta n \partial_\mu \delta u^\mu = -n_0 \partial_x \delta v_x \tag{89}$$

$$\dot{e} = \dot{e_0} + \dot{\delta e} = -(e_0 + \delta e + p_0 + \delta p)\partial_\mu u^\mu = -\left(e_0 + p_0\right)\partial_x \delta v_x \tag{90}$$

$$\begin{aligned}
(e_0 + \delta e + p_0 + \delta p)\dot{u}^\mu &= -\delta_x p(e, n) && (91)\\
(e_0 + p_0)\delta \dot{v}_x &= -\partial_x p(e_0 + \delta e, n_0 + \delta n) && (92)\\
(e_0 + p_0)\delta \dot{v}_x &= -\partial_x p(e_0 + \delta e, n_0 + \delta n) && (93)
\end{aligned}$$

$$p(e_0 + \delta e, n_0 + \delta n) = p_0(e_0, n_0) + \frac{\partial p(e,n)}{\partial e}\delta e + \frac{\partial p(e,n)}{\partial n}\delta n \tag{94}$$

$$(e_0 + p_0)\delta \dot{v}_x = \frac{\partial p(e,n)}{\partial e}\partial_x \delta e + \frac{\partial p(e,n)}{\partial n}\partial_x \delta n \tag{95}$$

Equations of motion for small perturbations:

$$\begin{aligned} \dot{\delta n} &= -n_0 \partial_x \delta v_x && (96) \\ \dot{\delta e} &= -(e_0 + p_0)\,\partial_x \delta v_x && (97) \\ (e_0 + p_0)\dot{\delta v_x} &= \frac{\partial p(e,n)}{\partial e}\partial_x \delta e + \frac{\partial p(e,n)}{\partial n}\partial_x \delta n && (98) \end{aligned}$$

$$\Longrightarrow \ddot{\delta e} = -(e_0 + p_0)\,\partial_x \dot{\delta v_x} \qquad (99)$$

$$\Longrightarrow \ddot{\delta e} = -\frac{\partial p(e,n)}{\partial e}\partial_x^2 \delta e - \frac{\partial p(e,n)}{\partial n}\partial_x^2 \delta n \qquad (100)$$

Thermodynamics:

$$\left.\begin{aligned} Tds &= de - \mu dn \\ e + p &= Ts + \mu n \end{aligned}\right\} \longrightarrow Td\left(\frac{s}{n}\right) = \frac{1}{n}de - \frac{e+p}{n}\frac{dn}{n} \qquad (101)$$

But in ideal fluid both N and S conserved $\longrightarrow d\left(\frac{s}{n}\right) = 0$

$$\Longrightarrow de = \frac{e+p}{n}dn \qquad (102)$$

$$\Longrightarrow \ddot{\delta e} = -\frac{\partial p(e,n)}{\partial e}\partial_x^2 \delta e - \frac{\partial p(e,n)}{\partial n}\partial_x^2 \delta n \tag{103}$$

and using

$$de = \frac{e+p}{n} dn \tag{104}$$

$$\Longrightarrow \ddot{\delta e} = -\frac{\partial p(e,n)}{\partial e}\partial_x^2 \delta e - \frac{n}{e+p}\frac{\partial p(e,n)}{\partial n}\partial_x^2 \delta e \tag{105}$$

But this is just wave equation:

$$\Longrightarrow \ddot{\delta e} = -c_s^2 \partial_x^2 \delta e \tag{106}$$

with wave propagation speed c_s:

$$c_s^2 = \frac{\partial p(e,n)}{\partial e} + \frac{n}{e+p}\frac{\partial p(e,n)}{\partial n} \tag{107}$$

Section 4

Ideal fluids in HI-collisions

Kinematics:

particle 4-momentum:

$$p^\mu = (E, \mathbf{p}) = (E, p_x, p_y, p_z) \tag{108}$$

$E = \sqrt{\mathbf{p}^2 + m^2}$
rapidity:

$$y = \frac{1}{2} \ln \left(\frac{E + p_z}{E - p_z} \right) \tag{109}$$

Show that

$$p^\mu = (m_T \cosh y, \mathbf{p}_T, m_T \sinh y) \,, \tag{110}$$

where $\mathbf{p}_T = (px, py)$ and $m_T = \sqrt{m^2 + \mathbf{p}_T^2}$
Pseudo-rapidity η_p

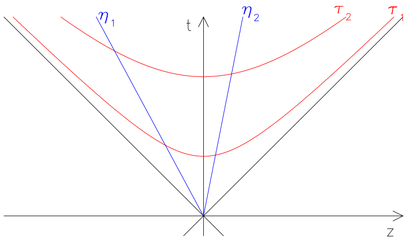
$$\eta_p = \frac{1}{2} \ln \left(\frac{p + p_z}{p - p_z} \right) \tag{111}$$

where $p = |\mathbf{p}|$
Show that

$$\eta_p = -\ln \left[\tan \left(\frac{\theta}{2} \right) \right] \tag{112}$$

where θ is the scattering angle of the particle, measured w.r.t positive direction of the beam axis.
$\theta = 90^o = \eta_p = 0$

Bjorken estimate for the initial energy density in HI collisions



Spacetime rapidity:

$$\eta_s = \frac{1}{2}\ln\left(\frac{t+z}{t-z}\right) = \frac{1}{2}\ln\left(\frac{1+z/t}{1-z/t}\right)$$

Longitudinal proper time

$$\tau = \sqrt{t^2 - z^2} = \sqrt{1-(z/t)^2}$$

τ is the time in the frame that moves with a velocity $v_z = \frac{z}{t}$ w.r.t. to the LAB frame.

- Particles produced at $z = t = 0$
- Free streaming particles: $\mathbf{v}$ =const.
- $\longrightarrow$ particle at (t, z) has $v_z = z/t$

$$\longrightarrow \frac{dE}{d\eta_s} = \text{const.}$$

Spacetime rapidity cannot be measured, but rapidity y can.

$$\begin{aligned} y &= \frac{1}{2}\ln\left(\frac{E+p_z}{E-p_z}\right) = \frac{1}{2}\ln\left(\frac{1+p_z/E}{1-p_z/E}\right) \\ &= \frac{1}{2}\ln\left(\frac{1+v_z}{1-v_z}\right) \end{aligned}$$

$$\longrightarrow \frac{dE}{dy} = \frac{dE}{d\eta_s} = \text{const.}$$

Bjorken estimate for the initial energy density in HI collisions

At $y \sim 0$

$$\frac{dE_T}{dy} \sim \frac{dE}{dy} \sim \frac{dE}{d\eta_s} \tag{113}$$

If we have a measured dE_T/dy we can estimate the energy density at τ_0
Volume at τ_0 (assume homogeneous matter in (x, y)-direction.

$$\Delta V = A\Delta z = \pi R_A^2 \tau_0 \Delta\eta_s \tag{114}$$

Energy density:

$$e\left(\tau = \tau_0\right) = \frac{\Delta E}{\Delta V} = \frac{1}{\pi R_A^2 \tau_0}\frac{\Delta E}{\Delta \eta_s} \sim \frac{1}{\pi R_A^2 \tau_0}\frac{dE_T}{dy} \tag{115}$$

Note that in this case (Free streaming particles)

$$e(\tau) = e\left(\tau_0\right)\frac{\tau_0}{\tau} \tag{116}$$

- SPS: $\frac{dE_T}{dy} \sim 400 GeV \longrightarrow e\left(\tau_0 = 1 \text{ fm}\right) \sim 3 \text{ GeV/fm}^3$
- RHIC: $\frac{dE_T}{dy} \sim 620 GeV \longrightarrow e\left(\tau_0 = 1 \text{ fm}\right) \sim 5 \text{ GeV/fm}^3$

Bjorken fluid dynamics

Dense systems! Neglecting interactions might not be a good idea...
Scaling flow/Bjorken (0+1)-D fluid dynamics, assume

- fluid velocity $v_z = \frac{z}{t}$
- $v_x = v_y = 0$
- Transversaly homogeneous matter
- initial conditions: $e_0(\tau_0, x, y, \eta_s) = e_0(\tau)$ and $n(\tau_0, x, y, \eta_s) = n_0(\tau)$ (independent of η_s: invariant under longitudinal boosts)

In this case

$$\theta = \frac{1}{\tau}, \tag{117}$$

and the equations of motion take a simple form: Charge conservation:

$$\frac{dn}{d\tau} = -\frac{n}{\tau} \tag{118}$$

Energy conservation:

$$\frac{de}{d\tau} = -\frac{e+p}{\tau} \tag{119}$$

Momentum conservation

$$\frac{\partial p}{\partial \eta_s} = 0 \tag{120}$$

guarantees that if the initial conditions are boost invariant, so are the solutions.

Bjorken fluid dynamics

If we assume a simple EoS: $p = c_s^2 e$ the solutions are

$$n = n_0 \left(\frac{\tau_0}{\tau}\right) \tag{121}$$

$$e = e_0 \left(\frac{\tau_0}{\tau}\right)^{1+c_s^2} \tag{122}$$

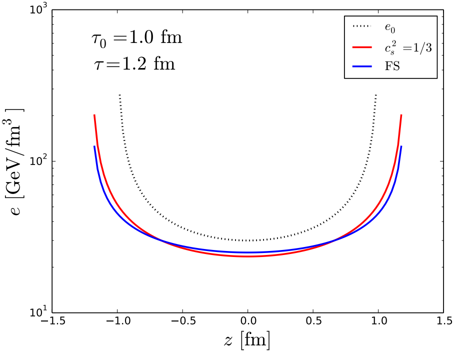
Remember: In the free streaming case $e = e_0 \left(\tau_0/\tau\right)$. Interacting system does work during the expansion, and e drops faster.

$dE_T/d\eta_s$ is not conserved!

Ex: Assume that the system behaves as free streming particles after $\tau = 10$ fm, and before that according to fluid dynamics. Recalculate the estimates for $e_0(\tau_0 = 1 \text{ fm})$ for SPS and RHIC.

So where does the energy go (it is conserved after all). Energy density profile at fixed t not τ.

$$e(t, z) = e(\tau = \sqrt{t^2 - z^2}) \tag{123}$$



Energy pushed into the longitudinal direction $\longrightarrow$ transverse energy decreases during the evolution.

Equation of state: simple model

Hadron gas: 3 massless pions

$$p_{\mathrm{HG}} = g_{\mathrm{HG}} \frac{\pi^2}{90} T^4, \; g_{\mathrm{HG}} = 3 \left(\pi^{\pm}, \pi^0\right) \tag{124}$$

Quark-Gluon plasma: massless gluons and quarks

$$p_{\mathrm{QGP}} = g_{\mathrm{QGP}} \frac{\pi^2}{90} T^4 - B \tag{125}$$

$$g_{\mathrm{QGP}} = 16 + \frac{21}{2} N_f = 37 \tag{126}$$

Gibbs phase coexistence condition (at a given temperature a phase with larger pressure is the stable phase)

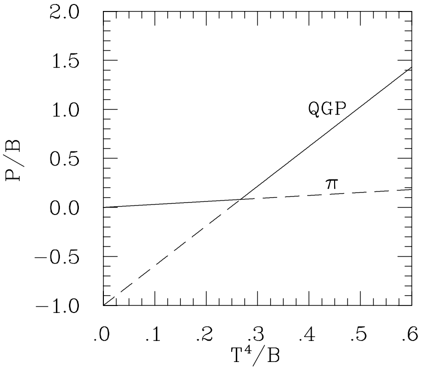
$$p_{\mathrm{HG}}(T_c) = p_{\mathrm{QGP}}(T_c) \tag{127}$$

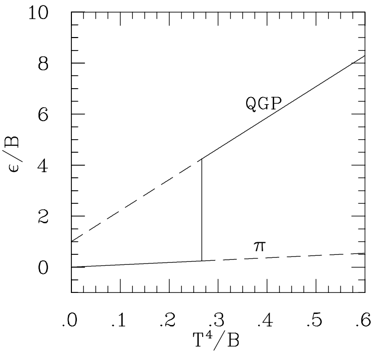
$$g_{\mathrm{HG}} \frac{\pi^2}{90} T_c^4 = g_{\mathrm{QGP}} \frac{\pi^2}{90} T_c^4 - B \tag{128}$$

$$B = \left(g_{\mathrm{QGP}} - g_{\mathrm{HG}}\right) \frac{\pi^2}{90} T_c^4 \tag{129}$$

Bag constant required to have HG as stable phase at low temperatures.

First order phase transition: latent heat & discontinuity in $\partial p/\partial T$





In mixed phase $T = T_c$ and p constant. Energy density changes by changing volume fraction of QGP, x_{QGP}

$$e = x_{\mathrm{QGP}}\, e_{\mathrm{QGP}}(T_c) + (1 - x_{\mathrm{QGP}})\, e_{\mathrm{HG}}(T_c) \qquad (130)$$

Figs: Pasi Huovinen

Equation of State

I. Hadron phase: approximate as non-interacting gas of hadrons & hadron resonance states: π, K, n, p, ρ, ω, ... Hadron resonance gas (HRG)

$$\begin{aligned} P(T,\mu) &= \sum_i g_i \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{\mathbf{p}^2}{3E} \frac{1}{e^{(E-\mu_i)/T} \pm 1} \\ &= \sum_i \frac{g_i}{2\pi^2} T^2 m_i^2 \sum_{n=1}^{\infty} \frac{(\mp 1)^{n+1}}{n^2} e^{n\frac{\mu_i}{T}} \mathrm{K}_2\left(n\frac{m_i}{T}\right) \end{aligned}$$

II. QGP

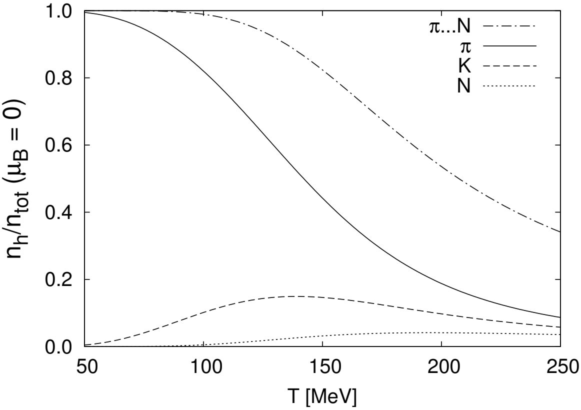
- Massless gas of quarks and gluons:

$$p_{\mathrm{QGP}} = g_{\mathrm{QGP}} \frac{\pi^2}{90} T^4 - B \tag{131}$$

  Connect with the HRG using Maxwell construction as we did with pion gas.
  $\longrightarrow$ Still 1st order transition.

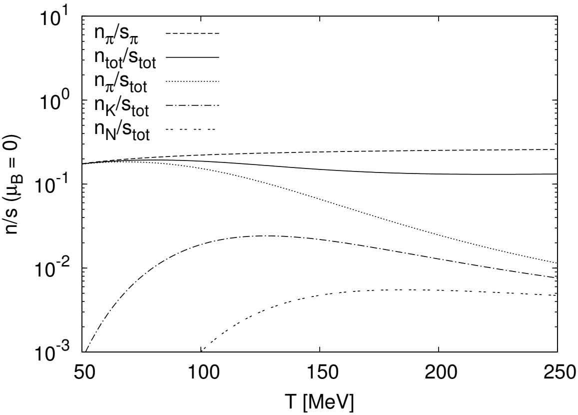
- Connect HRG to the high-temperature lattice QCD results
  $\longrightarrow$ "smooth cross-over"

Hadron gas: relative abundance of hadron species



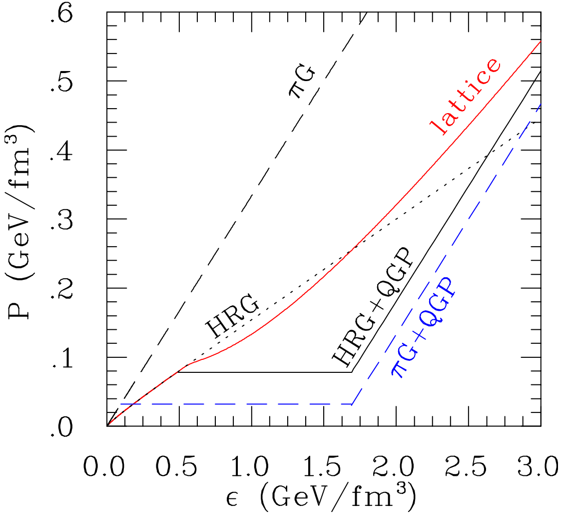
- Pions most abundant, but significant contribution from other hadrons.

Hadron gas: hadron density vs entropy density



- In boost-invariant ideal fluid: dS/η_s = constant
- In hadron gas $n_{\rm tot} \sim Cs$
- $\longrightarrow$ dN/η_s determined by the initial entropy dS/η_s

Comparison of different EoS models



Figs: Pasi Huovinen

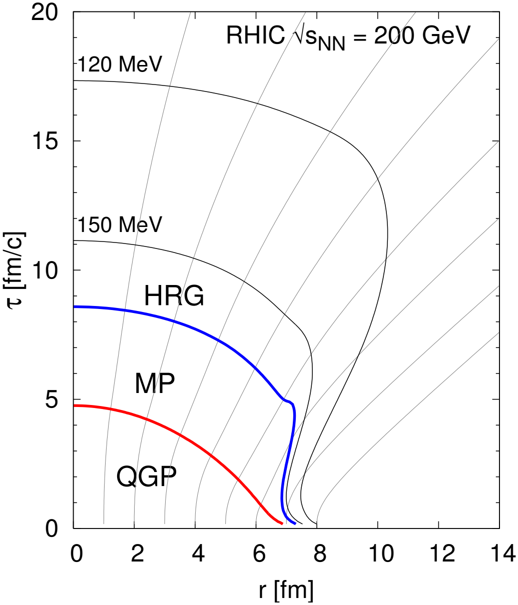
More realistic fluid-dynamical model: transverse expansion

- $(0+1)$D Bjorken model great for a simple estimates of the density evolution, but neglects completely the transverse flow!
- The most interesting and visible consequences of fluid-dynamical behaviour seen in effects of transverse flow on the p_T-spectra.
- Let's still assume boost invariance, i.e. $v_z = z/t$ and fluid dynamical variables independent of η_s.
- In this case more convenient to write the conservation laws directly in terms of $T^{\mu\nu}$

$$
\begin{aligned}
\partial_\tau T^{\tau\tau} &= -\partial_x(v_x T^{\tau\tau}) - \partial_y(v_y T^{\tau\tau}) - \frac{1}{\tau}\left(T^{\tau\tau} + p\right) - \partial_x\left(v_x p\right) - \partial_y\left(v_y p\right), \\
\partial_\tau T^{\tau x} &= -\partial_x(v_x T^{\tau x}) - \partial_y(v_y T^{\tau x}) - \frac{1}{\tau} T^{\tau x} - \partial_x p, \\
\partial_\tau T^{\tau y} &= -\partial_x(v_x T^{\tau y}) - \partial_y(v_y T^{\tau y}) - \frac{1}{\tau} T^{\tau y} - \partial_y p, \\
\partial_\tau N^{\tau} &= -\partial_x(v_x N^{\tau}) - \partial_y(v_y N^{\tau}) - \frac{1}{\tau} N^{\tau}.
\end{aligned}
\tag{132}
$$

Initial conditions: $e_0(\tau_0, x, y)$, $n_0(\tau_0, x, y)$, $\mathbf{v}_T(\tau_0, x, y)$

Example: spacetime evolution of temperature and velocity



Connecting fluid dynamics to the obervable hadron spectra

"Cooper-Frye decoupling procedure"[1]
Fluid dynamics:(u^μ, e, p) – not directly observable.
Fluid dynamical evolution ends at some point and the particles fly to the detector. Condition: e.g. when temperature drops below some $T_{\rm dec}$

- Find a decoupling surface σ from the spacetime evolution
  $\longrightarrow$ surface normal vector $d\sigma_\mu$
- Number of particles traveling through this surface

$$N = \int_\sigma N^\mu d\sigma_\mu \tag{133}$$

- Kinetic theory:

$$N^\mu = \int \frac{d^3\mathbf{p}}{p^0} p^\mu f(x,p) \tag{134}$$

$$\Longrightarrow$$

$$N = \int_\sigma \frac{d^3\mathbf{p}}{p^0} p^\mu f(x,p) d\sigma_\mu \tag{135}$$

[1] F. Cooper and G. Frye, Phys.Rev. **D10**, 186 (1974)

Cooper-Frye integral

$$E\frac{dN}{d^3\mathbf{p}} = \int_\sigma p^\mu f(x,p) d\sigma^\mu \tag{136}$$

- Needs to be calculated for all the particles included into HRG Equation of State.
- Most hadrons unstable and decay before they can be detected: also decays need to be calculated.
- Calculates the net-flux of particles through the surface (not emission from the surface)
- At some points of the surface with some values of **p** the net-flux can be inside the surface: negative contributions.

Cooper-Frye: Boost invariant flow

The freeze out hypersurface can be parametrized as $\tau = \tau(x, y)$, so that surface is given by $\sigma^\mu = (\tau(x, y), x, y, 0)$. Because the system is boost-invariant the surface is the same in any *boosted* frame, i.e. it is independent of η. However, we want the whole surface in the LAB frame. Therefore, we need to boost the surface vector at non-zero η to LAB frame, and this results in

$$\sigma^\mu = (\sigma^t, \sigma^x, \sigma^y, \sigma^z) = (\tau(x, y) \cosh \eta, x, y, \tau(x, y) \sinh \eta) \tag{137}$$

The surface element of the freeze out hypersurface can be generally given as:

$$d\sigma_\mu = \epsilon_{\mu\nu\lambda\rho} \frac{\partial \sigma^\nu}{\partial u} \frac{\partial \sigma^\lambda}{\partial v} \frac{\partial \sigma^\rho}{\partial w} dudvdw \,, \tag{138}$$

where $\epsilon_{\mu\nu\lambda\rho}$ is the antisymmetric fourth rank tensor i.e. permutation tensor, such that $\epsilon_{0123} = -1$, for and even permutation of the indices, and $(u, v, w) = (\eta, x, y)$, thus

$$d\sigma_\mu = -[\pm](\cosh \eta, -\frac{\partial \tau}{\partial x}, -\frac{\partial \tau}{\partial y}, -\sinh \eta)\tau d\eta dxdy \,, \tag{139}$$

where $[\pm] = \mathrm{Sign}(\partial T/\partial \tau)$ guarantees that the surface normal points outside of the surface (with $dxdyd\eta$ positive), i.e. towards smaller temperature.

Cooper-Frye: Boost invariant flow

The Cooper-Frye formula for ideal fluid is:

$$E\frac{dN}{d^3\mathbf{p}} = \int_\sigma d\sigma_\mu(x) p^\mu f_{\rm eq}(x,p) \tag{140}$$

where the distribution function is usually approximated by an equilibrium distribution,

$$f(x,p) = \frac{1}{(2\pi)^3}\frac{1}{\exp\left(\frac{p^\mu u_\mu - \mu}{T}\right) \pm 1} \,. \tag{141}$$

This can be simplified noting that the distribution function can be expanded (like $1/(1 \pm x)$),

$$\frac{1}{e^x \pm 1} \equiv \frac{e^{-x}}{1 \pm e^{-x}} = e^{-x}\sum_{n=0}^{\infty}(\mp 1)^n e^{-nx} = \sum_{n=1}^{\infty}(\mp 1)^{n-1} e^{-nx} \,. \tag{142}$$

$$p^\mu \equiv (m_T \cosh y, \mathbf{p}_T, m_T \sinh y) = (m_T \cosh y, p_x, p_y, m_T \sinh y)\,, \tag{143}$$

$$u^\mu = (u^t, u^x, u^y, u^z) = \gamma_T(\cosh\eta_s, \mathbf{v}_T, \sinh\eta_s)\,, \tag{144}$$

$$\Longrightarrow$$

$$p^\mu u_\mu = \gamma_T\left[m_T \cosh(y - \eta_s) - \mathbf{p}_T \mathbf{v}_T\right] \tag{145}$$

and

$$p^\mu d\sigma_\mu = -[\pm]\left[m_T \cosh(y - \eta_s) - p^x\frac{\partial\tau}{\partial x} - p^y\frac{\partial\tau}{\partial y}\right]\tau d\eta_s dx dy \,. \tag{146}$$

Cooper-Frye: Boost invariant flow

Using the above results we get,

$$E\frac{dN}{d^3p} = \frac{1}{(2\pi)^3}\sum_{n=1}^{\infty}(\mp 1)^{n-1}\int_\sigma d\sigma_\mu p^\mu \exp\left(\frac{n\,\mu - n\,p^\mu u_\mu}{T}\right). \tag{147}$$

Thus,

$$\begin{aligned} E\frac{dN}{d^3p} &= \frac{-1}{(2\pi)^3}\sum_{n=1}^{\infty}(\mp 1)^{n-1}\int_S [\pm]\tau dxdy\, e^{n\mu/T}\exp\left[n\,\frac{\mathbf{p}_T\cdot\mathbf{v}_T}{T}\right] \\ &\times \int_{-\infty}^{\infty} d\eta_s \exp\left[-n\frac{\gamma_T m_T}{T}\cosh(y-\eta_s)\right]\left[m_T\cosh(y-\eta_s) - p^x\frac{\partial\tau}{\partial x} - p^y\frac{\partial\tau}{\partial y}\right] \end{aligned} \tag{148}$$

The η_s integral can be done analytically:

$$\begin{aligned} E\frac{dN}{d^3p} &= \frac{-1}{(2\pi)^3}\sum_{n=1}^{\infty}(\mp 1)^{n-1}\int_S [\pm]\tau dxdy\, e^{n\mu/T}\exp\left[n\,\frac{\mathbf{p}_T\cdot\mathbf{v}_T}{T}\right] \\ &\times \left[m_T K_1\left(n\frac{\gamma_T m_T}{T}\right) - \left(p^x\frac{\partial\tau}{\partial x} + p^y\frac{\partial\tau}{\partial y}\right)K_0\left(n\frac{\gamma_T m_T}{T}\right)\right]. \end{aligned} \tag{149}$$

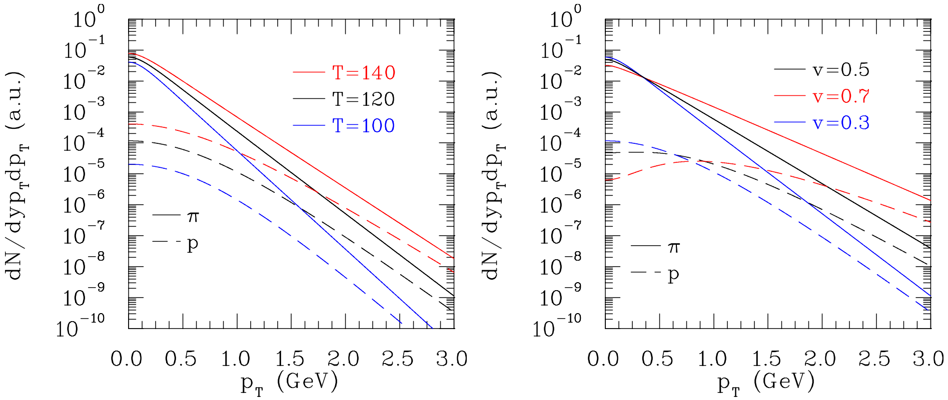
Cooper-Frye: Boost invariant flow

$$E\frac{dN}{d^3p} = \frac{dN}{dyd^2\mathbf{p}_T} = \frac{-1}{(2\pi)^3}\sum_{n=1}^{\infty}(\mp 1)^{n-1}\int_S [\pm]\tau dxdy\, e^{n\mu/T}\exp\left[n\frac{\mathbf{p}_T\cdot\mathbf{v}_T}{T}\right] \tag{150}$$
$$\times \int_{-\infty}^{\infty} d\eta_s \exp\left[-n\frac{\gamma_T m_T}{T}\cosh(y-\eta_s)\right]\left[m_T\cosh(y-\eta_s) - p^x\frac{\partial\tau}{\partial x} - p^y\frac{\partial\tau}{\partial y}\right]$$

Some consequences of the boost invariance:

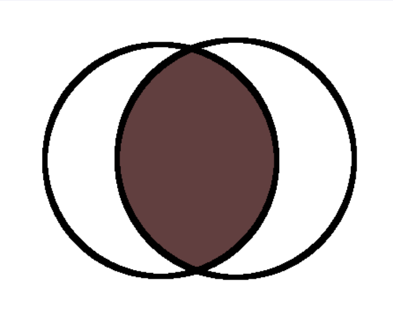
- η_s and y dependence only through $y-\eta_s$
- Spectrum is independent of y, i.e. dN/dy=const. (make a change of variable $\eta' = y - \eta_s$)
- We can change $y \leftrightarrow \eta_s \implies$
$$\frac{dN}{d\eta_s} = \frac{dN}{dy} \tag{151}$$
- spectra at rapidity y is dominated by contribution from $\eta_s \sim y$ (other regions suppressed by $\exp\left[-n\frac{\gamma_T m_T}{T}\cosh(y-\eta_s)\right]$
$\longrightarrow$ spectra at $y \sim 0$ really probes the $\eta_s \sim 0$ (or $z \sim 0$) region.

Effect of radial flow and temperature



- high-p_T behaviour: $e^{-u\cdot p/T} = e^{-(E\gamma-\gamma v_r\cdot p_T)/T} \longrightarrow e^{-p_T/T_{\rm eff}}$
- $T_{\rm eff} = T\sqrt{\frac{1+v_r}{1-v_r}}$
- Increasing temperature and increasing flow velocity have similar effects on the p_T-spectra.
- Fluid dynamics gives a connection between v_r and T

Constraints for shear viscosity from (elliptic) flow



- in non-central collisions pressure gradients asymmetric in transverse plane
- More particles flow into direction of larger gradient

$\longrightarrow$ azimuthally asymmetric particle spectra $dN/d\phi$

Quantify by Fourier coefficients (v_n)

$$\frac{1}{N}\frac{dN}{d\phi} = 1 + 2v_1 cos(\phi - \Psi_1) + 2v_2 cos(2(\phi - \Psi_2)) + 2v_3 cos(3(\phi - \Psi_3)) + \cdots$$

- v_2 = elliptic flow (Ψ_2 is its direction)

The v_n coefficients can depend on rapidity and transverse momentum.

$$\frac{dN}{dy dp_T^2 d\phi} = \frac{dN}{dy dp_T^2}\left[1 + 2v_1(y, p_T)\cos\phi + 2v_2(y, p_T)\cos(2\phi) + \cdots\right]. \tag{152}$$

The rapidity and transverse momentum dependent Fourier coefficients are given by

$$v_n(y, p_T) \equiv \left(\frac{dN}{dy dp_T^2}\right)^{-1} \int_{-\pi}^{\pi} d\phi \cos(n\phi) \frac{dN}{dy dp_T^2 d\phi} \tag{153}$$

Similarly, the p_T-integrated coefficients v_n are given by

$$v_n(y) \equiv \left(\frac{dN}{dy}\right)^{-1} \int_{-\pi}^{\pi} d\phi \cos(n\phi) \frac{dN(b)}{dy d\phi}. \tag{154}$$

In the boost-invariant approximation the coefficients v_n are rapidity independent.

Optical Glauber model and initial conditions

The optical Glauber model for nucleus-nucleus collisions is based on the assumption that each nucleon travels along straight-line trajectories and it is also assumed that the cross section for each nucleon-nucleon collision remains unchanged, even if the nucleons have already collided. In the Glauber model, the total cross section for an $A+B$ collisions is given by

$$\sigma_{AB} = \int d^2\mathbf{b} \left(1 - \left(1 - \frac{\sigma_{NN} T_{AB}(\mathbf{b})}{AB}\right)^{AB}\right) \simeq \int d^2\mathbf{b} \left(1 - e^{-\sigma_{NN} T_{AB}(\mathbf{b})}\right), \qquad (155)$$

where σ_{NN} is the cross section for inelastic nucleon-nucleon collisions, $\mathbf{b}$ is the impact parameter and T_{AB} is the standard nuclear overlap function, defined as

$$T_{AB}(\mathbf{b}) = \int d^2\mathbf{r}\ T_A(\mathbf{r} + \mathbf{b}/2) T_B(\mathbf{r} - \mathbf{b}/2), \qquad (156)$$

with $T_A(\mathbf{r})$ denoting the nuclear thickness function, which is an integral over the longitudinal coordinate z of the nuclear density function,

$$T_A(\mathbf{r}) = \int dz\ \rho_A(\mathbf{r}, z). \qquad (157)$$

The nuclear density can be parametrized with the Woods-Saxon profile

$$\rho_A(\mathbf{r}, z) = \frac{\rho_0}{\exp\left(\frac{r-R_A}{d}\right) + 1}, \tag{158}$$

In the Glauber model the transverse density of binary nucleon-nucleon collisions is given by

$$n_{BC}(\mathbf{r}, \mathbf{b}) = \sigma_{NN} T_A(\mathbf{r} + \mathbf{b}/2) T_B(\mathbf{r} - \mathbf{b}/2). \tag{159}$$

The number density of the nucleons participating in the nuclear collision, the wounded nucleon transverse density, is given by

$$\begin{aligned} n_{WN}(\mathbf{r}, \mathbf{b}) = T_A(\mathbf{r} + \mathbf{b}/2) &\left[1 - \left(1 - \sigma_{NN} \frac{T_B(\mathbf{r} - \mathbf{b}/2)}{B}\right)^B\right] \\ + T_B(\mathbf{r} - \mathbf{b}/2) &\left[1 - \left(1 - \sigma_{NN} \frac{T_A(\mathbf{r} + \mathbf{b}/2)}{A}\right)^A\right]. \end{aligned} \tag{160}$$

The integrals of $n_{WN}(\mathbf{r}, \mathbf{b})$ and $n_{BC}(\mathbf{r}, \mathbf{b})$ over the transverse plane, $\int d^2\mathbf{r}$, give the number of participants $N_{\rm part}(\mathbf{b}) = N_{WN}(\mathbf{b})$ and the number of binary collisions $N_{BC}(\mathbf{b})$, respectively.

Glauber model can be used in several ways to initialize the fluid dynamical evolution.

- $e \propto n_{\mathrm{BC}}$ (eBC)
- $e \propto n_{\mathrm{WN}}$ (eWN)
- $s \propto n_{\mathrm{BC}}$ (sBC)
- $s \propto n_{\mathrm{WN}}$ (eWN)

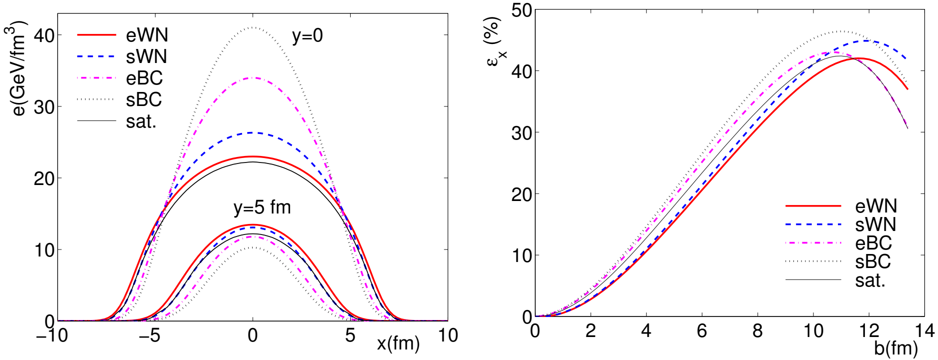
Each of these initializations give a slightly different initial energy density profile.
One way to characterize the shape of the initial profile is to calculate *eccentricity*

$$\varepsilon_x \equiv \frac{\langle y^2 - x^2 \rangle}{\langle y^2 + x^2 \rangle} \equiv \frac{\int dxdy\, \epsilon(x, y, \tau)(y^2 - x^2)}{\int dxdy\, \epsilon(x, y, \tau)(y^2 + x^2)}, \qquad (161)$$

- Non-zero initial eccentricity is converted by the fluid dynamical evolution to non-zero v_2 (elliptic flow)

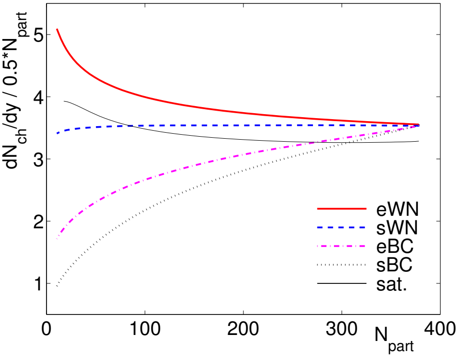
Initial energy density and eccentricity from different Glauber initializations

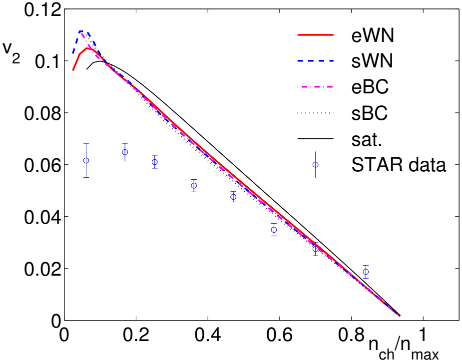
P. Kolb et al., Nucl.Phys. **A696**, 197–215 (2001)



centrality dependence of multiplicity and elliptic flow from different Glauber initializations

P. Kolb et al., Nucl.Phys. **A696**, 197–215 (2001)



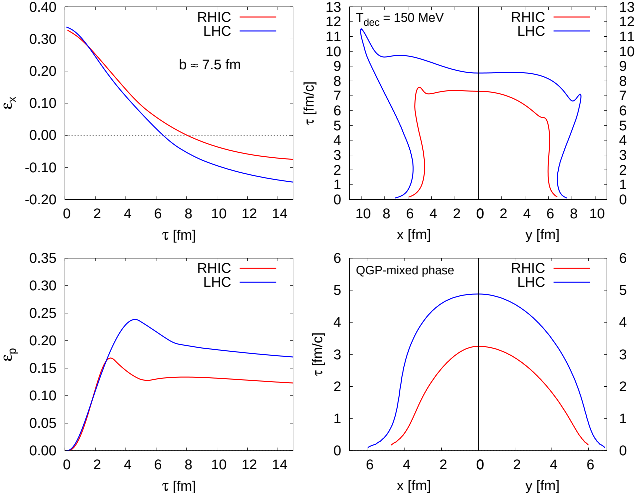


Section 5

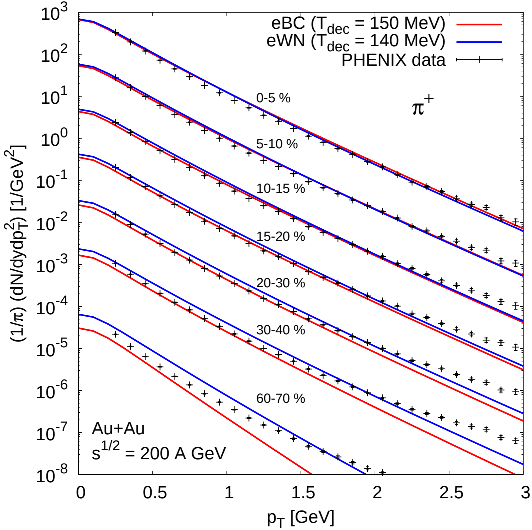
Some results from ideal fluid dynamics

Results from ideal fluid dynamics

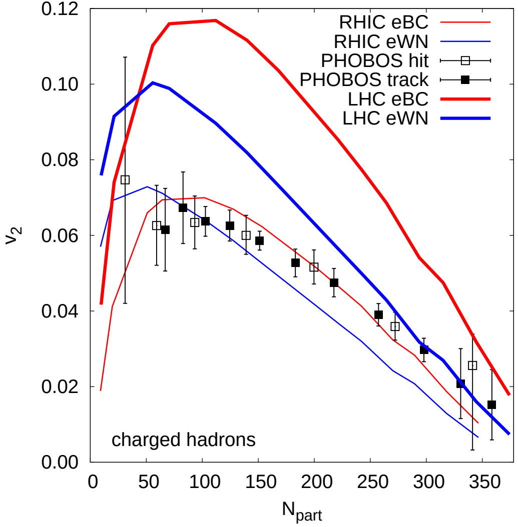
H. Niemi et al., Phys.Rev. **C79**, 024903 (2009)



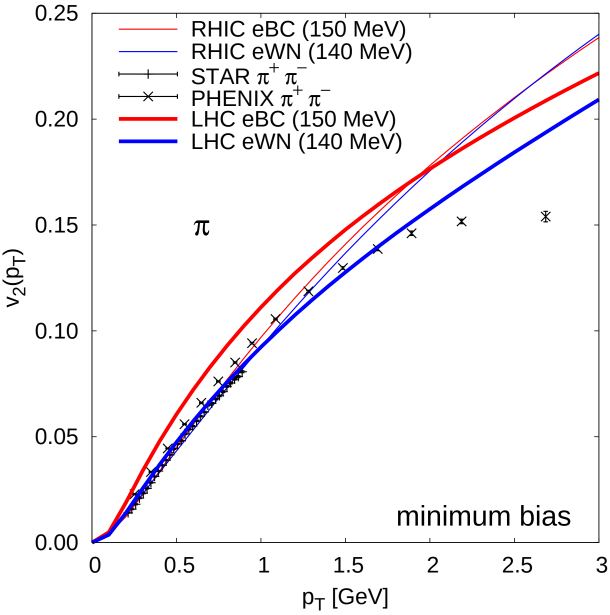
H. Niemi et al., Phys.Rev. **C79**, 024903 (2009)



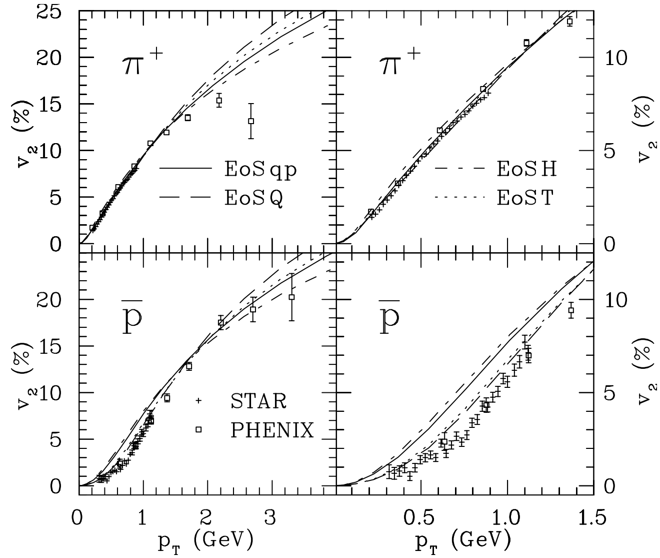
H. Niemi et al., Phys.Rev. **C79**, 024903 (2009)



H. Niemi et al., Phys.Rev. **C79**, 024903 (2009)



Pasi Huovinen: Effect of EoS lattice vs 1st order transition



Section 6

Dissipative fluid dynamics

Dissipative fluid dynamics: general structure of $T^{\mu\nu}$ and N^{μ}

In ideal fluid:

$$T_0^{\mu\nu} = e_0 u^\mu u^\nu - p_0 \Delta^{\mu\nu} \qquad N_0^\mu = n_0 u^\mu \tag{162}$$

This required that the system is in local thermal equilibrium, or (in kinetic theory) $f(x,p)$ is locally isotropic equilibrium distribution function. In this case, full $T^{\mu\nu}$ and N^μ could be described by 5 independent variables (e_0, n_0, u^μ). However, in general $T^{\mu\nu}$ contains 10 independent, and N^μ 4 independent components.

General case: decomposition of $T^{\mu\nu}$ and N^μ w.r.t. u^μ:

At this stage u^μ is an arbitrary, normalized ($u^\mu u_\mu = 1$), 4-vector. Later it will take the meaning of fluid velocity, but as we will see later, in dissipative fluid dynamics it is not uniquely determined. Charge/particle 4-current N^μ can be written as

$$N^\mu = n u^\mu + V^\mu = (n_0 + \delta n) u^\mu + V^\mu, \tag{163}$$

where $n = u_\mu N^\mu$, and $V^\mu = \Delta^\mu_\nu N^\nu$. Note that V^μ is orthogonal to the fluid velocity: $u_\mu V^\mu = 0$. It can also be further divided into equilibrium and off-equilibrium parts.
$\Rightarrow$ Two ways of transporting charge: by fluid flow (convection) nu^i or by diffusion V^μ.

$T^{\mu\nu}$ can be divided in a similar fashion into equilibrium and off-equilibrium parts:

$$T^{\mu\nu} = T_0^{\mu\nu} + \delta T^{\mu\nu} = e_0 u^\mu u^\nu - p_0 \Delta^{\mu\nu} + \delta T^{\mu\nu}. \quad (164)$$

The off-equilibrium part can be further decomposed into scalar, vector, and tensor parts:

$$\delta T^{\mu\nu} = \delta e u^\mu u^\nu - \delta p \Delta^{\mu\nu} + 2W^{(\mu} u^{\nu)} + \pi^{\mu\nu}, \quad (165)$$

where

$$\begin{aligned} e_0 + \delta e &= u_\mu u_\nu T^{\mu\nu} \qquad p_0 + \delta p = -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu} \\ W^\mu &= \Delta^\mu_\nu T^{\nu\alpha} u_\alpha \\ \pi^{\mu\nu} &= T^{\langle\mu\nu\rangle} = \left[\frac{1}{2}\left(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\nu_\alpha \Delta^\mu_\beta\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right] T^{\alpha\beta} \end{aligned} \quad (166)$$

These have properties: $u_\mu W^\mu = u_\mu \pi^{\mu\nu} = u_\nu \pi^{\mu\nu} = \pi^\mu_\mu = 0$

- W^μ = energy diffusion current (Energy flux orthogonal to the fluid velocity)
- $\pi^{\mu\nu}$ = Symmetric, traceless part of $T^{\mu\nu}$ that is orthogonal to the fluid velocity. = shear-stress tensor "Momentum diffusion currents"
- δe and δp are the off-equilibrium contributions to the energy density and isotropic pressure. (δp is the trace we substracted in the definition of $\pi^{\mu\nu}$)

Dissipative fluid dynamics: Landau matching conditions

$$
\begin{aligned}
N^\mu &= (n_0 + \delta n) u^\mu + V^\mu \\
T^{\mu\nu} &= (e_0 + \delta e) u^\mu u^\nu - (p_0 + \delta p)\Delta^{\mu\nu} + 2W^{(\mu} u^{\nu)} + \pi^{\mu\nu}
\end{aligned}
\tag{167}
$$

These contain

- 6 scalars $(e_0, \delta e, n_0, \delta n, p_0, \delta p)$ (6 components)
- 3 vectors (u^μ, V^μ, W^μ) (9 components)
- 1 2-rank tensor $\pi^{\mu\nu}$ (5 components)

These are 20 components in total, but N^μ and $T^{\mu\nu}$ together have only 14 independent components?

- For a general (off-equilibrium) state, the corresponding equilibrium state $(e_0, n_0, p_0(e_0, n_0))$ is an arbitrary choice.
- Without loss of generality we can choose the equilibrium state such that its energy and number density are the same as those of the actual state itself (Landau matching conditions)

$$
\begin{aligned}
e = e_0 = u_\mu u_\nu T^{\mu\nu}, \text{ i.e. } \delta e = 0 \\
n = n_0 = u_\mu N^\mu, \text{ i.e. } \delta n = 0
\end{aligned}
\tag{168}
$$

- Landau matching conditions $\longrightarrow p = p(e_0, n_0) = p(e, n)$. Note that EoS gives a relation between *equilibrium* quantities.
- It also follows that

$$\begin{aligned} T &= T(e_0, n_0) \\ \mu &= \mu(e_0, n_0) \end{aligned} \tag{169}$$

- Landau matching gives definitions of temperature and chemical potential for an arbitray off-equilibrium state!
- Once we have defined p_0 as $p_0 = p_0(e, n)$, δp can be identified as bulk viscous pressure

$$\delta p = \Pi = \text{ bulk viscous pressure} \tag{170}$$

- Π = off-equilibrium correction to the isotropic pressure

We still have 3 scalars, 3 vectors and 1 tensor = 17 components:

$$\begin{aligned} N^{\mu} &= (n)u^{\mu} + V^{\mu} \\ T^{\mu\nu} &= eu^{\mu}u^{\nu} - (p_0(e, n) + \Pi)\Delta^{\mu\nu} + 2W^{(\mu}u^{\nu)} + \pi^{\mu\nu} \end{aligned} \tag{171}$$

Dissipative fluid dynamics: Choise of fluid velocity

- Remember: so far u^μ was an arbitrary 4-vector
- To give u^μ a physical meaning as a fluid velocity, we must tie it to some physical current in the system.
- 2 common choises are Eckart and Landau definitions:

1. Eckart definition: u^μ defined such that it follows the particle/charge current

$$u_{\mathrm{E}} = \frac{N^\mu}{\sqrt{N_\mu N^\mu}} \tag{172}$$

   - In this case the particle/charge diffusion current vanishes: $V^\mu \equiv 0$

2. Landau definition: u^μ defined as the velocity of energy current

$$u^\mu_{\mathrm{L}} = \frac{T^{\mu\nu} u_{\mathrm{L}\,\nu}}{\sqrt{u_{\mathrm{L}\,\alpha} T^{\alpha\beta} T_{\beta\gamma} u^\gamma_{\mathrm{L}}}}, \text{ or equivalently } T^{\mu\nu} u_{\mathrm{L}\,\nu} = e u^\mu_{\mathrm{L}} \tag{173}$$

   - In this case the energy diffusion current vanishes: $W^\mu \equiv 0$

- u^μ can be chosen to replace the particle diffusion current V^μ, or the energy diffusion current W^μ as an independent variable.

- If no conserved charges: Only Landau choise is possible
- If more than one conserved charges: Only one of the diffusion currents V_i^μ can be made to vanish (or any linear combination of them)

In Landau frame:

$$\left.\begin{array}{lcl} N^\mu & = & nu^\mu + V^\mu \\ T^{\mu\nu} & = & eu^\mu u^\nu - (p_0(e,n) + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} \end{array}\right\} \text{14 variables} \tag{174}$$

$$\left.\begin{array}{lcl} \partial_\mu N^\mu & = & 0 \\ \partial_\mu T^{\mu\nu} & = & 0 \end{array}\right\} \text{5 equations} \tag{175}$$

- 9 more constraints needed in order to close the system of equations $\longleftarrow$ These constraints define the dissipative fluid dynamics
- Similarly to the ideal case eom's can also be written as:

$$\begin{array}{rcl} Dn & = & -n\theta - \partial_\mu V^\mu \\ De & = & -(e + p_0 + \Pi)\theta - \pi^{\mu\nu}\partial_\mu u_\nu \\ (e + P + \Pi)Du^\mu & = & \nabla^\mu(p_0 + \Pi) - \Delta^{\mu\nu}\partial_\lambda \pi_\nu^\lambda \end{array}$$

where

$$D = u^\mu \partial_\mu, \qquad \text{and} \qquad \nabla^\mu = \Delta^{\mu\nu}\partial_\nu.$$

Covariant thermodynamics

In general dissipative fluids deviate from the local thermal equilibrium
$\longrightarrow$ Entropy is no longer conserved, but should increase during the evolution.

Thermodynamical relations:

$$\begin{aligned} e + p &= Ts + \mu n \\ ds &= \beta de - \alpha dn \\ dp &= sdT + nd\mu \end{aligned} \tag{176}$$

where $\beta = 1/T$ and $\alpha = \mu/T$.

Let's write these in more suitable form for us (use $\beta^\mu = \beta u^\mu$):

$$\begin{aligned} S_0^\mu &= p\beta^\mu + T_0^{\mu\nu}\beta_\nu - \alpha N_0^\mu \\ dS_0^\mu &= \beta_\nu dT_0^{\mu\nu} - \alpha dN_0^\mu \\ d(p\beta^\mu) &= N_0^\mu d\alpha - T_0^{\mu\nu} d\beta_\nu \end{aligned} \tag{177}$$

Contracting these with u_μ gives back the usual relations.
Note: Here $T_0^{\mu\nu}$ and N_0^μ are the equilibrium parts of the energy-momentum tensor and particle 4-current.

Now we can write for the equilibrium entropy current S_0^μ

$$\partial_\mu S_0^\mu = \beta_\nu \partial_\mu T_0^{\mu\nu} - \alpha \partial_\mu N_0^\mu \tag{178}$$

For ideal fluids

$$\partial_\mu T_0^{\mu\nu} = 0 \qquad \partial_\mu N_0^\mu = 0 \tag{179}$$

and one gets the usual result $\partial_\mu S_0^\mu = 0$.

However in dissipative fluids:

$$\begin{aligned} \partial_\mu T_0^{\mu\nu} &= \partial_\mu \left[\Pi \Delta^{\mu\nu} - \pi^{\mu\nu}\right] \neq 0 \\ \partial_\mu N_0^\mu &= -\partial_\mu V^\mu \neq 0 \end{aligned} \tag{180}$$

Now

$$\begin{aligned} \partial_\mu S_0^\mu &= \beta_\nu \left[\partial_\mu \left(\Pi \Delta^{\mu\nu} - \pi^{\mu\nu}\right)\right] - \alpha \partial_\mu V^\mu \\ &= \partial_\mu (\alpha V^\mu) - V^\mu \partial_\mu \alpha - \beta \Pi \Delta^{\mu\nu} \partial_\mu u_\nu + \beta \pi^{\mu\nu} \partial_\mu u_\nu \end{aligned} \tag{181}$$

Define $S^\mu = S_0^\mu - \alpha V^\mu$ as the non-equilibrium entropy.

$$\partial_\mu S^\mu = -V^\mu \partial_\mu \alpha - \beta \Pi \Delta^{\mu\nu} \partial_\mu u_\nu + \beta \pi^{\mu\nu} \partial_\mu u_\nu \tag{182}$$

The derivative of the four-velocity can be generally decomposed as

$$\partial_\mu u_\nu = u_\mu D u_\nu + \sigma_{\mu\nu} + \frac{1}{3}\Delta_{\mu\nu}\theta - \omega_{\mu\nu}, \tag{183}$$

where the shear tensor $\sigma_{\mu\nu}$, and the vorticity tensor $\omega_{\mu\nu}$ are defined as

$$\begin{aligned} \sigma_{\mu\nu} &\equiv \nabla_{<\mu} u_{\nu>} = \frac{1}{2}\left(\nabla_\mu u_\nu + \nabla_\nu u_\mu\right) - \frac{1}{3}\Delta^{\mu\nu}\theta \\ \omega_{\mu\nu} &\equiv \frac{1}{2}\Delta^\alpha_\mu \Delta^\beta_\nu \left(\partial_\beta u_\alpha - \partial_\alpha u_\beta\right) \end{aligned} \tag{184}$$

where $\sigma^{\mu\nu} u_\nu = \sigma^\mu_\mu = 0$ and $\omega^{\mu\nu} u_\nu = \omega^\mu{}_\mu = 0$. The entropy production can be written as

$$\partial_\mu S^\mu = -V^\mu \partial_\mu \alpha - \beta \Pi \Delta^{\mu\nu} \partial_\mu u_\nu + \beta \pi^{\mu\nu} \partial_\mu u_\nu \tag{185}$$

$\Longrightarrow$

$$\partial_\mu S^\mu = -V^\mu \partial_\mu \alpha - \beta \Pi \Delta^{\mu\nu} \theta + \beta \pi^{\mu\nu} \sigma_{\mu\nu} = Q \tag{186}$$

Q is the entropy production rate.

Entropy production rate must be positive

$$Q = -V^{\mu}\partial_{\mu}\alpha - \beta\Pi\Delta^{\mu\nu}\theta + \beta\pi^{\mu\nu}\sigma_{\mu\nu} > 0 \tag{187}$$

In general this can be satisfied only if

$$\begin{aligned} V^{\mu} &= \kappa\nabla^{\mu}\alpha \\ \Pi &= -\zeta\theta \\ \pi^{\mu\nu} &= 2\eta\sigma^{\mu\nu} \end{aligned} \tag{188}$$

where κ, ζ and η are positive.

$$\begin{aligned} \kappa &= \text{ particle diffusion or heat conductivity} \\ \zeta &= \text{ bulk viscosity} \\ \eta &= \text{ shear viscosity} \end{aligned} \tag{189}$$

These are properties of the matter similarly to the Equation of state, and in general depend on T and μ, or e and n

Relativistic Navier-Stokes equations

The set of equations

$$\begin{aligned} &\partial_\mu N^\mu = 0 \\ &\partial_\mu T^{\mu\nu} = 0 \\ &V^\mu = \kappa \nabla^\mu \alpha \\ &\Pi = -\zeta\theta \\ &\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} \end{aligned} \tag{190}$$

with $p_0 = p_0(e, n)$, $\kappa = \kappa(e, n)$, $\zeta = \zeta(e, n)$, $\eta = \eta(e, n)$ form a closed set of equations.
Physical interpretation of different terms:

- $\kappa\nabla^\mu\alpha$: drives the diffusion current
- -$\zeta\theta$: resistance to the volume changes of the fluid element
- $2\eta\sigma^{\mu\nu}$: resistace to the deformations of the fluid element

This theory has, however, some problems.

- The equations of motion are parabolic: support infinite signal propagation speed (fluid velocity is still bounded $|\mathbf{v}| < 1$)
- As a consequence: Even hydrostatic equilibrium state is unstable (small perturbations grow exponentially).

This theory is not good for relativistic fluids!

Second-order fluid dynamics

In order to cure the problems with relativistic Navier-Stokes, need to go step further in approximations.
Add terms second order in dissipative currents to the entropy current

$$S^\mu = S_0^\mu + \frac{\mu}{T}\frac{q^\mu}{h} - \left(\beta_0\Pi^2 - \beta_1 V_\nu V^\nu + \beta_2 \pi_{\lambda\nu}\pi^{\lambda\nu}\right)\frac{u^\mu}{2T} - \frac{\alpha_0 V^\mu \Pi}{T} + \frac{\alpha_1 V_\nu \pi^{\nu\mu}}{T}$$

Essentially the same steps as below (requiring positive definite entropy production) lead to the equations of motion for the dissipative quantities that are of the form

$$D\Pi = -\frac{1}{\tau_\Pi}\left(\Pi + \zeta\nabla_\mu u^\mu\right) + \cdots \tag{191}$$

$$DV^\mu = -\frac{1}{\tau_V}\left[V^\mu - \kappa\nabla^\mu\left(\frac{\mu}{T}\right)\right] + \cdots \tag{192}$$

$$D\pi^{\mu\nu} = -\frac{1}{\tau_\pi}\left(\pi^{\mu\nu} - 2\eta\nabla^{\langle\mu}u^{\nu\rangle}\right) + \cdots \tag{193}$$

- Israel-Stewart equations for dissipative quantities.

- The Israel-Stewart equations describe exponential decay of dissipative quantities into NS-values. e.g. $\pi^{\mu\nu} \longrightarrow 2\eta\sigma^{\mu\nu}$ within timescale τ_π
- Note that in the absence of spatial gradients, the dissipative currents vanish exponentially

$$\tau_\Pi D\Pi + \Pi = 0 \Longrightarrow \Pi = \Pi_0 e^{-t/\tau_\Pi} \tag{194}$$

- $\Pi = V^\mu = \pi^{\mu\nu} = 0 \Longleftrightarrow$ equilibrium
- $\longrightarrow \tau_i$ are thermalization time scales, i.e. timescales of the microscopic processes that thermalize the system.
- In IS theory signal propagation speeds are finite and theory is stable and causal.

Second-order fluid dynamics

$$\begin{aligned} D\Pi &= -\frac{1}{\tau_\Pi}\left(\Pi + \zeta\nabla_\mu u^\mu\right) - \frac{1}{2}\Pi\left(\nabla_\mu u^\mu + D\ln\frac{\beta_0}{T}\right) \\ &\quad + \frac{\alpha_0}{\beta_0}\partial_\mu q^\mu - \frac{a_0'}{\beta_0} q^\mu D u_\mu \,, \end{aligned} \tag{195}$$

$$\begin{aligned} Dq^\mu &= -\frac{1}{\tau_q}\left[q^\mu + \kappa_q \frac{T^2 n}{e+p}\nabla^\mu\left(\frac{\mu}{T}\right)\right] - u^\mu q_\nu D u^\nu \\ &\quad - \frac{1}{2} q^\mu \left(\nabla_\lambda u^\lambda + D\ln\frac{\beta_1}{T}\right) - \omega^{\mu\lambda} q_\lambda - \frac{\alpha_0}{\beta_1}\nabla^\mu\Pi \\ &\quad + \frac{\alpha_1}{\beta_1}\left(\partial_\lambda \pi^{\lambda\mu} + u^\mu \pi^{\lambda\nu}\partial_\lambda u_\nu\right) + \frac{a_0}{\beta_1}\Pi D u^\mu - \frac{a_1}{\beta_1}\pi^{\lambda\mu} D u_\lambda \,, \end{aligned} \tag{196}$$

$$\begin{aligned} D\pi^{\mu\nu} &= -\frac{1}{\tau_\pi}\left(\pi^{\mu\nu} - 2\eta\nabla^{\langle\mu}u^{\nu\rangle}\right) - \left(\pi^{\lambda\mu}u^\nu + \pi^{\lambda\nu}u^\mu\right) D u_\lambda \\ &\quad - \frac{1}{2}\pi^{\mu\nu}\left(\nabla_\lambda u^\lambda + D\ln\frac{\beta_2}{T}\right) - 2\pi_\lambda{}^{\langle\mu}\omega^{\nu\rangle\lambda} - \frac{\alpha_1}{\beta_2}\nabla^{\langle\mu}q^{\nu\rangle} + \frac{a_1'}{\beta_2} q^{\langle\mu} D u^{\nu\rangle} \,, \end{aligned} \tag{197}$$

$$q^\mu = W^\mu - \frac{e+p}{n} V^\mu \tag{198}$$

Section 7

Dissipative fluid dynamics from kinetic theory

Boltzmann equation

$$k^{\mu}\partial_{\mu}f_{\mathbf{k}} = C[f]\,, \tag{199}$$

Evolution equation for single particle distribution function $f_{\mathbf{k}} = f(x, \mathbf{k})$ Collision integral for elastic two-to-two collisions with incoming momenta k, k', and outgoing momenta p, p',

$$C[f] = \frac{1}{\nu}\int dK' dP dP' W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} \times \left(f_{\mathbf{p}} f_{\mathbf{p}'} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k}'} - f_{\mathbf{k}} f_{\mathbf{k}'} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p}'} \right), \tag{200}$$

The Lorentz-invariant phase volume:

$$dK \equiv \frac{g d^3\mathbf{k}}{(2\pi)^3 k^0}, \tag{201}$$

with g the number of internal degrees of freedom, e.g. spin degeneracy.
The Lorentz-invariant transition rate $W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'}$ is symmetric with respect to the exchange of the outgoing momentum, as well as for time-reversed processes,

$$W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} \equiv W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}'\mathbf{p}} = W_{\mathbf{p}\mathbf{p}'\to\mathbf{k}\mathbf{k}'}. \tag{202}$$

Here we also take into account quantum statistics and introduced the notation $\tilde{f}_{\mathbf{k}} \equiv 1 - a f_{\mathbf{k}}$, where $a = 1$ ($a = -1$) for fermions (bosons) and $a = 0$ in the limiting case of classical Boltzmann-Gibbs statistics.

Conservation laws

The particle four-flow and the energy-momentum tensor are defined as the first and second moments of the single-particle distribution function,

$$N^{\mu} = \langle k^{\mu} \rangle , \tag{203}$$
$$T^{\mu\nu} = \langle k^{\mu} k^{\nu} \rangle , \tag{204}$$

where we adopted the following notation for the averages

$$\langle \ldots \rangle = \int dK \ (\ldots) \ f_{\mathbf{k}}. \tag{205}$$

If the microscopic scatterings conserve energy and momentum it follows that the particle four-flow and the energy-momentum tensor satisfy the conservation equations for any solution of the Boltzmann equation,

$$\partial_{\mu} \langle k^{\mu} \rangle \equiv \int dKC\,[f] = 0, \tag{206}$$
$$\partial_{\mu} \langle k^{\mu} k^{\nu} \rangle \equiv \int dKk^{\nu} C\,[f] = 0. \tag{207}$$

Macroscopic variables (moments)

As before N^μ and $T^{\mu\nu}$ can be decomposed w.r.t. u^μ as before

$$\begin{aligned} N^\mu &= n_0 u^\mu + V^\mu \\ T^{\mu\nu} &= e_0 u^\mu u^\nu - (p_0 + \Pi)\Delta^{\mu\nu} + 2W^{(\mu} u^{\nu)} + \pi^{\mu\nu} \end{aligned} \tag{208}$$

where the coefficients of the decomposition can be expressed in terms of single particle distribution function

$$\begin{aligned} n_0 &= \langle E_{\mathbf{k}} \rangle \,,\; e_0 = \langle E_{\mathbf{k}}^2 \rangle \,,\; p_0 + \Pi = -\frac{1}{3}\langle \Delta^{\mu\nu} k_\mu k_\nu \rangle \,, \\ V^\mu &= \left\langle k^{\langle\mu\rangle} \right\rangle \,,\; W^\mu = \left\langle E_{\mathbf{k}} k^{\langle\mu\rangle} \right\rangle \,,\; \pi^{\mu\nu} = \left\langle k^{\langle\mu} k^{\nu\rangle} \right\rangle \,. \end{aligned} \tag{209}$$

Here $E_{\mathbf{k}} = u_\mu k^\mu$, $k^{\langle\mu\rangle} = \Delta^\mu_\nu k^\nu$, and $k^{\langle\mu} k^{\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta} k^\alpha k^\beta$

Note that $k^\mu = E_{\mathbf{k}} u^\mu + k^{\langle\mu\rangle}$
We can further define infinitely many macroscopic moments of $f_{\mathbf{k}}$:

$$\langle k^{\mu_1} \cdots k^{\mu_\ell} \rangle \tag{210}$$

Equilibrium state

The local equilibrium distribution is

$$f_{0\mathbf{k}}(x^\mu, k^\mu) = \left[\exp\left(\beta_0 E_\mathbf{k} - \alpha_0\right) + a\right]^{-1}. \tag{211}$$

For $f_{0\mathbf{k}}$ the collision integral vanishes $C\left[f_{0\mathbf{k}}\right] = 0$, and it is a solution of the Boltzmann equation if all the gradients vanish (but only then).
We can introduce the average with respect to the local equilibrium distribution function as

$$\langle \ldots \rangle_0 = \int dK \left(\ldots\right) f_{0\mathbf{k}}, \tag{212}$$

Note that in kinetic theory the equation of state is not a choise, but for example

$$p_0(\alpha_0, \beta_0) = -\frac{1}{3}\Delta_{\mu\nu}\langle k^\mu k^\nu \rangle_0 =^{LRF} \frac{1}{3}\langle \mathbf{k}^2 \rangle_0 \tag{213}$$

In equilibrium all the dissipative quantities vanish:

$$\left\langle k^{\langle\mu\rangle} \right\rangle_0 = \left\langle E_\mathbf{k} k^{\langle\mu\rangle} \right\rangle_0 = \left\langle k^{\langle\mu} k^{\nu\rangle} \right\rangle_0 = 0 \tag{214}$$

Expansion around equilibrium (method of moments)

Since we are interested in near-equilibrium solutions of the Boltzmann equation, we start by expanding $f_{\mathbf{k}}$ around a local equilibrium distribution function $f_{0\mathbf{k}}$:

$$f_{\mathbf{k}} \equiv f_{0\mathbf{k}} + \delta f_{\mathbf{k}} = f_{0\mathbf{k}} \left(1 + \tilde{f}_{0\mathbf{k}} \phi_{\mathbf{k}}\right), \tag{215}$$

where $\phi_{\mathbf{k}}$ represents a general non-equilibrium correction.
As before the equilibrium state (α_0, β_0) is not unique, but needs to be determined through the matching conditions.

$$e = e_0(\alpha_0, \beta_0) \qquad n = n_0(\alpha_0, \beta_0) \tag{216}$$

Israel and Stewart: expansion of $\phi_{\mathbf{k}}$ in basis:

$$1, k^\mu, k^\mu k^\nu, k^\mu k^\nu k^\lambda, \ldots \tag{217}$$

Truncated expansion:

$$\phi_{\mathbf{k}} = \varepsilon + \varepsilon^\mu k_\mu + \varepsilon^{\mu\nu} k_\mu k_\nu \tag{218}$$

The coefficients ε, ε^μ, $\varepsilon^{\mu\nu}$ can be determined by matching to N^μ and $T^{\mu\nu}$

Dissipative fluid dynamics: 14-moment approximation

Shear-stress tensor $\pi^{\mu\nu}$ as an example:

$$\begin{aligned}\pi^{\mu\nu} &= T^{\langle\mu\nu\rangle} = \int dK k^{\langle\mu} k^{\nu\rangle} f_0(T,\mu) \left[1 + \epsilon + \epsilon_\alpha k^\alpha + \epsilon_{\alpha\beta} k^\alpha k^\beta\right] \\ &= \int dK k^{\langle\mu} k^{\nu\rangle} f_0(T,\mu) \epsilon_{\alpha\beta} k^\alpha k^\beta \\ &= \epsilon_{\alpha\beta} \Delta^{\mu\nu}_{\gamma\delta} \underbrace{\int dK k^\gamma k^\delta k^\alpha k^\beta f_0(T,\mu)}_{=I^{\gamma\delta\alpha\beta}}\end{aligned}$$

$$I^{\gamma\delta\alpha\beta} = J_{40} u^\gamma u^\delta u^\alpha u^\beta + 6 J_{41} u^{(\gamma} u^\delta \Delta^{\alpha\beta)} + 3 J_{42} \Delta^{\gamma(\delta} \Delta^{\alpha\beta)} \tag{219}$$

$$\Delta_{\gamma\delta} \Delta_{\alpha\beta} I^{\gamma\delta\alpha\beta} = 3 J_{42} \Delta_{\gamma\delta} \Delta_{\alpha\beta} \Delta^{\gamma(\delta} \Delta^{\alpha\beta)} = 15 J_{42} \tag{220}$$

$$\Longrightarrow J_{42} = \frac{1}{15} \Delta_{\gamma\delta} \Delta_{\alpha\beta} I^{\gamma\delta\alpha\beta} = \frac{1}{15} \int dK \left(\Delta_{\alpha\beta} k^\alpha k^\beta\right)^2 f_0(T,\mu) \tag{221}$$

$$
\begin{aligned}
\pi^{\mu\nu} &= \epsilon_{\alpha\beta}\Delta^{\mu\nu}_{\gamma\delta}\int dK k^{\gamma}k^{\delta}k^{\alpha}k^{\beta}f_0(T,\mu) \\
&= \epsilon_{\alpha\beta}\Delta^{\mu\nu}_{\gamma\delta}3J_{42}\Delta^{\gamma(\delta}\Delta^{\alpha\beta)} \\
&= 2J_{42}\epsilon_{\alpha\beta}\Delta^{\mu\nu\alpha\beta} \\
&= 2J_{42}\epsilon^{\langle\mu\nu\rangle}
\end{aligned}
$$

$$
f = f_0(T,\mu) + \delta f = f_0(T,\mu)\left(1 + \frac{1}{2J_{42}}\pi^{\mu\nu}p_{\mu}p_{\nu}\right) \tag{222}
$$

More convenient to use irreducible tensors:

$$1, k^{\langle\mu\rangle}, k^{\langle\mu} k^{\nu\rangle}, k^{\langle\mu} k^{\nu} k^{\lambda\rangle}, \cdots \tag{223}$$

These irreducible tensors are defined by using the symmetrized traceless projections as

$$k^{\langle\mu_1} \cdots k^{\mu_m\rangle} = \Delta^{\mu_1 \cdots \mu_m \nu_1 \cdots \nu_m} k_{\nu_1} \cdots k_{\nu_m}, \tag{224}$$

Tensors $k^{\langle\mu_1} \cdots k^{\mu_m\rangle}$ satisfy the orthogonality condition:

$$\begin{aligned} & \int dK F_{\mathbf{k}} k^{\langle\mu_1} \cdots k^{\mu_m\rangle} k^{\langle\nu_1} \cdots k^{\nu_n\rangle} \\ = \quad & \frac{m! \delta_{mn}}{(2m+1)!!} \Delta^{\mu_1 \cdots \mu_m \nu_1 \cdots \nu_m} \int dK F_{\mathbf{k}} \left(\Delta^{\alpha\beta} k_\alpha k_\beta \right)^m . \end{aligned} \tag{225}$$

Here $m, n = 0, 1, 2, \cdots$, $F_{\mathbf{k}}$ is an arbitrary scalar function of $E_{\mathbf{k}}$

Using these tensors as the basis of the expansion, the non-equilibrium correction can be written as,

$$\phi_{\mathbf{k}} = \sum_{\ell=0}^{\infty} \lambda_{\mathbf{k}}^{\langle\mu_1\cdots\mu_\ell\rangle} \, k_{\langle\mu_1} \cdots k_{\mu_\ell\rangle}, \tag{226}$$

ℓ is the rank of the tensor $\lambda_{\mathbf{k}}^{\langle\mu_1\cdots\mu_\ell\rangle}$ ($\ell = 0$ is scalar λ). The coefficients $\lambda_{\mathbf{k}}^{\langle\mu_1\cdots\mu_\ell\rangle}$ may still be arbitrary functions of $E_{\mathbf{k}}$, and can be further expanded with another orthogonal basis of functions $P_{\mathbf{k}n}^{(\ell)}$,

$$\lambda_{\mathbf{k}}^{\langle\mu_1\cdots\mu_\ell\rangle} = \sum_{n=0}^{N_\ell} c_n^{\langle\mu_1\cdots\mu_\ell\rangle} P_{\mathbf{k}n}^{(\ell)}, \tag{227}$$

where $c_n^{\langle\mu_1\cdots\mu_\ell\rangle}$ are as of yet undetermined coefficients. The polynomial $P_{\mathbf{k}n}^{(\ell)}$ is a linear combination of powers of $E_{\mathbf{k}}$, while N_ℓ is the number of functions $P_{\mathbf{k}n}^{(\ell)}$ considered to describe the ℓ-th rank tensor $\lambda_{\mathbf{k}}^{\langle\mu_1\cdots\mu_\ell\rangle}$.

The functions $P_{\mathbf{k}n}^{(\ell)}$ are chosen to be polynomials of order n in energy,

$$P_{\mathbf{k}n}^{(\ell)} = \sum_{r=0}^{n} a_{nr}^{(\ell)} E_{\mathbf{k}}^{r}, \tag{228}$$

and are constructed using the following orthonormality condition,

$$\int dK \, \omega^{(\ell)} \, P_{\mathbf{k}m}^{(\ell)} P_{\mathbf{k}n}^{(\ell)} = \delta_{mn}, \tag{229}$$

where the weight $\omega^{(\ell)}$ is defined as

$$\omega^{(\ell)} = \frac{\mathcal{N}^{(\ell)}}{(2\ell+1)!!} \left(\Delta^{\alpha\beta} k_\alpha k_\beta\right)^{\ell} f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \,. \tag{230}$$

The coefficients $a_{nr}^{(\ell)}$ and the normalization constants $\mathcal{N}^{(\ell)}$ can be found via Gram-Schmidt orthogonalization using the orthonormality condition (229).

Finally, the single-particle distribution can be expressed including all moments of the non-equilibrium corrections,

$$f_{\mathbf{k}} = f_{0\mathbf{k}} \left(1 + \tilde{f}_{0\mathbf{k}} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \mathcal{H}_{\mathbf{k}n}^{(\ell)} \rho_n^{\mu_1 \cdots \mu_\ell} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle} \right), \tag{231}$$

where we introduced the following energy-dependent coefficients,

$$\mathcal{H}_{\mathbf{k}n}^{(\ell)} = \frac{\mathcal{N}^{(\ell)}}{\ell!} \sum_{m=n}^{N_\ell} a_{mn}^{(\ell)} P_{\mathbf{k}m}^{(\ell)}. \tag{232}$$

The generalized irreducible moment of $\delta f_{\mathbf{k}}$ is defined as

$$\rho_r^{\mu_1 \cdots \mu_\ell} = \left\langle E_{\mathbf{k}}^r \; k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \right\rangle_\delta, \tag{233}$$

where

$$\langle \ldots \rangle_\delta = \int dK \, (\ldots) \, \delta f_{\mathbf{k}}. \tag{234}$$

Using this notation, the expansion coefficients in Eq. (227) can be immediately determined using Eqs. (225) and (229). For $n \leq N_\ell$ they are given by

$$\begin{aligned} c_n^{\langle \mu_1 \cdots \mu_\ell \rangle} &\equiv \frac{\mathcal{N}^{(\ell)}}{\ell!} \left\langle P_{\mathbf{k}n}^{(\ell)} \; k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \right\rangle_\delta \\ &= \frac{\mathcal{N}^{(\ell)}}{\ell!} \sum_{r=0}^{n} \rho_r^{\mu_1 \cdots \mu_\ell} a_{nr}^{(\ell)}. \end{aligned} \tag{235}$$

So far we have written the single particle distribution function in terms of macroscopic moments

$$\rho_r^{\mu_1\cdots\mu_\ell} = \left\langle E_{\mathbf{k}}^r \; k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} \right\rangle_\delta , \tag{236}$$

Some of these moments can be immediately recognized as dissipative quantities

$$\rho_0 = -3\Pi/m^2, \tag{237}$$
$$\rho_0^\mu = V^\mu, \tag{238}$$
$$\rho_1^\mu = W^\mu, \tag{239}$$
$$\rho_0^{\mu\nu} = \pi^{\mu\nu}. \tag{240}$$

Furthermore, the Landau matching conditions $e = e_0$ and $n = n_0$ imply

$$\rho_1 = \rho_2 = 0. \tag{241}$$

Note that, by definition $\Pi = -\frac{1}{3}\left(m^2\rho_0 - \rho_2\right)$, therefore to match ρ_0 to Π we made use of the above fitting conditions. The definition of the fluid velocity u^μ via Landau's choice ($W^\mu = 0$) implies

$$\rho_1^\mu = 0, \tag{242}$$

while Eckart's definition ($V^\mu = 0$) leads to

$$\rho_0^\mu = 0. \tag{243}$$

Equations of motion

So far, the single-particle distribution function was expressed in terms of the irreducible tensors $\rho_n^{\mu_1\cdots\mu_\ell}$. The time-evolution equations for these tensors can be obtained directly from the Boltzmann equation by applying the comoving derivative to Eq. (236), together with the symmetrized traceless projection as,

$$\dot{\rho}_r^{\langle\mu_1\cdots\mu_\ell\rangle} = \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell} \frac{d}{d\tau} \int dK E_{\mathbf{k}}^r k^{\langle\nu_1} \cdots k^{\nu_\ell\rangle} \delta f_{\mathbf{k}}. \qquad (244)$$

Boltzmann equation $k^\mu \partial_\mu f_{\mathbf{k}} = C[f]$ can be written in the form

$$\delta \dot{f}_{\mathbf{k}} = -\dot{f}_{0\mathbf{k}} - E_{\mathbf{k}}^{-1} k_\nu \nabla^\nu f_{0\mathbf{k}} - E_{\mathbf{k}}^{-1} k_\nu \nabla^\nu \delta f_{\mathbf{k}} + E_{\mathbf{k}}^{-1} C[f]. \qquad (245)$$

using $f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}}$, $k^\mu = E_{\mathbf{k}} u^\mu + k^{\langle\mu\rangle}$, and $\partial_\mu = u^\mu \frac{d}{d\tau} + \nabla^\mu$.
Substituting into Eq. (244), we obtain the *exact* equations for $\dot{\rho}_r^{\langle\mu_1\cdots\mu_\ell\rangle}$.

Equations of motion: second rank tensors

$$\dot{\rho}_r^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta} \frac{d}{d\tau} \int dK E^r_{\mathbf{k}} k^{\langle\alpha} k^{\beta\rangle} \delta f_{\mathbf{k}}. \tag{246}$$

$$\begin{aligned} \dot{\rho}_r^{\langle\mu\nu\rangle} &= \Delta^{\mu\nu}_{\alpha\beta} \frac{d}{d\tau} \int dK E^r_{\mathbf{k}} k^{\langle\alpha} k^{\beta\rangle} \delta f_{\mathbf{k}} \\ &= \Delta^{\mu\nu}_{\alpha\beta} \int dK \frac{d}{d\tau} \left(E^r_{\mathbf{k}} k^{\langle\alpha} k^{\beta\rangle} \right) \delta f_{\mathbf{k}} + \Delta^{\mu\nu}_{\alpha\beta} \int dK E^r_{\mathbf{k}} k^{\langle\alpha} k^{\beta\rangle} \delta \dot{f}_{\mathbf{k}} \end{aligned} \tag{247}$$

Substitute Boltzmann equation

$$= \Delta^{\mu\nu}_{\alpha\beta} \int dK \frac{d}{d\tau} \left(E^r_{\mathbf{k}} k^{\langle\alpha} k^{\beta\rangle} \right) \delta f_{\mathbf{k}} \tag{248}$$

$$+ \Delta^{\mu\nu}_{\alpha\beta} \int dK E^r_{\mathbf{k}} k^{\langle\alpha} k^{\beta\rangle} \left(-\dot{f}_{0\mathbf{k}} - E^{-1}_{\mathbf{k}} k_\lambda \nabla^\lambda f_{0\mathbf{k}} - E^{-1}_{\mathbf{k}} k_\lambda \nabla^\lambda \delta f_{\mathbf{k}} + E^{-1}_{\mathbf{k}} C[f] \right) \tag{249}$$

First term

$$\Delta^{\mu\nu}_{\alpha\beta} \int dK E^r_{\mathbf{k}} k^{\langle\alpha} k^{\beta\rangle} \dot{f}_{0\mathbf{k}} = 0 \tag{250}$$

Second term

$$\Delta^{\mu\nu}_{\alpha\beta} \int dK E^r_{\mathbf{k}} k^{\langle\alpha} k^{\beta\rangle} E^{-1}_{\mathbf{k}} k_\lambda \nabla^\lambda f_{0\mathbf{k}} \tag{251}$$

$$= \Delta^{\mu\nu}_{\alpha\beta} \nabla^\lambda \int dK E^{r-1}_{\mathbf{k}} k^\alpha k^\beta k_\lambda f_{0\mathbf{k}} - (r-1)\Delta^{\mu\nu}_{\alpha\beta}\nabla^\lambda u_\rho \int dK E^{r-2}_{\mathbf{k}} k^\alpha k^\beta k_\rho k_\lambda f_{0\mathbf{k}} \tag{252}$$

Decompose the moments of equilibrium distribution

$$\int dK E^{r-1}_{\mathbf{k}} k^\alpha k^\beta k^\lambda f_{0\mathbf{k}} = I_{r+2,0} u^\alpha u^\beta u^\lambda - 3 I_{r+2,1} u^{(\alpha} \Delta^{\beta\lambda)} \tag{253}$$

$$\int dK E^{r-2}_{\mathbf{k}} k^\alpha k^\beta k^\lambda k^\rho f_{0\mathbf{k}} = I_{r+2,0} u^\alpha u^\beta u^\lambda u^\rho - 6 I_{r+2,1} u^{(\alpha} u^\rho \Delta^{\beta\lambda)} + 3 I_{r+2,2} \Delta^{\alpha(\beta} \Delta^{\lambda\rho)} \tag{254}$$

Only the last terms (with $I_{r+2,1}$ and $I_{r+2,2}$) survive the projection operator $\Delta^{\mu\nu}_{\alpha\beta}$.

$$\Delta^{\mu\nu}_{\alpha\beta} \nabla_\lambda \left(-I_{r+2,1} u^{(\alpha} \Delta^{\beta\lambda)} \right) = -2 I_{r+2,1} \Delta^{\mu\nu}_{\alpha\beta} \nabla_\lambda \left(u^\alpha \Delta^{\beta\lambda} \right) \tag{255}$$

$$= -2 I_{r+2,1} \Delta^{\mu\nu}_{\alpha\beta} \nabla^\beta u^\alpha = -2 I_{r+2,1} \sigma^{\mu\nu} \tag{256}$$

$$(r-1)\Delta^{\mu\nu}_{\alpha\beta} \nabla_\lambda u_\rho \left(I_{r+2,2} \Delta^{\alpha(\beta} \Delta^{\lambda\rho)} \right) = 2 I_{r+2,2} (r-1) \sigma^{\mu\nu} \tag{257}$$

$$\rightarrow \Delta^{\mu\nu}_{\alpha\beta} \int dK E^r_{\mathbf{k}} k^{\langle\alpha} k^{\beta\rangle} E^{-1}_{\mathbf{k}} k_\lambda \nabla^\lambda f_{0\mathbf{k}} = -2 \left[I_{r+2,1} + (r-1) I_{r+2,2} \right] \sigma^{\mu\nu} \tag{258}$$

Collision integral

$$C_{r-1}^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta} \int dK E_{\mathbf{k}}^{r} k^{\langle\alpha} k^{\beta\rangle} \left(E_{\mathbf{k}}^{-1} C\left[f\right] \right) \tag{259}$$

The first step is to linearize the collision operator,

$$C\left[f\right] = \frac{1}{\nu} \int dK' dP dP' W_{\mathbf{kk'}\to\mathbf{pp'}} \left(f_{\mathbf{p}} f_{\mathbf{p'}} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k'}} - f_{\mathbf{k}} f_{\mathbf{k'}} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p'}} \right), \tag{260}$$

in the deviations from the equilibrium. Keeping only terms of first order in ϕ:

$$f_{\mathbf{p}} f_{\mathbf{p'}} = f_{0\mathbf{p}} f_{0\mathbf{p'}} \left(1 + \tilde{f}_{0\mathbf{p'}} \phi_{\mathbf{p'}} + \tilde{f}_{0\mathbf{p}} \phi_{\mathbf{p}} \right) + \mathcal{O}\left(\phi^2\right), \tag{261}$$

$$\tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p'}} = \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p'}} \left(1 - a f_{0\mathbf{p'}} \phi_{\mathbf{p'}} - a f_{0\mathbf{p}} \phi_{\mathbf{p}} \right) + \mathcal{O}\left(\phi^2\right) . \tag{262}$$

Substituting Eqs. (261) and (262) into Eq. (260), we obtain,

$$C\left[f\right] = \frac{1}{\nu} \int dK' dP dP' W_{\mathbf{kk'}\to\mathbf{pp'}} f_{0\mathbf{k}} f_{0\mathbf{k'}} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p'}} \left(\phi_{\mathbf{p}} + \phi_{\mathbf{p'}} - \phi_{\mathbf{k}} - \phi_{\mathbf{k'}} \right) + \mathcal{O}\left(\phi^2\right) , \tag{263}$$

where following equalities were used:

$$\tilde{f}_{0\mathbf{p}} = f_{0\mathbf{p}} \exp\left(\beta_0 E_{\mathbf{p}} - \alpha_0 \right) , \tag{264}$$

$$f_{0\mathbf{p}} f_{0\mathbf{p'}} \tilde{f}_{0\mathbf{k}} \tilde{f}_{0\mathbf{k'}} = f_{0\mathbf{k}} f_{0\mathbf{k'}} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p'}} . \tag{265}$$

Irreducible collision term can be written as

$$C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = \frac{1}{\nu}\int dKdK'dPdP'W_{\mathbf{kk'}\to\mathbf{pp'}}f_{0\mathbf{k}}f_{0\mathbf{k'}}\tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p'}} \times E_{\mathbf{k}}^{r-1}k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\left(\phi_{\mathbf{p}}+\phi_{\mathbf{p'}}-\phi_{\mathbf{k}}-\phi_{\mathbf{k'}}\right)+\mathcal{O}\left(\phi^2\right). \tag{266}$$

$$C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\sum_{n=0}^{N_\ell}\mathcal{A}_{rn}^{(\ell)}\rho_n^{\mu_1\cdots\mu_\ell}. \tag{267}$$

The coefficients $\mathcal{A}_{rn}^{(\ell)}$ can be written as

$$\mathcal{A}_{rn}^{(\ell)} = \frac{1}{\nu(2\ell+1)}\int dKdK'dPdP'W_{\mathbf{kk'}\to\mathbf{pp'}}f_{0\mathbf{k}}f_{0\mathbf{k'}}\tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p'}}E_{\mathbf{k}}^{r-1}k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} \times\left(\mathcal{H}_{\mathbf{k}n}^{(\ell)}k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle}+\mathcal{H}_{\mathbf{k'}n}^{(\ell)}k'_{\langle\mu_1}\cdots k'_{\mu_\ell\rangle}-\mathcal{H}_{\mathbf{p}n}^{(\ell)}p_{\langle\mu_1}\cdots p_{\mu_\ell\rangle}-\mathcal{H}_{\mathbf{p'}n}^{(\ell)}p'_{\langle\mu_1}\cdots p'_{\mu_\ell\rangle}\right) \tag{268}$$

All the information about the microscopic scattering are in the coefficients $\mathcal{A}_{rn}^{(\ell)}$.

The equation of motion for the second rank tensor

$$\dot{\rho}_r^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta} \int dK \frac{d}{d\tau} \left(E_{\mathbf{k}}^r k^{\langle\alpha} k^{\beta\rangle} \right) \delta f_{\mathbf{k}} \tag{269}$$

$$+ \Delta^{\mu\nu}_{\alpha\beta} \int dK E_{\mathbf{k}}^r k^{\langle\alpha} k^{\beta\rangle} \left(-\dot{f}_{0\mathbf{k}} - E_{\mathbf{k}}^{-1} k_\lambda \nabla^\lambda f_{0\mathbf{k}} - E_{\mathbf{k}}^{-1} k_\lambda \nabla^\lambda \delta f_{\mathbf{k}} + E_{\mathbf{k}}^{-1} C\left[f\right] \right) \tag{270}$$

can be written as

$$\dot{\rho}_r^{\langle\mu\nu\rangle} = 2\left[I_{r+2,1} + (r-1) I_{r+2,2}\right] \sigma^{\mu\nu} - \sum_{n=0}^{N_\ell} \mathcal{A}_{rn}^{(2)} \rho_n^{\mu\nu}$$

$$+ \Delta^{\mu\nu}_{\alpha\beta} \int dK \frac{d}{d\tau} \left(E_{\mathbf{k}}^r k^{\langle\alpha} k^{\beta\rangle} \right) \delta f_{\mathbf{k}} - \Delta^{\mu\nu}_{\alpha\beta} \int dK E_{\mathbf{k}}^r k^{\langle\alpha} k^{\beta\rangle} E_{\mathbf{k}}^{-1} k_\lambda \nabla^\lambda \delta f_{\mathbf{k}} \tag{271}$$

The rest of the integrals can be written similarly in terms of moment $\rho_n^{\mu_1\cdots\mu_\ell}$, and lead to non-linear terms, e.g. $\rho_n^{\mu\nu}\theta$, etc.
This already resembles the equation of motion for shear stress tensor

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + (\text{higher order terms}), \tag{272}$$

but is still an infinite set of coupled equations for the moments.

In their full glory the equations of motion take the form: The equation for an arbitrary scalar moment is

$$\dot{\rho}_r = C_{r-1} + \alpha_r^{(0)}\theta + \left(r\rho_{r-1}^{\mu} + \frac{G_{2r}}{D_{20}} W^{\mu} \right) \dot{u}_{\mu} - \nabla_{\mu}\rho_{r-1}^{\mu} + \frac{G_{3r}}{D_{20}} \partial_{\mu} V^{\mu} - \frac{G_{2r}}{D_{20}} \partial_{\mu} W^{\mu}$$
$$+ \frac{1}{3}\left[(r-1)\, m^2 \rho_{r-2} - (r+2)\, \rho_r - 3\frac{G_{2r}}{D_{20}} \Pi \right] \theta + \left[(r-1)\, \rho_{r-2}^{\mu\nu} + \frac{G_{2r}}{D_{20}} \pi^{\mu\nu} \right] \sigma_{\mu\nu} \qquad (273)$$

Similarly, the time-evolution equation for the vector moment is

$$\dot{\rho}_r^{\langle\mu\rangle} = C_{r-1}^{\langle\mu\rangle} + \alpha_r^{(1)} \nabla^{\mu}\alpha_0 - \alpha_r^h \dot{W}^{\mu} + r\rho_{r-1}^{\mu\nu}\dot{u}_{\nu} + \frac{1}{3}\left[rm^2\rho_{r-1} - (r+3)\,\rho_{r+1} - 3\alpha_r^h \Pi \right] \dot{u}^{\mu}$$
$$- \frac{1}{3}\nabla^{\mu}\left(m^2\rho_{r-1} - \rho_{r+1} \right) + \alpha_r^h \nabla^{\mu}\Pi - \Delta_{\nu}^{\mu}\left(\nabla_{\lambda}\rho_{r-1}^{\nu\lambda} + \alpha_r^h \partial_{\lambda}\pi^{\nu\lambda} \right)$$
$$+ \frac{1}{3}\left[(r-1)\, m^2 \rho_{r-2}^{\mu} - (r+3)\,\rho_r^{\mu} - 4\alpha_r^h W^{\mu} \right] \theta$$
$$+ \frac{1}{5}\left[(2r-2)\, m^2 \rho_{r-2}^{\nu} - (2r+3)\,\rho_r^{\nu} - 5\alpha_r^h W^{\nu} \right] \sigma_{\nu}^{\mu}$$
$$+ \left(\rho_r^{\nu} + \alpha_r^h W^{\nu} \right) \omega^{\mu}{}_{\nu} + (r-1)\,\rho_{r-2}^{\mu\nu\lambda}\sigma_{\nu\lambda}, \qquad (274)$$

The equation for $\rho_r^{\mu\nu}$ is

$$\dot{\rho}_r^{\langle\mu\nu\rangle} = C_{r-1}^{\langle\mu\nu\rangle} + 2\alpha_r^{(2)}\sigma^{\mu\nu} + \frac{2}{15}\left[(r-1)\, m^4 \rho_{r-2} \; - (2r+3)\, m^2 \rho_r + (r+4)\rho_{r+2}\right] \sigma^{\mu\nu}$$
$$+ \frac{2}{5}\left[r m^2 \rho_{r-1}^{\langle\mu} - (r+5)\, \rho_{r+1}^{\langle\mu}\right] \dot{u}^{\nu\rangle} + r\rho_{r-1}^{\mu\nu\lambda}\dot{u}_\lambda - \frac{2}{5}\nabla^{\langle\mu}\left(m^2 \rho_{r-1}^{\nu\rangle} - \rho_{r+1}^{\nu\rangle}\right)$$
$$+ \frac{1}{3}\left[(r-1)\, m^2 \rho_{r-2}^{\mu\nu} - (r+4)\, \rho_r^{\mu\nu}\right]\theta + \frac{2}{7}\left[(2r-2)\, m^2 \rho_{r-2}^{\lambda\langle\mu} - (2r+5)\, \rho_r^{\lambda\langle\mu}\right]\sigma_\lambda^{\nu\rangle}$$
$$+ 2\rho_r^{\lambda\langle\mu}\omega^{\nu\rangle}_{\lambda} - \Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda \rho_{r-1}^{\alpha\beta\lambda} + (r-1)\rho_{r-2}^{\mu\nu\lambda\kappa}\sigma_{\lambda\kappa}, \tag{275}$$

All derivatives of α_0 and β_0 that appear in the above equations were replaced using the following equations, obtained from the conservation laws,

$$\dot{\alpha}_0 = \frac{1}{D_{20}}\left[-J_{30}\left(n_0\theta + \partial_\mu V^\mu\right) + J_{20}\left(\varepsilon_0 + P_0 + \Pi\right)\theta \; + J_{20}\left(\partial_\mu W^\mu - W^\mu \dot{u}_\mu - \pi^{\mu\nu}\sigma_{\mu\nu}\right)\right], \tag{276}$$

$$\dot{\beta}_0 = \frac{1}{D_{20}}\left[-J_{20}\left(n_0\theta + \partial_\mu V^\mu\right) + J_{10}\left(\varepsilon_0 + P_0 + \Pi\right)\theta \; + J_{10}\left(\partial_\mu W^\mu - W^\mu \dot{u}_\mu - \pi^{\mu\nu}\sigma_{\mu\nu}\right)\right], \tag{277}$$

$$\dot{u}^\mu = \beta_0^{-1}\left(h_0^{-1}\nabla^\mu \alpha_0 - \nabla^\mu \beta_0\right) - \frac{h_0^{-1}}{n_0}\left(\Pi \dot{u}^\mu - \nabla^\mu \Pi\right)$$
$$- \frac{h_0^{-1}}{n_0}\left[\frac{4}{3}W^\mu \theta + W_\nu\left(\sigma^{\mu\nu} - \omega^{\mu\nu}\right) + \dot{W}^\mu + \Delta^\mu_\nu \partial_\lambda \pi^{\nu\lambda}\right], \tag{278}$$

$h_0 = (\varepsilon_0 + P_0)/n_0$

The coefficients α_r are functions of thermodynamic variables,

$$\alpha_r^{(0)} = (1-r)\, I_{r1} - I_{r0} - \frac{n_0}{D_{20}}\,(h_0 G_{2r} - G_{3r})\,, \tag{279}$$

$$\alpha_r^{(1)} = J_{r+1,1} - h_0^{-1} J_{r+2,1}, \tag{280}$$

$$\alpha_r^{(2)} = I_{r+2,1} + (r-1)\, I_{r+2,2}, \tag{281}$$

$$\alpha_r^{h} = -\frac{\beta_0}{\varepsilon_0 + P_0} J_{r+2,1}, \tag{282}$$

where we used the notation

$$I_{nq} = \frac{(-1)^q}{(2q+1)!!} \int dK E_{\mathbf{k}}^{n-2q} \left(\Delta^{\alpha\beta} k_\alpha k_\beta\right)^q f_{0\mathbf{k}}, \tag{283}$$

$$J_{nq} = \frac{(-1)^q}{(2q+1)!!} \int dK E_{\mathbf{k}}^{n-2q} \left(\Delta^{\alpha\beta} k_\alpha k_\beta\right)^q f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}}, \tag{284}$$

$$G_{nm} = J_{n0} J_{m0} - J_{n-1,0} J_{m+1,0}, \tag{285}$$

$$D_{nq} = J_{n+1,q} J_{n-1,q} - (J_{nq})^2\,. \tag{286}$$

Thus, we obtained an infinite set of coupled equations containing all moments of the distribution function, but the derivation of these equations is independent of the form of the expansion we introduced in the previous subsection.

Section 8

Reduction of the degrees of freedom

So far we have just written Boltzmann equation in terms of the moments of the distribution function. In order to obtain fluid dynamical equations of motion, we must somehow reduce the infinite (momentum-space) degrees of freedom into the fluid dynamical ones, i.e. n_0, e_0, u^μ, Π, V^μ, W^μ, and $\pi^{\mu\nu}$.

Israel and Stewart: 14-moment approximation

This was already introduced before, but now in irreducible basis. The degrees of freedom can be reduced into the 14 fluid dynamical ones by directly truncating the expansion of the distribution function.

$$\phi_{\mathbf{k}} = \sum_{\ell=0}^{\infty} \lambda_{\mathbf{k}}^{\langle \mu_1 \cdots \mu_\ell \rangle} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle}, \qquad \lambda_{\mathbf{k}}^{\langle \mu_1 \cdots \mu_\ell \rangle} = \sum_{n=0}^{N_\ell} c_n^{\langle \mu_1 \cdots \mu_\ell \rangle} P_{\mathbf{k}n}^{(\ell)}, \tag{287}$$

$$\lambda_{\mathbf{k}} \equiv \sum_{n=0}^{N_0} c_n P_{\mathbf{k}n}^{(0)} \simeq c_0 P_{\mathbf{k}0}^{(0)} + c_1 P_{\mathbf{k}1}^{(0)} + c_2 P_{\mathbf{k}2}^{(0)}, \tag{288}$$

$$\lambda_{\mathbf{k}}^{\langle \mu \rangle} \equiv \sum_{n=0}^{N_1} c_n^{\langle \mu \rangle} P_{\mathbf{k}n}^{(1)} \simeq c_0^{\langle \mu \rangle} P_{\mathbf{k}0}^{(1)} + c_1^{\langle \mu \rangle} P_{\mathbf{k}1}^{(1)}, \tag{289}$$

$$\lambda_{\mathbf{k}}^{\langle \mu\nu \rangle} \equiv \sum_{n=0}^{N_2} c_n^{\langle \mu\nu \rangle} P_{\mathbf{k}n}^{(2)} \simeq c_0^{\langle \mu\nu \rangle} P_{\mathbf{k}0}^{(2)}, \tag{290}$$

where the tensors $c_n^{\langle \mu_1 \cdots \mu_\ell \rangle}$ are given by Eq. (235), while those which do not appear in the above equations are set to zero.

According to Eq. (235) the scalars c_0, c_1, and c_2 are proportional to the bulk viscous pressure,

$$c_0 = -\frac{3\Pi}{m^2} a_{00}^{(0)} \mathcal{N}^{(0)}, \tag{291}$$

$$c_1 = -\frac{3\Pi}{m^2} a_{10}^{(0)} \mathcal{N}^{(0)}, \tag{292}$$

$$c_2 = -\frac{3\Pi}{m^2} a_{20}^{(0)} \mathcal{N}^{(0)}. \tag{293}$$

The vectors $c_0^{\langle\mu\rangle}$ and $c_1^{\langle\mu\rangle}$ are given by a linear combination of particle and energy-momentum diffusion currents,

$$c_0^{\langle\mu\rangle} = V^\mu a_{00}^{(1)} \mathcal{N}^{(1)}, \tag{294}$$

$$c_1^{\langle\mu\rangle} = V^\mu a_{10}^{(1)} \mathcal{N}^{(1)} + W^\mu a_{11}^{(1)} \mathcal{N}^{(1)}, \tag{295}$$

while $c_0^{\langle\mu\nu\rangle}$ is proportional to the shear-stress tensor,

$$c_0^{\langle\mu\nu\rangle} = \pi^{\mu\nu} a_{00}^{(2)} \frac{\mathcal{N}^{(2)}}{2}. \tag{296}$$

orthogonal polynomials

For any $\ell \geq 0$ we set

$$P^{(\ell)}_{\mathbf{k}0} \equiv a^{(\ell)}_{00} = 1, \tag{297}$$

while

$$P^{(0)}_{\mathbf{k}1} = a^{(0)}_{11} E_{\mathbf{k}} + a^{(0)}_{10}, \tag{298}$$

$$P^{(1)}_{\mathbf{k}1} = a^{(1)}_{11} E_{\mathbf{k}} + a^{(1)}_{10}, \tag{299}$$

$$P^{(0)}_{\mathbf{k}2} = a^{(0)}_{22} E^2_{\mathbf{k}} + a^{(0)}_{21} E_{\mathbf{k}} + a^{(0)}_{20}. \tag{300}$$

The orthonormality condition (229) gives

$$\mathcal{N}^{(\ell)} = \left(J_{2\ell,\ell}\right)^{-1}, \tag{301}$$

$$\frac{a^{(0)}_{10}}{a^{(0)}_{11}} = -\frac{J_{10}}{J_{00}}, \ \left(a^{(0)}_{11}\right)^2 = \frac{J^2_{00}}{D_{10}}, \tag{302}$$

$$\frac{a^{(0)}_{21}}{a^{(0)}_{22}} = \frac{G_{12}}{D_{10}}, \ \frac{a^{(0)}_{20}}{a^{(0)}_{22}} = \frac{D_{20}}{D_{10}}, \tag{303}$$

$$\left(a^{(0)}_{22}\right)^2 = \frac{J_{00} D_{10}}{J_{20} D_{20} + J_{30} G_{12} + J_{40} D_{10}}, \tag{304}$$

$$\frac{a^{(1)}_{10}}{a^{(1)}_{11}} = -\frac{J_{31}}{J_{21}}, \ \left(a^{(1)}_{11}\right)^2 = \frac{J^2_{21}}{D_{31}}. \tag{305}$$

Using the orthogonality relation (225) together with Eqs. (226-228)

$$\rho_r^{\mu_1\cdots\mu_\ell} = \ell! \sum_{n=0}^{N_\ell} \sum_{m=0}^{n} c_n^{\langle\mu_1\cdots\mu_\ell\rangle} a_{nm}^{(\ell)} \, J_{r+m+2\ell,\ell}. \tag{306}$$

Applying the truncation scheme we obtain that all scalar moments, ρ_r, become proportional to the bulk viscous pressure Π,

$$\rho_r \equiv \sum_{n=0}^{N_0} \sum_{m=0}^{n} c_n a_{nm}^{(0)} \, J_{r+m,0} = \gamma_r^{\Pi} \Pi. \tag{307}$$

Similarly, all vector moments, ρ_r^μ, are proportional to a linear combination of V^μ and W^μ,

$$\rho_r^\mu \equiv \sum_{n=0}^{N_1} \sum_{m=0}^{n} c_n^{\langle\mu\rangle} a_{nm}^{(1)} \, J_{r+m+2,1} = \gamma_r^V V^\mu + \gamma_r^W W^\mu, \tag{308}$$

and, finally, $\rho_r^{\mu\nu}$ is proportional to $\pi^{\mu\nu}$,

$$\rho_r^{\mu\nu} \equiv \sum_{n=0}^{N_2} \sum_{m=0}^{n} c_n^{\langle\mu\nu\rangle} a_{nm}^{(2)} \, J_{r+m+4,2} = \gamma_r^{\pi} \pi^{\mu\nu}. \tag{309}$$

The proportionality coefficients are

$$\gamma_r^{\Pi} = \mathcal{A}_{\Pi} J_{r0} + \mathcal{B}_{\Pi} J_{r+1,0} + \mathcal{C}_{\Pi} J_{r+2,0}, \tag{310}$$

$$\gamma_r^{V} = \mathcal{A}_{V} J_{r+2,1} + \mathcal{B}_{V} J_{r+3,1}, \tag{311}$$

$$\gamma_r^{W} = \mathcal{A}_{W} J_{r+2,1} + \mathcal{B}_{W} J_{r+3,1}, \tag{312}$$

$$\gamma_r^{\pi} = 2\mathcal{A}_{\pi} J_{r+4,2}. \tag{313}$$

where

$$\mathcal{A}_{\Pi} = -\frac{3}{m^2}\frac{D_{30}}{J_{20}D_{20} + J_{30}G_{12} + J_{40}D_{10}}, \tag{314}$$

$$\mathcal{B}_{\Pi} = -\frac{3}{m^2}\frac{G_{23}}{J_{20}D_{20} + J_{30}G_{12} + J_{40}D_{10}}, \tag{315}$$

$$\mathcal{C}_{\Pi} = -\frac{3}{m^2}\frac{D_{20}}{J_{20}D_{20} + J_{30}G_{12} + J_{40}D_{10}}, \tag{316}$$

$$\mathcal{A}_{V} = \frac{J_{41}}{D_{31}},\ \mathcal{A}_{W} = -\frac{J_{31}}{D_{31}}, \tag{317}$$

$$\mathcal{B}_{V} = -\frac{J_{31}}{D_{31}},\ \mathcal{B}_{W} = \frac{J_{21}}{D_{31}}, \tag{318}$$

$$\mathcal{A}_{\pi} = \frac{1}{2J_{42}}. \tag{319}$$

The matching conditions $\rho_1 = \rho_2 = 0$ require that $\gamma_1^{\Pi} = \gamma_2^{\Pi} = 0$.

Now we are ready to write down the fluid dynamical e.o.m. Recall

$$\dot{\rho}_r^{\langle\mu\nu\rangle} = C_{r-1}^{\langle\mu\nu\rangle} + 2\alpha_r^{(2)}\sigma^{\mu\nu} + \frac{2}{15}\left[(r-1)\,m^4\rho_{r-2} - (2r+3)\,m^2\rho_r + (r+4)\rho_{r+2}\right]\sigma^{\mu\nu}$$
$$+\frac{2}{5}\left[rm^2\rho_{r-1}^{\langle\mu} - (r+5)\,\rho_{r+1}^{\langle\mu}\right]\dot{u}^{\nu\rangle} + r\rho_{r-1}^{\mu\nu\lambda}\dot{u}_\lambda - \frac{2}{5}\nabla^{\langle\mu}\left(m^2\rho_{r-1}^{\nu\rangle} - \rho_{r+1}^{\nu\rangle}\right)$$
$$+\frac{1}{3}\left[(r-1)\,m^2\rho_{r-2}^{\mu\nu} - (r+4)\,\rho_r^{\mu\nu}\right]\theta + \frac{2}{7}\left[(2r-2)\,m^2\rho_{r-2}^{\lambda\langle\mu} - (2r+5)\,\rho_r^{\lambda\langle\mu}\right]\sigma_\lambda^{\nu\rangle}$$
$$+2\rho_r^{\lambda\langle\mu}\omega^{\nu\rangle}_{\ \lambda} - \Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda\rho_{r-1}^{\alpha\beta\lambda} + (r-1)\rho_{r-2}^{\mu\nu\lambda\kappa}\sigma_{\lambda\kappa}, \tag{320}$$

and plug in the 14-moment approximation

$$\rho_r \equiv \sum_{n=0}^{N_0}\sum_{m=0}^{n} c_n a_{nm}^{(0)}\, J_{r+m,0} = \gamma_r^\Pi \Pi. \tag{321}$$

$$\rho_r^\mu \equiv \sum_{n=0}^{N_1}\sum_{m=0}^{n} c_n^{\langle\mu\rangle} a_{nm}^{(1)}\, J_{r+m+2,1} = \gamma_r^V V^\mu + \gamma_r^W W^\mu, \tag{322}$$

$$\rho_r^{\mu\nu} \equiv \sum_{n=0}^{N_2}\sum_{m=0}^{n} c_n^{\langle\mu\nu\rangle} a_{nm}^{(2)}\, J_{r+m+4,2} = \gamma_r^\pi \pi^{\mu\nu}. \tag{323}$$

$$C_{r-1}^{\langle\mu\nu\rangle} = -\tau_r^{-1}\pi^{\mu\nu} \tag{324}$$

Fluid dynamical equations of motion

$$\frac{d}{d\tau}\left(\gamma_r^{\pi}\pi^{\langle\mu\nu\rangle}\right) = -\tau_r^{-1}\pi^{\mu\nu} + 2\alpha_r^{(2)}\sigma^{\mu\nu} + \text{non-linear terms} \tag{325}$$

$$\gamma_r^{\pi}\tau_r\frac{d}{d\tau}\left(\pi^{\langle\mu\nu\rangle}\right) + \pi^{\mu\nu} = 2\tau_r\alpha_r^{(2)}\sigma^{\mu\nu} + \text{non-linear terms} \tag{326}$$

These are the equations of motion for shear-stress tensor in 14-moment approximation, with relaxation time $\tau_\pi = \gamma_r^{\pi}\tau_r$, and shear viscosity $\eta = \tau_r\alpha_r^{(2)}$.
Now the problem is that we can actually choose any value of r to derive the equations, and all of them give different transport coefficients!
For example Israel and Stewart choose second moment of the Boltzmann equation to close the system (corresponds $r = 2$), and in DKR the choise corresponds $r = 1$.
Obviously something is still missing from the derivation, i.e. when reducing the degrees of freedom it is not sufficient to simply truncate the moment expansion.
Solution: resum all the moments (or the relevant ones)

Section 9

Power counting and the reduction of dynamical variables

So far, we have derived a general expansion of the distribution function in terms of the irreducible moments of $\delta f_{\mathbf{k}}$, as well as exact equations of motion for these moments.
Fluid dynamical limit:

- Evolution described by the conserved currents N^μ and $T^{\mu\nu}$ alone.
- Separation of microscopic and macroscopic scales, quantify by Knudsen numbers

$$\mathrm{Kn} \equiv \frac{\ell_{\mathrm{micr}}}{L_{\mathrm{macr}}} \ll 1 \tag{327}$$

- System close to local thermal equilibrium, quantify by inverse Reynolds numbers

$$\mathrm{R}_{\Pi}^{-1} \equiv \frac{|\Pi|}{P_0} \ll 1\ , \ \mathrm{R}_{n}^{-1} \equiv \frac{|n^\mu|}{n_0} \ll 1, \ \mathrm{R}_{\pi}^{-1} \equiv \frac{|\pi^{\mu\nu}|}{P_0} \ll 1 \tag{328}$$

- In transient fluid dynamics these are two independent quantities
- 14-moment approximation is not truncation in either of these

Microscopic scales: τ_Π, τ_n, τ_π
Macroscopic scales: θ, $\nabla^\mu \alpha_0$, $\nabla^\mu e_0$, ...

A general structure of the equations of motion is (for second rank tensors)

$$\begin{aligned}\dot{\rho}_r &= C_{r-1} + \alpha_r^{(0)}\theta + \text{higher-order terms}\\ \dot{\rho}_r^{\langle\mu\rangle} &= C_{r-1}^{\langle\mu\rangle} + \alpha_r^{(1)}\nabla^\mu\alpha_0 + \text{higher-order terms}\\ \dot{\rho}_r^{\langle\mu\nu\rangle} &= C_{r-1}^{\langle\mu\nu\rangle} + 2\alpha_r^{(2)}\sigma^{\mu\nu} + \text{higher-order terms}\end{aligned} \tag{329}$$

Where the linearized collision terms $C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle}$ are

$$C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\sum_{n=0}^{N_\ell} \mathcal{A}_{rn}^{(\ell)}\,\rho_n^{\mu_1\cdots\mu_\ell} \tag{330}$$

The coefficient $\mathcal{A}_{rn}^{(\ell)}$ is the (rn) element of an $(N_\ell+1)\times(N_\ell+1)$ matrix $\mathcal{A}^{(\ell)}$ and contains all the information of the underlying microscopic theory.
These are not relaxation equations for $\rho_r^{\mu_1\cdots\mu_\ell}$ because the collision term couples all the different r.

Fluid dynamics is expected to emerge when the microscopic degrees of freedom are integrated out, and the system can be described solely by the conserved currents. The exact equations of motion contain infinitely many degrees of freedom, and also infinitely many microscopic time scales, related to the coefficients $\mathcal{A}_{rn}^{(\ell)}$.

The slowest microscopic time scale should dominate the dynamics at long times.

For this purpose, we shall introduce the matrix $\Omega^{(\ell)}$ which diagonalizes $\mathcal{A}^{(\ell)}$,

$$\left(\Omega^{-1}\right)^{(\ell)} \mathcal{A}^{(\ell)}\Omega^{(\ell)} = \mathrm{diag}\left(\chi_0^{(\ell)}, \ldots, \chi_j^{(\ell)}, \ldots\right), \tag{331}$$

where $\chi_j^{(\ell)}$ are the eigenvalues of $\mathcal{A}^{(\ell)}$. Above, $\left(\Omega^{-1}\right)^{(\ell)}$ is defined as the matrix inverse of $\Omega^{(\ell)}$.

We further define the tensors $X_i^{\mu_1\cdots\mu_\ell}$ as

$$X_i^{\mu_1\cdots\mu_\ell} \equiv \sum_{j=0}^{N_\ell} \left(\Omega^{-1}\right)_{ij}^{(\ell)} \rho_j^{\mu_1\cdots\mu_\ell}. \tag{332}$$

These are eigenmodes of the linearized Boltzmann equation.

Multiplying Eq. (330) with $\left(\Omega^{-1}\right)^{(\ell)}$ from the left and using Eqs. (331) and (332) we obtain

$$\sum_{j=0}^{N_\ell} \left(\Omega^{-1}\right)_{ij}^{(\ell)} C_{j-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\chi_i^{(\ell)} X_i^{\mu_1\cdots\mu_\ell} + (\text{terms nonlinear in } \delta f) \ . \tag{333}$$

where we do not sum over the index i on the right-hand side of the equation. Then we multiply Eqs. (329) with $\left(\Omega^{-1}\right)_{ir}^{(\ell)}$ and sum over r. Using Eq. (333), we obtain the equations of motion for the variables $X_i^{\mu_1\cdots\mu_\ell}$,

$$\begin{aligned} \dot{X}_i + \chi_i^{(0)} X_i &= \beta_i^{(0)} \theta + (\text{higher-order terms})\,, \\ \dot{X}_i^{\langle\mu\rangle} + \chi_i^{(1)} X_i^{\mu} &= \beta_i^{(1)} I^{\mu} + (\text{higher-order terms})\,, \\ \dot{X}_i^{\langle\mu\nu\rangle} + \chi_i^{(2)} X_i^{\mu\nu} &= \beta_i^{(2)} \sigma^{\mu\nu} + (\text{higher-order terms})\,, \end{aligned} \tag{334}$$

where the coefficients are

$$\beta_i^{(0)} = \sum_{j=0,\neq 1,2}^{N_0} \left(\Omega^{-1}\right)_{ij}^{(0)} \alpha_j^{(0)}, \quad \beta_i^{(1)} = \sum_{j=0,\neq 1}^{N_1} \left(\Omega^{-1}\right)_{ij}^{(1)} \alpha_j^{(1)}, \quad \beta_i^{(2)} = 2\sum_{j=0}^{N_2} \left(\Omega^{-1}\right)_{ij}^{(2)} \alpha_j^{(2)}. \tag{335}$$

The equations of motion for the tensors $X_i^{\mu_1\cdots\mu_\ell}$ decouple in the linear regime. We can order the tensors $X_r^{\mu_1\cdots\mu_\ell}$ according to increasing $\chi_r^{(\ell)}$, e.g., in such a way that $\chi_r^{(\ell)} < \chi_{r+1}^{(\ell)}$, $\forall\ \ell$.

By diagonalizing Eqs. (329)) we were able to identify the microscopic time scales of the Boltzmann equation given by the inverse of the coefficients $\chi_r^{(\ell)}$.

If the nonlinear terms in Eqs. (334) are small enough, each tensor $X_r^{\mu_1 \cdots \mu_\ell}$ relaxes independently to its respective asymptotic value, given by the first term on the right-hand sides of Eqs. (334) (divided by the corresponding $\chi_r^{(\ell)}$), on a time scale $\sim 1/\chi_r^{(\ell)}$.

The asymptotic solutions ("Navier-Stokes values"): Neglecting all the relaxation timescales, i.e., taking the limit $\chi_r^{(\ell)} \to \infty$ with $\beta_r^{(\ell)}/\chi_r^{(\ell)}$ fixed, all irreducible moments $\rho_r^{\mu_1 \cdots \mu_\ell}$ become proportional to gradients of α_0, β_0, and u^μ.

Assuming that only the slowest modes with rank 2 and smaller, X_0, X_0^{μ}, and $X_0^{\mu\nu}$, remain in the transient regime and satisfy the partial differential equations (334),

$$\begin{aligned}
\dot{X}_0 + \chi_0^{(0)} X_0 &= \beta_0^{(0)} \theta + (\text{higher-order terms}) \ , \\
\dot{X}_0^{\langle\mu\rangle} + \chi_0^{(1)} X_0^{\mu} &= \beta_0^{(1)} I^{\mu} + (\text{higher-order terms}) \ , \\
\dot{X}_0^{\langle\mu\nu\rangle} + \chi_0^{(2)} X_0^{\mu\nu} &= \beta_0^{(2)} \sigma^{\mu\nu} + (\text{higher-order terms}) \ ,
\end{aligned} \qquad (336)$$

The modes described by faster relaxation scales, i.e., X_r, X_r^{μ}, and $X_r^{\mu\nu}$, for any r *larger than* 0, will be approximated by their asymptotic solution,

$$\begin{aligned}
X_r &\simeq \frac{\beta_r^{(0)}}{\chi_r^{(0)}} \theta + (\text{higher-order terms}) \, , \\
X_r^{\mu} &\simeq \frac{\beta_r^{(1)}}{\chi_r^{(1)}} I^{\mu} + (\text{higher-order terms}) \, , \\
X_r^{\mu\nu} &\simeq \frac{\beta_r^{(2)}}{\chi_r^{(2)}} \sigma^{\mu\nu} + (\text{higher-order terms}) \, ,
\end{aligned} \qquad (337)$$

This is the reduction of the degrees of freedom, but still need to express everything in terms of fluid dynamical variables, Π, V^{μ}, $\pi^{\mu\nu}$

First invert Eq. (332),

$$\rho_i^{\mu_1\cdots\mu_\ell} = \sum_{j=0}^{N_\ell} \Omega_{ij}^{(\ell)} X_j^{\mu_1\cdots\mu_\ell} \ , \tag{338}$$

then, using Eqs. (337), we obtain

$$\begin{aligned}
\rho_i &\simeq \Omega_{i0}^{(0)} X_0 + \sum_{j=3}^{N_0} \Omega_{ij}^{(0)} \frac{\beta_j^{(0)}}{\chi_j^{(0)}}\, \theta = \Omega_{i0}^{(0)} X_0 + \mathcal{O}(\mathrm{Kn}) \ , \\
\rho_i^{\mu} &\simeq \Omega_{i0}^{(1)} X_0^{\mu} + \sum_{j=2}^{N_1} \Omega_{ij}^{(1)} \frac{\beta_j^{(1)}}{\chi_j^{(1)}}\, I^{\mu} = \Omega_{i0}^{(1)} X_0^{\mu} + \mathcal{O}(\mathrm{Kn}) \ , \\
\rho_i^{\mu\nu} &\simeq \Omega_{i0}^{(2)} X_0^{\mu\nu} + \sum_{j=1}^{N_2} \Omega_{ij}^{(2)} \frac{\beta_j^{(2)}}{\chi_j^{(2)}}\, \sigma^{\mu\nu} = \Omega_{i0}^{(2)} X_0^{\mu\nu} + \mathcal{O}(\mathrm{Kn}) \ .
\end{aligned} \tag{339}$$

The contribution from the modes X_r, X_r^{μ},and $X_r^{\mu\nu}$ for $r \geq 1$ is of order $\mathcal{O}(\mathrm{Kn})$.

Taking $i = 0$, setting $\Omega_{00}^{(\ell)} = 1$, and remembering that $\rho_0 = -\frac{3}{m^2}\Pi$, $\rho_0^\mu = n^\mu$, $\rho_0^{\mu\nu} = \pi^{\mu\nu}$. we obtain the relations

$$X_0 \simeq -\frac{3}{m^2}\Pi - \sum_{j=3}^{N_0} \Omega_{0j}^{(0)} \frac{\beta_j^{(0)}}{\chi_j^{(0)}} \theta ,$$
$$X_0^\mu \simeq n^\mu - \sum_{j=2}^{N_1} \Omega_{0j}^{(1)} \frac{\beta_j^{(1)}}{\chi_j^{(1)}} I^\mu ,$$
$$X_0^{\mu\nu} \simeq \pi^{\mu\nu} - \sum_{j=1}^{N_2} \Omega_{0j}^{(2)} \frac{\beta_j^{(2)}}{\chi_j^{(2)}} \sigma^{\mu\nu} . \quad (340)$$

Substituting Eqs. (340) into Eqs. (339),

$$\frac{m^2}{3}\rho_i \simeq -\Omega_{i0}^{(0)}\Pi - \left(\zeta_i - \Omega_{i0}^{(0)}\zeta_0\right)\theta = -\Omega_{i0}^{(0)}\Pi + \mathcal{O}(\mathrm{Kn}),$$
$$\rho_i^\mu \simeq \Omega_{i0}^{(1)} n^\mu + \left(\kappa_{n\,i} - \Omega_{i0}^{(1)}\kappa_{n\,0}\right) I^\mu = \Omega_{i0}^{(1)} n^\mu + \mathcal{O}(\mathrm{Kn}) ,$$
$$\rho_i^{\mu\nu} \simeq \Omega_{i0}^{(2)}\pi^{\mu\nu} + 2\left(\eta_i - \Omega_{i0}^{(2)}\eta_0\right)\sigma^{\mu\nu} = \Omega_{i0}^{(2)}\pi^{\mu\nu} + \mathcal{O}(\mathrm{Kn}) , \quad (341)$$

The coefficients are defined as

$$\zeta_i = \frac{m^2}{3} \sum_{r=0,\neq 1,2}^{N_0} \tau_{ir}^{(0)} \alpha_r^{(0)} \,, \quad \kappa_{n\,i} = \sum_{r=0,\neq 1}^{N_1} \tau_{ir}^{(1)} \alpha_r^{(1)} \,, \quad \eta_i = \sum_{r=0}^{N_2} \tau_{ir}^{(2)} \alpha_r^{(2)} \,, \tag{342}$$

where $\tau^{(\ell)} \equiv \left(\mathcal{A}^{-1}\right)^{(\ell)}$ and used the relation,

$$\tau_{in}^{(\ell)} = \sum_{m=0}^{N_\ell} \Omega_{im}^{(\ell)} \frac{1}{\chi_m^{(\ell)}} \left(\Omega^{-1}\right)_{mn}^{(\ell)} \,.$$

We can identify the coefficients ζ_0, $\kappa_{n\,0}$, and η_0 as the bulk-viscosity, particle-diffusion, and shear-viscosity coefficients, respectively.

The relations (341) are only valid for the moments $\rho_r^{\mu\nu\lambda\cdots}$ with positive r, but in the full equations of motion also moments with $r < 0$ also appear.
We expect the expansion (**??**) to be complete and, therefore, any moment that does not appear in this expansion must be linearly related to those that do appear. This means that, using the moment expansion, Eq. (**??**), it is possible to express the moments with negative r in terms of the ones with positive r. Substituting Eq. (**??**) into Eq. (236) and using Eq. (225), we obtain

$$\rho_{-r}^{\nu_1\cdots\nu_\ell} = \sum_{n=0}^{N_\ell} \mathcal{F}_{rn}^{(\ell)} \rho_n^{\nu_1\cdots\nu_\ell}, \tag{343}$$

where we defined the following thermodynamic integral

$$\mathcal{F}_{rn}^{(\ell)} = \frac{\ell!}{(2\ell+1)!!} \int dK \; f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} E_{\mathbf{k}}^{-r} \mathcal{H}_{\mathbf{k}n}^{(\ell)} \left(\Delta^{\alpha\beta} k_\alpha k_\beta\right)^\ell . \tag{344}$$

Therefore, Eqs. (341) lead to

$$\begin{aligned} \rho_{-r} &= -\frac{3}{m^2}\, \gamma_r^{(0)} \Pi + \mathcal{O}(\mathrm{Kn}) \,, \\ \rho_{-r}^{\mu} &= \gamma_r^{(1)} n^\mu + \mathcal{O}(\mathrm{Kn}) \,, \\ \rho_{-r}^{\mu\nu} &= \gamma_r^{(2)} \pi^{\mu\nu} + \mathcal{O}(\mathrm{Kn}) \,, \end{aligned} \tag{345}$$

where we introduced the coefficients

$$\gamma_r^{(0)} = \sum_{n=0,\neq 1,2}^{N_0} \mathcal{F}_{rn}^{(0)} \Omega_{n0}^{(0)} \,, \quad \gamma_r^{(1)} = \sum_{n=0,\neq 1}^{N_1} \mathcal{F}_{rn}^{(1)} \Omega_{n0}^{(1)} \,, \quad \gamma_r^{(2)} = \sum_{n=0}^{N_2} \mathcal{F}_{rn}^{(2)} \Omega_{n0}^{(2)} \,. \tag{346}$$

The resulting equations of motion take the form

$$
\begin{aligned}
\tau_{\Pi}\dot{\Pi} + \Pi &= -\zeta\theta + \mathcal{J} + \mathcal{K} + \mathcal{R} , \\
\tau_n \dot{n}^{\langle\mu\rangle} + n^{\mu} &= \kappa_n I^{\mu} + \mathcal{J}^{\mu} + \mathcal{K}^{\mu} + \mathcal{R}^{\mu} , \\
\tau_{\pi}\dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} &= 2\eta\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{K}^{\mu\nu} + \mathcal{R}^{\mu\nu} .
\end{aligned}
\tag{347}
$$

The tensors $\mathcal{J}$, $\mathcal{J}^{\mu}$, and $\mathcal{J}^{\mu\nu}$ contain all terms of first order in Knudsen and inverse Reynolds numbers,

$$
\begin{aligned}
\mathcal{J} &= -\ell_{\Pi n}\nabla\cdot n - \tau_{\Pi n} n\cdot F - \delta_{\Pi\Pi}\Pi\theta - \lambda_{\Pi n} n\cdot I + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu} , \\
\mathcal{J}^{\mu} &= -n_{\nu}\omega^{\nu\mu} - \delta_{nn}n^{\mu}\theta - \ell_{n\Pi}\nabla^{\mu}\Pi + \ell_{n\pi}\Delta^{\mu\nu}\nabla_{\lambda}\pi^{\lambda}_{\nu} + \tau_{n\Pi}\Pi F^{\mu} - \tau_{n\pi}\pi^{\mu\nu}F_{\nu} \\
&\quad - \lambda_{nn}n_{\nu}\sigma^{\mu\nu} + \lambda_{n\Pi}\Pi I^{\mu} - \lambda_{n\pi}\pi^{\mu\nu}I_{\nu} , \\
\mathcal{J}^{\mu\nu} &= 2\pi^{\langle\mu}_{\lambda}\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma^{\nu\rangle}_{\lambda} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} - \tau_{\pi n}n^{\langle\mu}F^{\nu\rangle} \\
&\quad + \ell_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} + \lambda_{\pi n}n^{\langle\mu}I^{\nu\rangle} .
\end{aligned}
\tag{348}
$$

where $F^{\mu} = \nabla^{\mu}p_0$ and $I^{\mu} = \nabla^{\mu}\alpha_0$

The tensors $\mathcal{K}$, $\mathcal{K}^\mu$, and $\mathcal{K}^{\mu\nu}$ contain all terms of second order in Knudsen number,

$$\begin{aligned}
\mathcal{K} &= \zeta_1\, \omega_{\mu\nu}\omega^{\mu\nu} + \zeta_2\, \sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta_3\, \theta^2 + \zeta_4\, I \cdot I + \zeta_5\, F \cdot F + \zeta_6\, I \cdot F + \zeta_7\, \nabla \cdot I + \zeta_8\, \nabla \cdot F, \\
\mathcal{K}^\mu &= \kappa_1 \sigma^{\mu\nu} I_\nu + \kappa_2 \sigma^{\mu\nu} F_\nu + \kappa_3 I^\mu \theta + \kappa_4 F^\mu \theta + \kappa_5 \omega^{\mu\nu} I_\nu + \kappa_6 \Delta^\mu_\lambda \partial_\nu \sigma^{\lambda\nu} + \kappa_7 \nabla^\mu \theta, \\
\mathcal{K}^{\mu\nu} &= \eta_1 \omega_\lambda{}^{\langle\mu}\, \omega^{\nu\rangle\lambda} + \eta_2 \theta \sigma^{\mu\nu} + \eta_3 \sigma^{\lambda\langle\mu}\, \sigma^{\nu\rangle}_\lambda + \eta_4 \sigma^{\langle\mu}_\lambda\, \omega^{\nu\rangle\lambda} \\
&\quad + \eta_5 I^{\langle\mu}\, I^{\nu\rangle} + \eta_6 F^{\langle\mu}\, F^{\nu\rangle} + \eta_7 I^{\langle\mu}\, F^{\nu\rangle} + \eta_8 \nabla^{\langle\mu}\, I^{\nu\rangle} + \eta_9 \nabla^{\langle\mu}\, F^{\nu\rangle} .
\end{aligned} \tag{349}$$

The tensors $\mathcal{R}$, $\mathcal{R}^\mu$, and $\mathcal{R}^{\mu\nu}$ contain all terms of second order in inverse Reynolds number,

$$\begin{aligned}
\mathcal{R} &= \varphi_1 \Pi^2 + \varphi_2 n \cdot n + \varphi_3 \pi_{\mu\nu}\pi^{\mu\nu}, \\
\mathcal{R}^\mu &= \varphi_4 n_\nu \pi^{\mu\nu} + \varphi_5 \Pi n^\mu, \\
\mathcal{R}^{\mu\nu} &= \varphi_6 \Pi \pi^{\mu\nu} + \varphi_7 \pi^{\lambda\langle\mu}\, \pi^{\nu\rangle}_\lambda + \varphi_8 n^{\langle\mu}\, n^{\nu\rangle} .
\end{aligned} \tag{350}$$

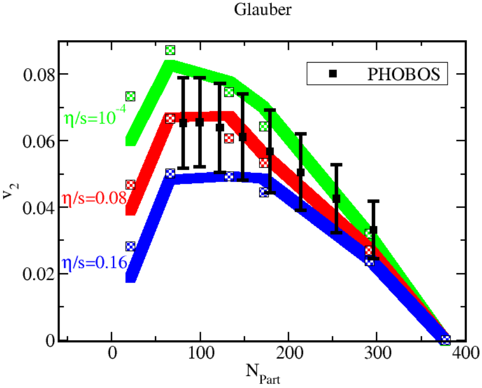
These equations contain all the contributions up to order $\mathcal{O}(\mathrm{Kn}^2)$, $\mathcal{O}(\mathrm{R}_i^{-1}\mathrm{R}_j^{-1})$, and $\mathcal{O}(\mathrm{Kn}\mathrm{R}_i^{-1})$.

Section 10

Results from Dissipative fluid dynamics

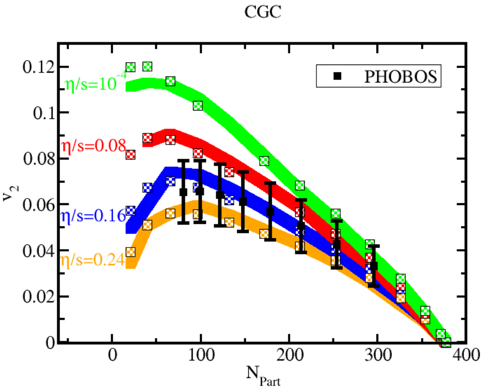
Shear viscosity over entropy density ratio is small

P. Romatschke and U. Romatschke, Phys.Rev.Lett. **99**, 172301 (2007)



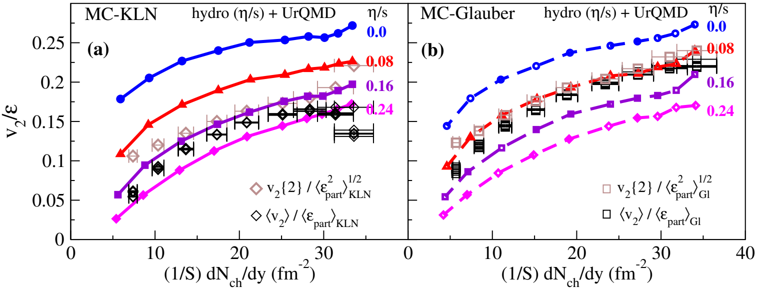
. . . , but the extracted value depends on the initial conditions

P. Romatschke and U. Romatschke, Phys.Rev.Lett. **99**, 172301 (2007)



Extracting η/s from experimental data: Initial conditions

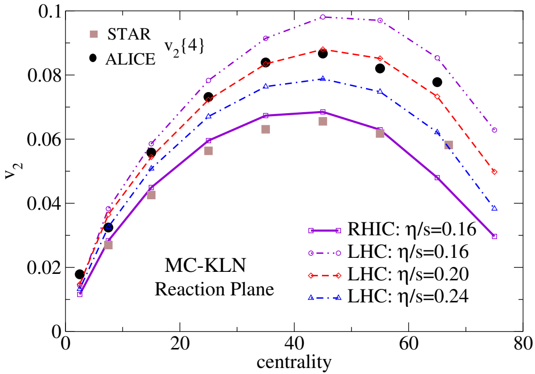
C. Shen, S. A. Bass, T. Hirano, P. Huovinen, Z. Qiu, H. Song and U. Heinz, J. Phys. G **38**, 124045 (2011) [arXiv:1106.6350 [nucl-th]]



- $\eta/s \sim 0.08 - 0.24$
- Large uncertainty from the initial conditions (MC-Glauber vs. MC-KLN)

Extracting η/s from experimental data: RHIC vs LHC

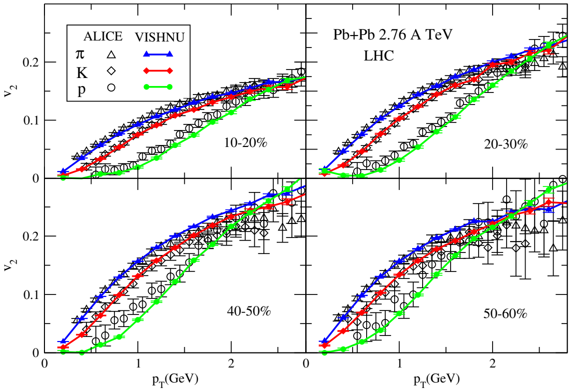
C. Shen, S. A. Bass, T. Hirano, P. Huovinen, Z. Qiu, H. Song and U. Heinz, J. Phys. G **38**, 124045 (2011) [arXiv:1106.6350 [nucl-th]]



- $\eta/s \sim 0.16$ (RHIC) $\neq \eta/s \sim 0.20$ (LHC)?
- Temperature dependent η/s ?

Extracting η/s from experimental data: identified hadrons

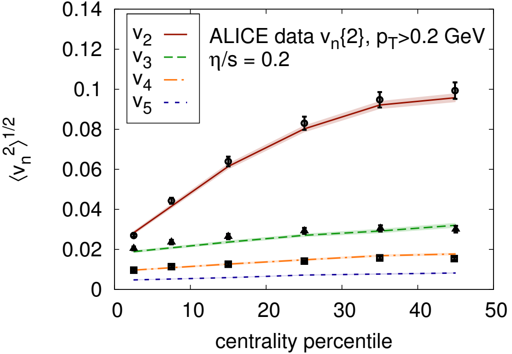
H. Song, S. Bass and U. W. Heinz, Phys. Rev. C **89**, 034919 (2014) [arXiv:1311.0157 [nucl-th]]



- UrQMD + (2+1)-D viscous hydro (hybrid)
- Mass ordering of elliptic flow (definite prediction of hydrodynamics)
- Details of hadronic evolution matter

Extracting η/s from experimental data: higher harmonics

C. Gale, S. Jeon, B. Schenke, P. Tribedy and R. Venugopalan, Phys. Rev. Lett. **110**, 012302 (2013) [arXiv:1209.6330 [nucl-th]]

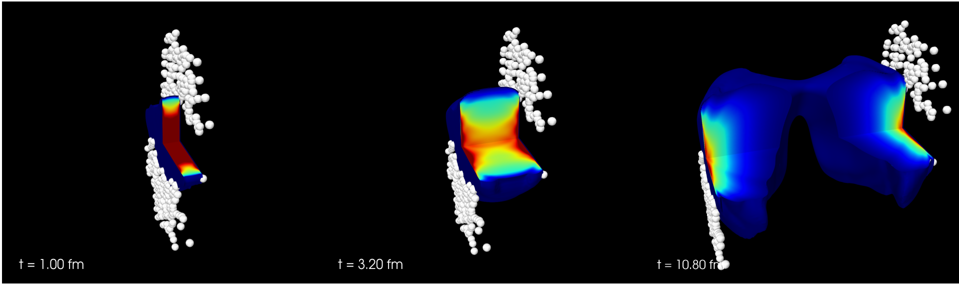


- Higher harmonics (v_n's)
- Initial density fluctuations make all the difference
- IP-Glasma + viscous hydrodynamics $\longrightarrow$ v_n's well described with $\eta/s = 0.20$

Section 11

How to constrain $\eta/s(T)$

Search for QCD matter properties



Relativistic heavy ion collisions:

- Create small droplet of QCD fluid
- Extract limits for η/s, ζ/s, ... from experimental data

Need a complete model:

- Initial particle production
- Fluid dynamical evolution
- Convert fluid to particle spectra

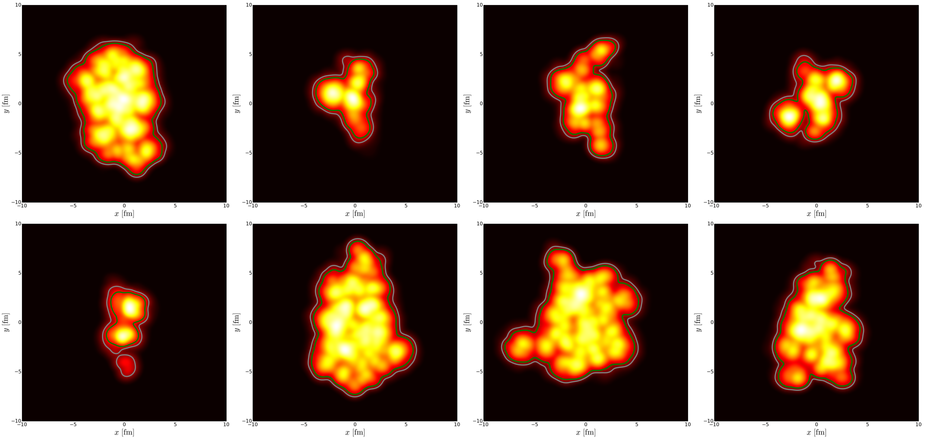
How to proceed:

- Take your favorite initial condition model/parametrizations, and EoS
- Tune your initial state to reproduce the multiplicities (also centrality dependence)
- Tune your chemical/kinetic freeze-out parameters to reproduce the p_T spectra of hadrons.
- Tune your $\eta/s(T)$ parametrization to reproduce the v_2 data.
- Check against v_3, v_4, correlations, fluctuations etc.
- If the parameter tuning done for the LHC, retune the model for RHIC (but keeping the properties of the matter unchanged, i.e. EoS and $\eta/s(T)$)
- Check the consistency with the RHIC data.
- If unable to get all the data simultaneously, make a new $\eta/s(T)$ parametrization, and repeat. . .

Collisions come with all centralities

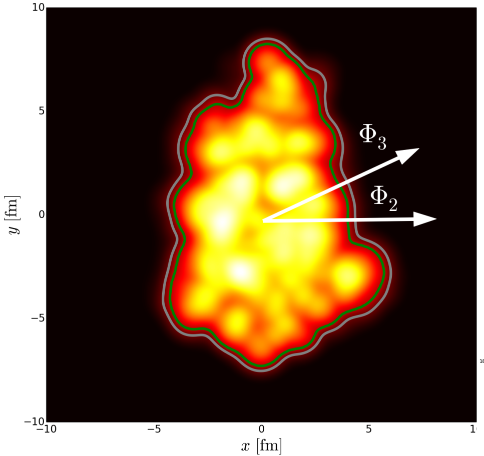
Classify events according to the number of produced particles (multiplicity): 0-5 % centrality class contains 5 % of events with largest multiplicity,
5-10 % centrality class the next 5 %, and so on

Initial states come in all shapes



Average over all events

Characterizing initial conditions

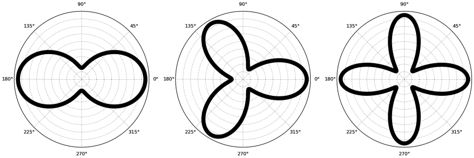


$$\varepsilon_n e^{in\Phi_n} = \{r^n e^{in\phi}\}$$

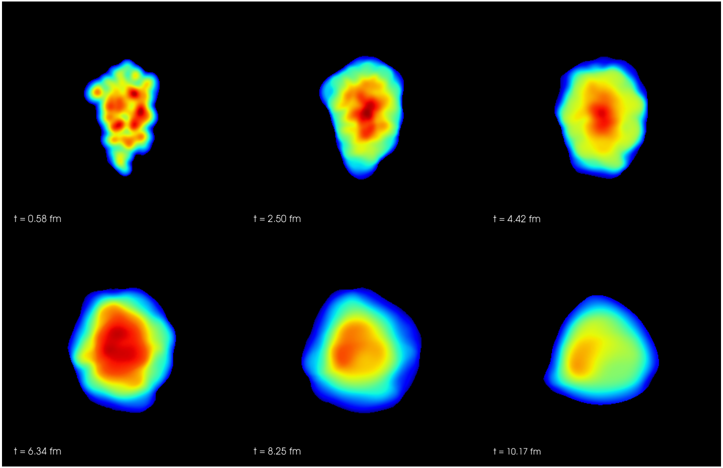
$$\{\cdots\} = \int dxdy\, e(x, y, \tau_0)(\cdots)$$

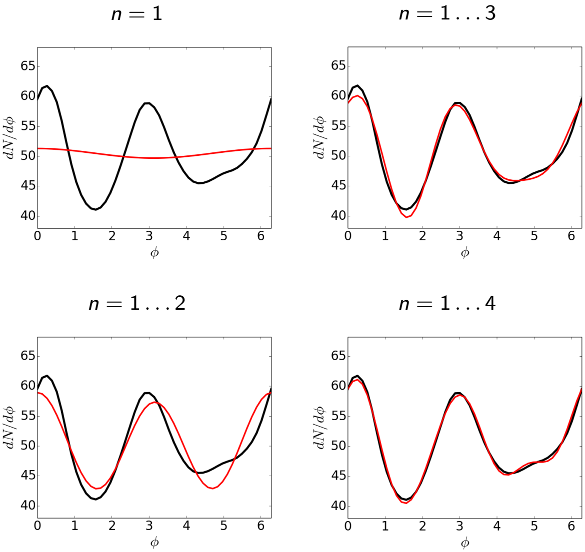
- ε_n eccentricity
- Φ_n "participant plane" angle

n=2 n=3 n=4



Fluid dynamical evolution





- Example: azimuthal spectrum of charged hadrons $dN/d\phi$ in one collision
- **Black: full result**
- **Red: Fourier decomposition**
- Typically v_2 dominant
- Because initial state fluctuates (esp. its eccentricities), also v_n coefficients fluctuate.
- $\longrightarrow$ Even for a fixed centrality class: distribution of v_n, $P(v_n)$.

Flow coefficients as correlators

In a single event define:

$$v_n(p_T, y)e^{in\Psi_n(p_T,y)} = \langle e^{in\phi}\rangle_\phi, \qquad (351)$$

where the angular brackets $\langle\cdots\rangle_\phi$ denote the average

$$\langle\cdots\rangle_\phi = \left(\frac{dN}{dydp_T^2}\right)^{-1} \int d\phi \frac{dN}{dydp_T^2 d\phi}\,(\cdots)\,. \qquad (352)$$

Similarly, the p_T-integrated flow coefficients are defined as

$$v_n(y)e^{in\Psi_n(y)} = \langle e^{in\phi}\rangle_{\phi,p_T}, \qquad (353)$$

where the average is defined as

$$\langle\cdots\rangle_{\phi,p_T} = \left(\frac{dN}{dy}\right)^{-1} \int d\phi dp_T^2 \frac{dN}{dydp_T^2 d\phi}\,(\cdots)\,. \qquad (354)$$

n-particle cumulants

Two-particle cumulant is defined as the correlation

$$v_n\{2\}^2 = \langle e^{in(\phi_1-\phi_2)}\rangle_\phi \equiv \frac{1}{N_2}\int d\phi_1 d\phi_2 \frac{dN_2}{d\phi_1 d\phi_2} e^{in(\phi_1-\phi_2)}, \quad (355)$$

where $dN_2/d\phi_1 d\phi_2$ is the two-particle spectrum (suppressing the possible rapidity and p_T dependence), which can in general be decomposed as a sum of a product of single-particle spectra and a "direct" two-particle correlation $\delta_2(\phi_1, \phi_2)$,

$$\frac{dN_2}{d\phi_1 d\phi_2} = \frac{dN}{d\phi_1}\frac{dN}{d\phi_2} + \delta_2(\phi_1, \phi_2). \quad (356)$$

The direct correlations can result e.g. from a ρ-meson decaying into two pions, and these correlations are usually referred to as non-flow contributions. The event-averaged two-particle cumulant can be written as

$$v_n\{2\} = \langle v_n^2 + \delta_2\rangle_{ev}^{1/2} \overset{\text{flow}}{=} \langle v_n^2\rangle_{ev}^{1/2}, \quad (357)$$

where the last equality follows in the absence of the non-flow contributions, i.e. assuming that all the azimuthal correlations are due to the collective flow only.

Similarly, the event-averaged p_T-integrated 4-particle cumulant flow coefficients are defined as

$$v_n\{4\} \equiv \left(2\langle v_n^2\rangle_{ev}^2 - \langle v_n^4\rangle_{ev}\right)^{1/4}. \tag{358}$$

In addition to the $v_n\{2\}$ and $v_n\{4\}$, we can also define a three-particle cumulant $v_4\{3\}$

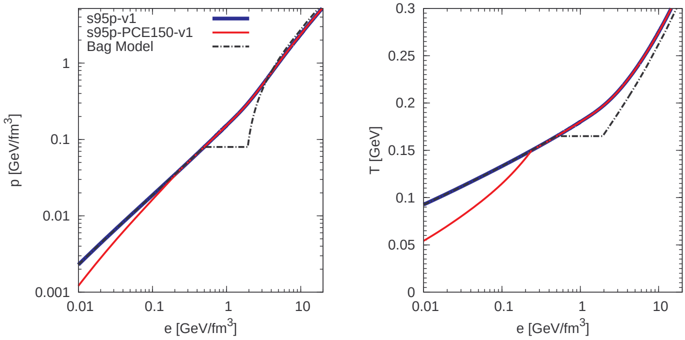
$$v_4\{3\} \equiv \frac{\langle v_2^2 v_4 \cos(4\,[\Psi_2 - \Psi_4])\rangle_{ev}}{\langle v_2^2\rangle_{ev}}. \tag{359}$$

Originally, the higher-order cumulants were introduced to suppress the non-flow correlations, but after the full realization of the importance of the event-by-event fluctuations it has become clear that different cumulants do not only have different sensitivity to non-flow correlations, but also measure different moments of the underlying probability distributions of the flow coefficients. v_n have not only distributions, but they can also have correlations

$$\langle\cos(k_1\Psi_1 + \cdots + nk_n\Psi_n)\rangle_{\mathrm{SP}} \equiv \frac{\langle v_1^{|k_1|} \cdots v_n^{|k_n|} \cos(k_1\Psi_1 + \cdots + nk_n\Psi_n)\rangle_{ev}}{\sqrt{\langle v_1^{2|k_1|}\rangle_{ev} \cdots \langle v_n^{2|k_n|}\rangle_{ev}}}, \tag{360}$$

where the k_n's are integers with the property $\sum_n nk_n = 0$.

Equation of State



- Lattice parametrization by Petreczky/Huovinen:
  Nucl. Phys. **A837**, 26-53 (2010), [arXiv:0912.2541 [hep-ph]].
- Chemical equilibrium (s95p-v1)
- **(partial) chemical freeze-out at** $T_{\text{chem}} = 175$ **MeV (s95p-PCE175-v1)**
- for comparison bag-model EoS
- Hadron Resonance Gas (HRG) includes all hadronic states up to $m \sim 2$ GeV

Converting fluid to particles

$$e,\ u^\mu,\ \pi^{\mu\nu} \longrightarrow E\frac{dN}{d^3\mathbf{p}}$$

- Standard Cooper-Frye freeze-out for particle i

$$E\frac{dN}{d^3\mathbf{p}} = \frac{g_i}{(2\pi)^3}\int d\sigma^\mu p_\mu f_i(\mathbf{p},x),$$

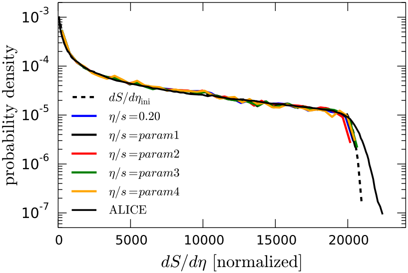
where

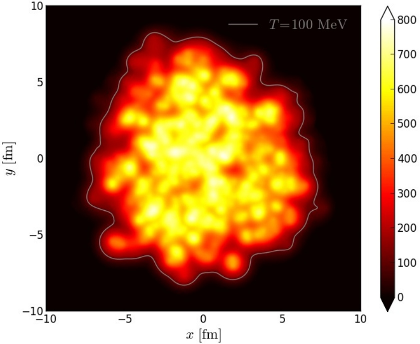
$$f_i(\mathbf{p},x) = f_{i,\mathrm{eq}}(\mathbf{p},u^\mu,T,\{\mu_i\})\left[1+\frac{\pi^{\mu\nu}p_\mu p_\nu}{2T^2(e+p)}\right]$$

- Integral over constant temperature hypersurface
- 2- and 3-body decays of unstable hadrons included
- Here $T_{\mathrm{dec}} = 100$ MeV

Initial conditions and centrality selection

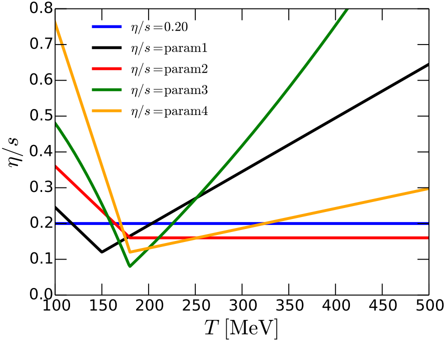
Example: H. Niemi, K. J. Eskola and R. Paatelainen, Phys. Rev. C **93**, no. 2, 024907 (2016) [arXiv:1505.02677 [hep-ph]].





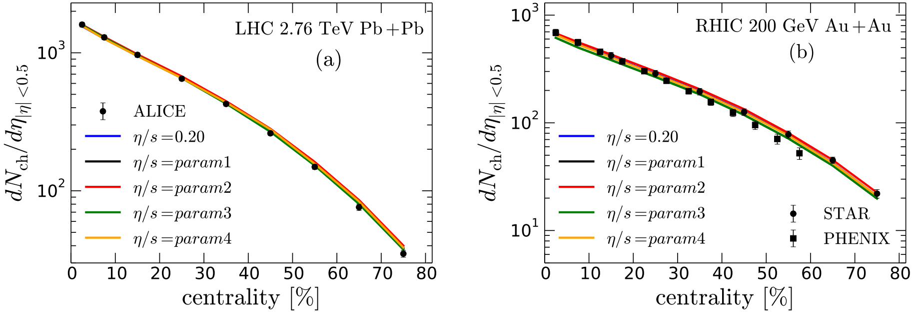
- Calculate ensemble of random initial conditions: random impact parameter, random positions of nucleons inside the nuclei.
- Calculate hydrodynamical evolution and spectra for each initial conditions
- Divide events into centrality classes according to hadron multiplicity.

Temperature dependent η/s



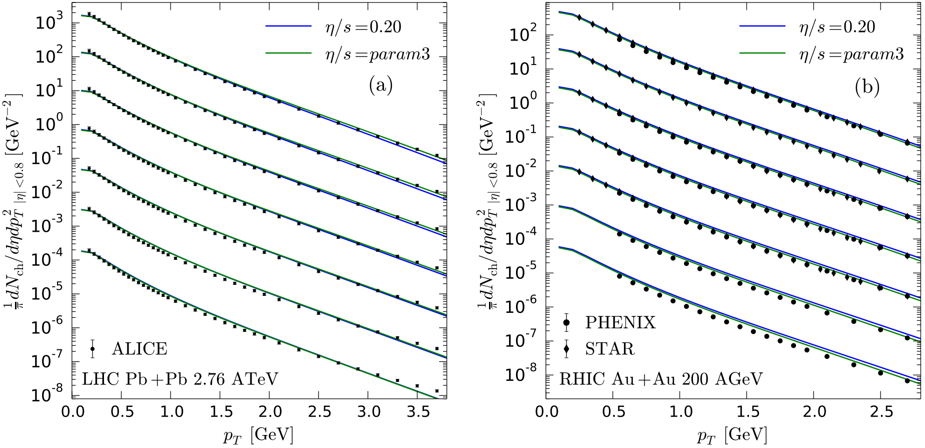
- Test different temperature dependencies.

multiplicity



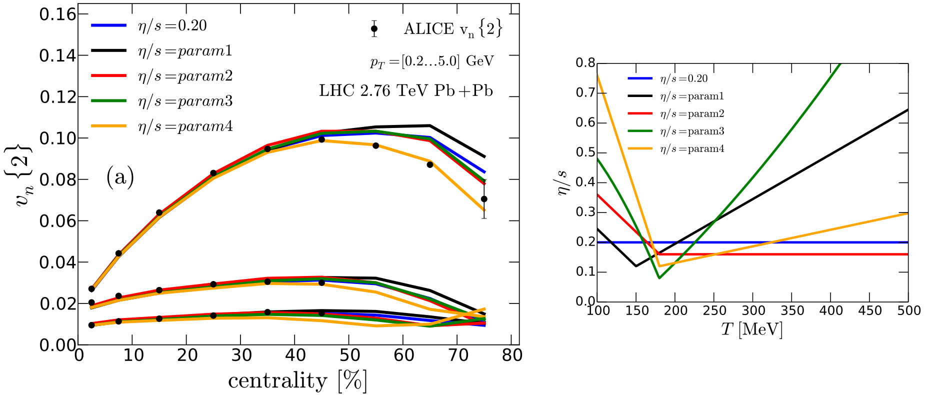
- Fix the parameters of the initial conditions to reproduce the centrality dependence of the hadron multiplicity.
- Entropy production depends on the viscosity.
- If the particle number conservation is not solved explicitly: entropy production = particle production.

Transverse momentum spectra



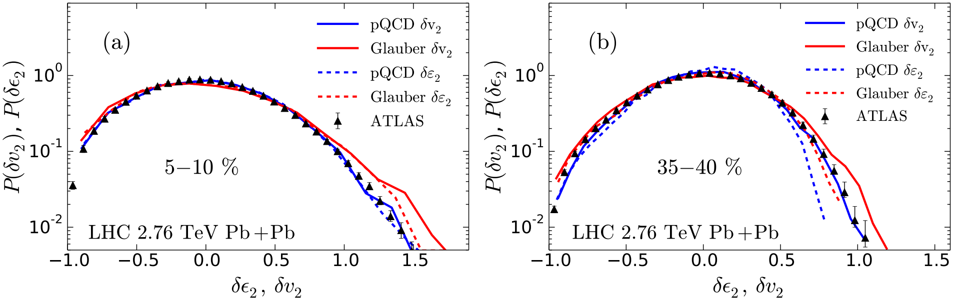
- kinetic ($T_{\rm dec}$) and chemical ($T_{\rm chem}$) decoupling temperatures are the most important parameters that determine the shape of p_T-spectra.
- $T_{\rm dec} = 100$ MeV
- $T_{\rm chem} = 175$ MeV

$\eta/s(T)$ from v_n data



- $\eta/s(T)$ parametrizations tuned to reproduce the v_n data at the LHC.
- No strong constraints to the temperature dependence (all give equally good agreement)
- Deviations mainly in peripheral collisions, where the applicability of hydro most uncertain.

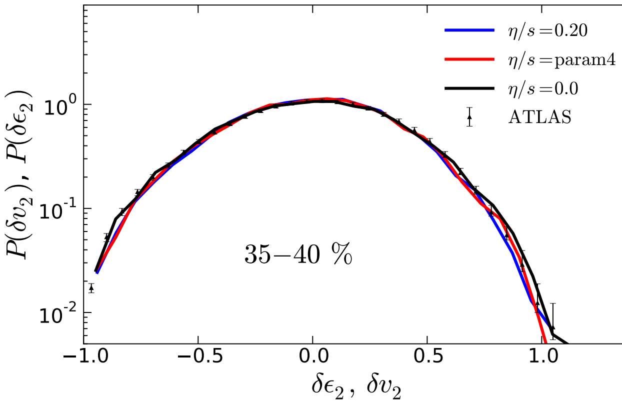
Flow fluctuations



$$\delta v_n = \frac{v_n - \langle v_n \rangle_{ev}}{\langle v_n \rangle_{ev}}$$

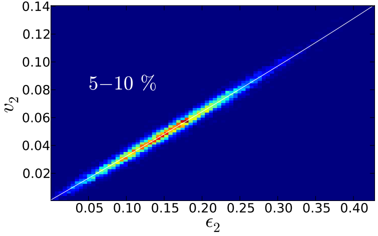
- Even in one centrality class ϵ_n fluctuates from event to event $\longrightarrow$ v_n fluctuates.
- Event-by-event models should also reproduce the v_n fluctuation spectra.
- Turns out, if the average v_n is scaled out, that v_n fluctuations mainly sensitive to initial conditions.

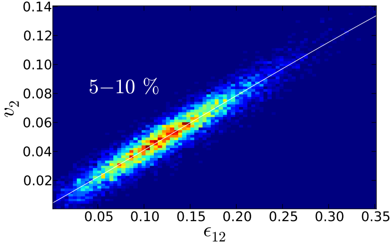
Sensitivity of $P(\delta v_2)$ to viscosity

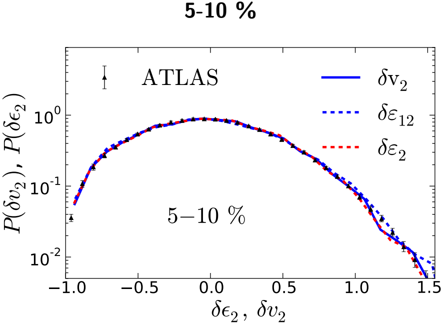


- Scaled fluctuations show no sensitivity to η/s.

(non)linear-response?





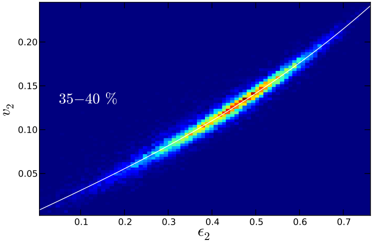


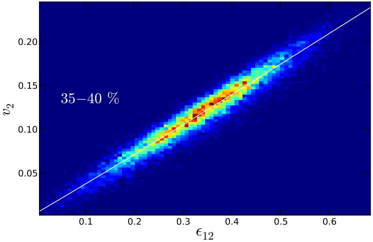
$$\varepsilon_{m,n} e^{in\Psi_{m,n}} = -\{r^m e^{in\phi}\}/\{r^m\},$$

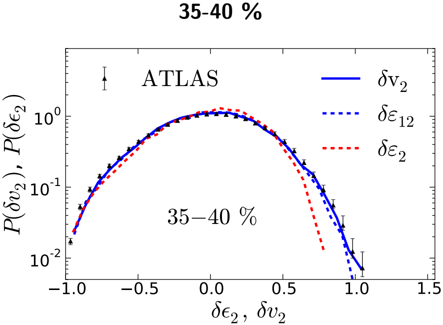
$$\varepsilon_2 \equiv \varepsilon_{2,2} \text{ vs } \varepsilon_{1,2}$$

Full azimuthal structure: $m = 0, \ldots, \infty$

(non)linear-response?





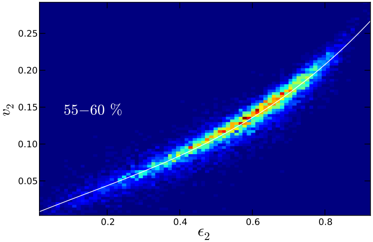


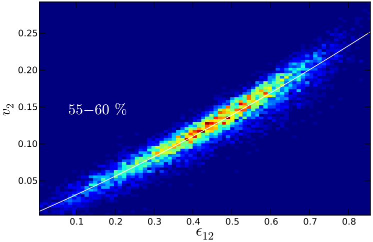
$$\varepsilon_{m,n} e^{in\Psi_{m,n}} = -\{r^m e^{in\phi}\}/\{r^m\},$$

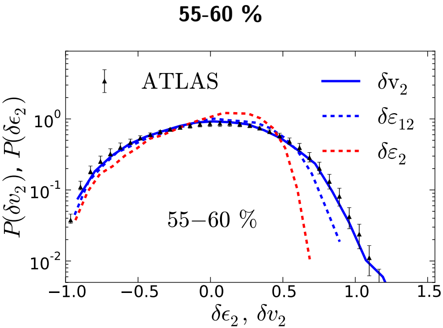
$\varepsilon_2 \equiv \varepsilon_{2,2}$ vs $\varepsilon_{1,2}$

Full azimuthal structure: $m = 0, \ldots, \infty$

(non)linear-response?





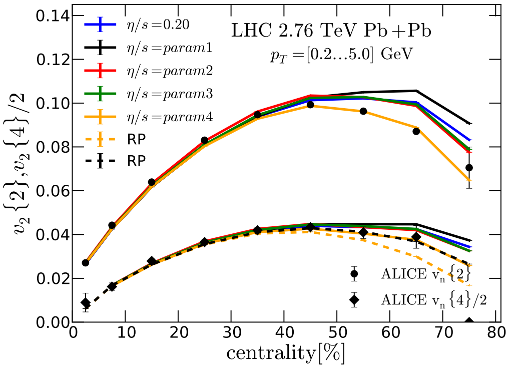


$$\varepsilon_{m,n} e^{in\Psi_{m,n}} = -\{r^m e^{in\phi}\}/\{r^m\},$$

$\varepsilon_2 \equiv \varepsilon_{2,2}$ vs $\varepsilon_{1,2}$

Full azimuthal structure: $m = 0, \ldots, \infty$

Flow fluctuations from $v_n\{2\}$ and $v_n\{4\}$



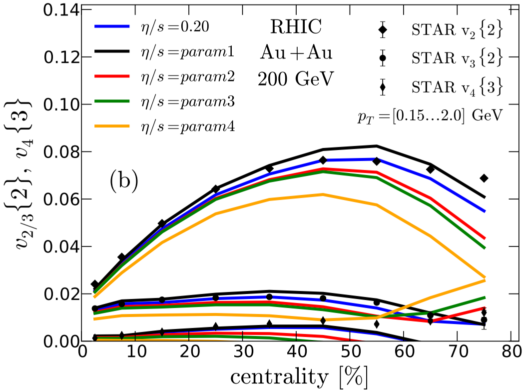
$v_n\{2\}$ and $v_n\{4\}$ measure different moments of the v_n–fluctuation spectrum:

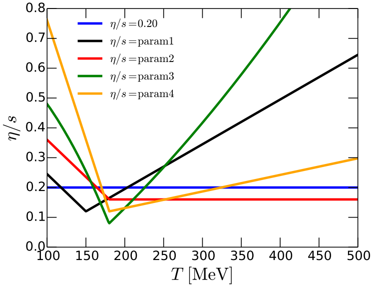
$$v_n\{2\} \stackrel{\text{flow}}{=} \langle v_n^2 \rangle_{ev}^{1/2}$$

$$v_n\{4\} \stackrel{\text{flow}}{=} \left(2\langle v_n^2\rangle_{ev}^2 - \langle v_n^4\rangle_{ev}\right)^{1/4}$$

- If v_n fluctuation spectra and average $v_n\{2\}$ are reproduced $\longrightarrow$ also $v_n\{4\}$ should come out right (It is just a different moment of the full distribution)
- non-flow effects?
- v_n w.r.t. reaction plane $\sim v_n\{4\}$

Constraints for $\eta/s(T)$ from RHIC v_n data

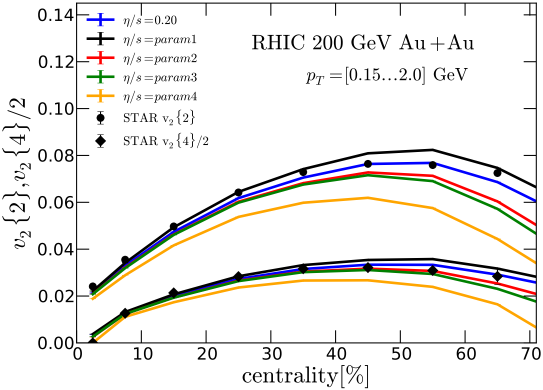




$$v_4\{3\} \equiv \frac{\langle v_2^2 v_4 \cos(4\,[\Psi_2 - \Psi_4])\rangle_{ev}}{\langle v_2^2\rangle_{ev}}.$$

- At RHIC different sensitivity to $\eta/s(T)$: parametrizations that fit nicely LHC v_n's start to deviate from each other at RHIC.

Flow fluctuations



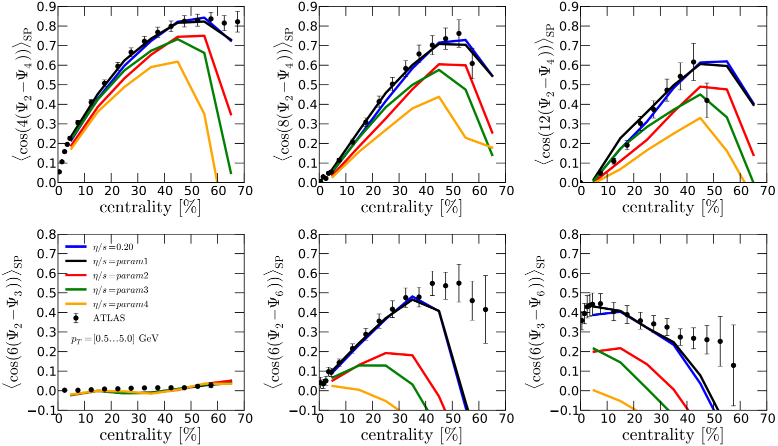
- No direct measurements of the v_n fluctuation spectra, but different cumulants.
- Simultaneous fit of $v_2\{2\}$ and $v_2\{4\}$ minimal condition to describe the full spectra.

Event-plane correlations

Luzum & Ollitrault: one should calculate:

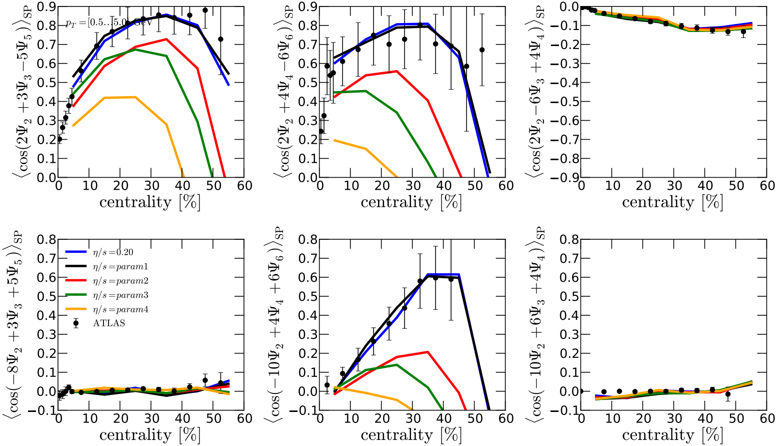
$$\langle \cos(k_1\Psi_1 + \cdots + nk_n\Psi_n)\rangle_{\mathrm{SP}} \equiv \frac{\langle v_1^{|k_1|} \cdots v_n^{|k_n|} \cos(k_1\Psi_1 + \cdots + nk_n\Psi_n)\rangle_{ev}}{\sqrt{\langle v_1^{2|k_1|}\rangle_{ev} \cdots \langle v_n^{2|k_n|}\rangle_{ev}}},$$

Event-plane correlations: 2 angles



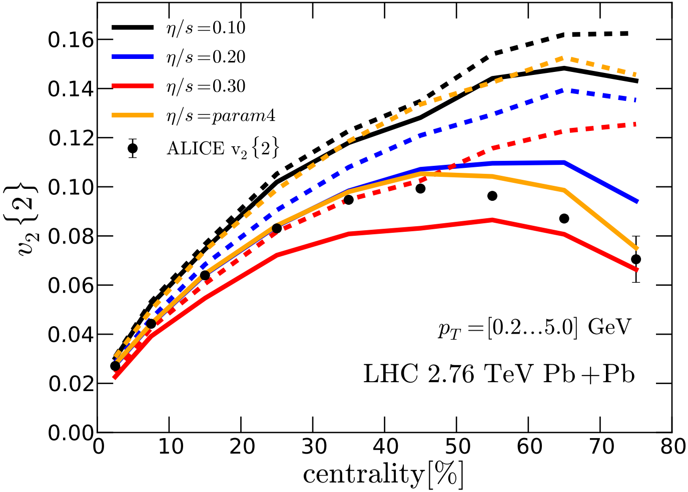
- Already from the LHC data more constraints to $\eta/s(T)$.

Event-plane correlations: 3 angles



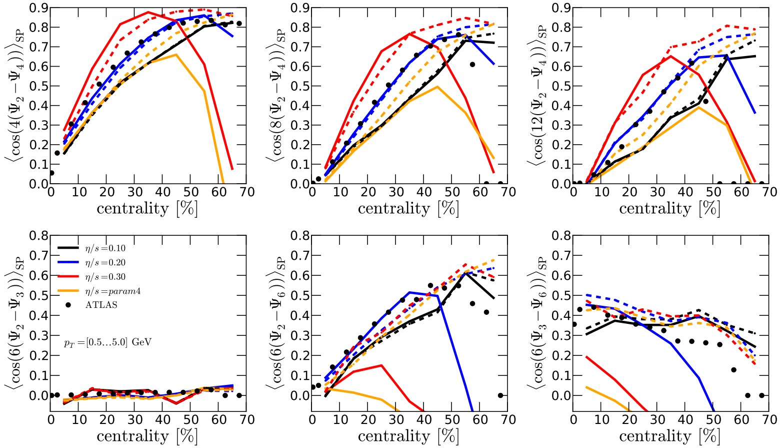
- Equally well described by the same parametrizations that describe 2-angle correlations.

δf in v_2



- Relative magnitude of δf depends on the η/s parametrization
- δf also larger at lower energy collisions.

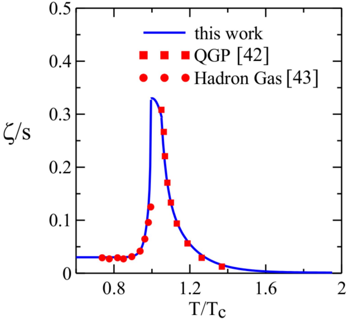
δf in event-plane correlations



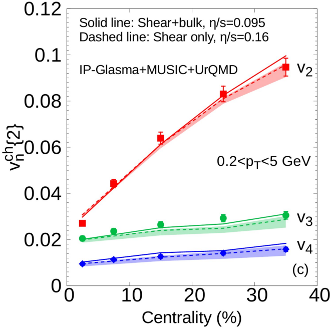
- In (Ψ_2, Ψ_4)-correlators, in central to mid-peripheral collisions: δf corrections small.
- but note the correlatos involving Ψ_6: δf can destroy the correlation completely.

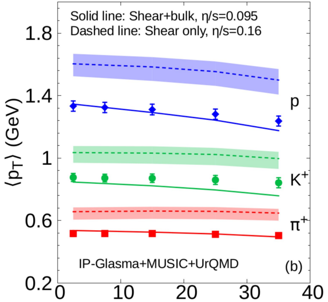
Bulk viscosity

- Bulk viscosity can be large near the QCD transition
- Large bulk viscosity affects the determination of η/s
- Helps to reduce average p_T (important especially at LHC energies)



Ryu, Paquet, Shen, Denicol, Schenke, Jeon, Gale, arXiv:1502.01675 [nucl-th]



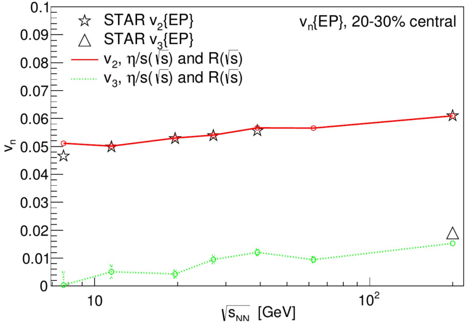


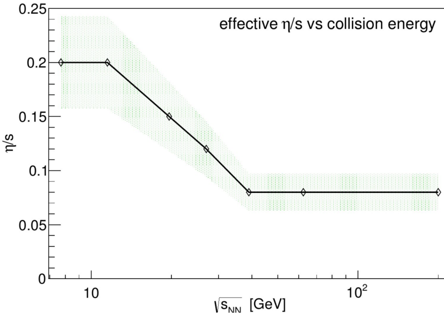
Beam Energy scan

- More constraints to the hadronic properties of the matter
- Important background in determining the QGP properties
- Here constant η/s fitted separately for each $\sqrt{s}$

Evidence for temperature and/or net-baryon density dependence of η/s?

Iu.A. Karpenko, P. Huovinen, H. Petersen, M. Bleicher, Phys.Rev. C91 (2015) 6, 064901



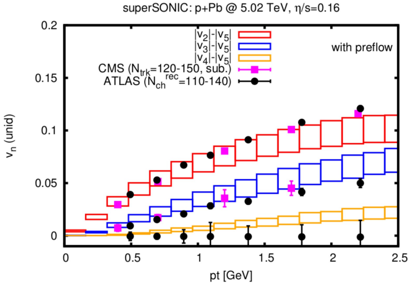


p+Pb collisions

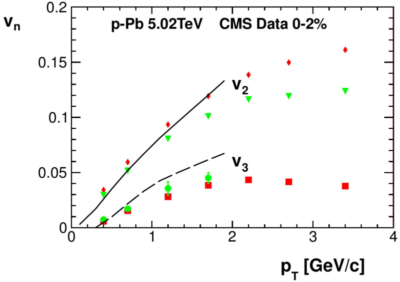
- Can be described by using hydrodynamics
- Typically η/s small $O(0.08)$
- Inconsistency with AA results with saturation based initial conditions $\eta/s \sim 0.20$

Is hydrodynamics valid?

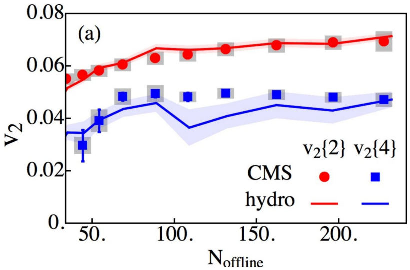
Romatschke, Eur.Phys.J. C75 (2015) no.7, 305



P. Bozek, W. Broniowski and G. Torrieri, Phys. Rev. Lett. 111, 172303 (2013)



Kozlov, Luzum, Denicol, Jeon, Gale, Nucl. Phys. A931 (2014)



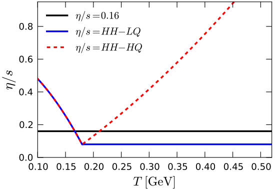
Section 12

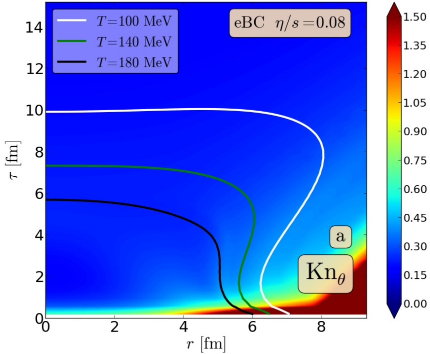
Applicability of fluid dynamics: Knudsen numbers

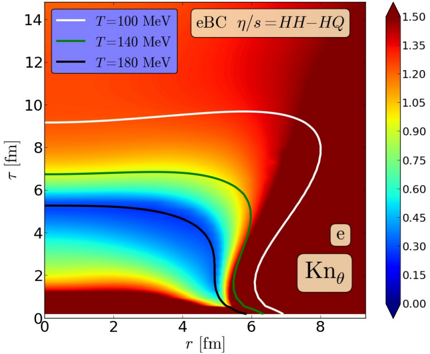
Knudsen numbers: applicability of fluid dynamics

H. Niemi and G. S. Denicol, arXiv:1404.7327

- $\mathrm{Kn} = \tau_\pi \theta$, $\theta =$ expansion rate, $\tau_\pi =$ relaxation/thermalization time.
- fluid dynamics: $\mathrm{Kn} < 0.5$ blue
- change $\eta/s = 0.08 \longrightarrow$ *HH-HQ*
- Applicability: whole spacetime $\longrightarrow$ small region around T_c



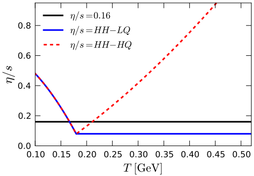




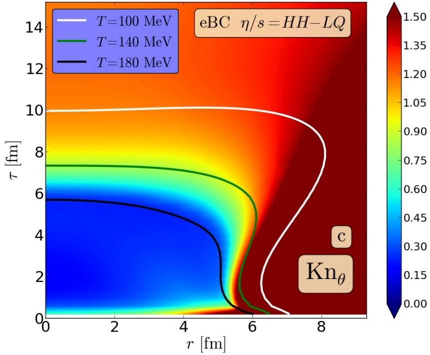
Knudsen numbers: applicability of fluid dynamics in pA

H. Niemi and G. S. Denicol, arXiv:1404.7327 [nucl-th]

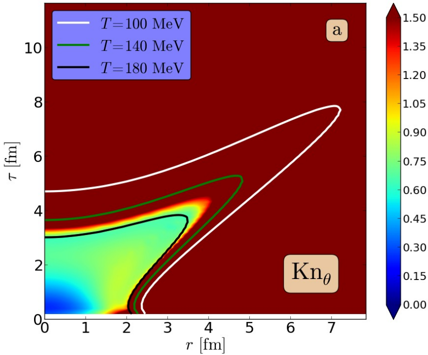
- AA collisions $\longrightarrow$ pA collisions
- $\mathrm{Kn} > 0.5$ almost everywhere
- even with small QGP $\eta/s = 0.08$



Pb+Pb



p+Pb



Summary

- QCD properties near thermal equlibrium (EoS, transport coefficients) direct input to fluid dynamics $\longrightarrow$ Fluid dynamics convenient tool in extracting the properties from the data.
- Ideal fluid dynamics: assume local thermal equilibrium (LTE)
- Relativistic Navier-Stokes: deviations from LTE (**unstable, acausal**) $\longrightarrow$ viscosity
- Israel-Stewart or transient fluid dynamics: include effects of non-zero relaxation/thermalization time.
- In HI-collisions: in addition model initial sate and freeze-out
- QGP shear viscosity (η/s) small (at least near the QCD phase transition)
- Applicability of fluid dynamics in small systems? (especially pA and peripheral AA collisions)

Some further references

- Good introduction to relativistic fluid dynamics:
  D. H. Rischke, Lect. Notes Phys. **516**, 21 (1999) [nucl-th/9809044].
- The Bible of Relativistic fluid dynamics:
  Relativistic Kinetic Theory. Principles and Applications - De Groot, S.R. et al.
- The Israel-Stewart paper:
  W. Israel and J. M. Stewart, Annals Phys. **118**, 341 (1979).
- Resummed transient fluid dynamics:
  G. S. Denicol, H. Niemi, E. Molnar and D. H. Rischke, Phys. Rev. D **85**, 114047 (2012) [arXiv:1202.4551 [nucl-th]].
  G. S. Denicol, H. Niemi, I. Bouras, E. Molnar, Z. Xu, D. H. Rischke and C. Greiner, Phys. Rev. D **89**, no. 7, 074005 (2014) [arXiv:1207.6811 [nucl-th]].
- Reviews of hydrodynamics and flow in heavy-ion collisions:
  U. Heinz and R. Snellings, Ann. Rev. Nucl. Part. Sci. **63**, 123 (2013) [arXiv:1301.2826 [nucl-th]].
  M. Luzum and H. Petersen, J. Phys. G **41**, 063102 (2014) [arXiv:1312.5503 [nucl-th]].