

FRACTIONAL POINCARÉ AND LOCALIZED HARDY INEQUALITIES ON METRIC SPACES

BARTŁOMIEJ DYDA, JUHA LEHRBÄCK, AND ANTTI V. VÄHÄKANGAS

ABSTRACT. We prove fractional Sobolev–Poincaré inequalities, capacity versions of fractional Poincaré inequalities, and pointwise and localized fractional Hardy inequalities in a metric space equipped with a doubling measure. Our results generalize and extend earlier work where such inequalities have been considered in the Euclidean spaces or in the non-fractional setting in metric spaces. The results concerning pointwise and localized variants of fractional Hardy inequalities are new even in the Euclidean case.

1. INTRODUCTION

Let $X = (X, d, \mu)$ be a metric measure space and let $1 \leq p, q, t < \infty$ and $0 < s < 1$. The fractional (s, q, p, t) -Poincaré (or Sobolev–Poincaré) inequality on X reads as

$$\left(\int_B |u(x) - u_B|^q dx \right)^{1/q} \leq c_P r^s \left(\int_B \left(\int_{\lambda B} \frac{|u(x) - u(y)|^t}{d(x, y)^{st} \mu(B(x, d(x, y)))} dy \right)^{p/t} dx \right)^{1/p}, \quad (1.1)$$

where $dx = d\mu(x)$ and $dy = d\mu(y)$. We say that X supports a (s, q, p, t) -Poincaré inequality if there are constants $c_P > 0$ and $\lambda \geq 1$ such that inequality (1.1) holds for every ball $B = B(x_0, r) \subset X$ and for all functions $u: X \rightarrow \mathbb{R}$ that are integrable on balls.

If $q \leq \min\{p, t\}$ and the measure μ is doubling, then it is straightforward to show that the space X supports a (s, q, p, t) -Poincaré inequality; see Lemma 2.2. This is quite different compared to the usual (i.e. non-fractional) Poincaré inequalities, whose validity in a metric measure space is usually an indication of the existence of a rich geometric structure in the space; we refer to the monographs [1, 15] for more explanation and examples.

The main goal in this work is to prove stronger variants of fractional inequalities, such as (Sobolev–)Poincaré inequalities for $q > p$, capacity versions of Poincaré inequalities, and pointwise and localized Hardy inequalities. The validity of these stronger variants often requires additional assumptions on the space and the functions and sets in the inequalities. For example, in the so-called boundary Poincaré inequalities the mean value u_B on the left-hand side of (1.1) can be omitted if the set where $u = 0$ (i.e. the “boundary”) is large enough.

The parameter $1 \leq t < \infty$ in inequality (1.1) allows certain flexibility in the applications, for instance in the proof of the localized fractional Hardy inequality

$$\int_{B \setminus E} \frac{|u(x)|^p}{d(x, E)^{sp}} dx \leq C \int_{\lambda B} \int_{\lambda B} \frac{|u(x) - u(y)|^p}{d(x, y)^{sp} \mu(B(x, d(x, y)))} dy dx, \quad (1.2)$$

where $0 < s < 1$, $1 < p < \infty$, $\lambda \geq 1$, $E \subset X$ is a closed set, $B = B(w, r)$ for some $w \in E$ and $0 < r < \text{diam}(E)$, and $u: X \rightarrow \mathbb{R}$ is a continuous function with $u = 0$ on E . As one of our main results we show that the validity of inequality (1.2) is essentially characterized by dimensional information related to the set E . More precisely, $\overline{\text{codim}}_A(E) < sp$ is sufficient and $\overline{\text{codim}}_A(E) \leq sp$ is necessary for (1.2), where $\overline{\text{codim}}_A(E)$ is the upper Assouad codimension of E , see Definition 4.4. The upper bound for this codimension means that the set E must be sufficiently large in comparison to the size of the ambient space X . In the Euclidean case $X = \mathbb{R}^n$ we have

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$\overline{\text{codim}}_A(E) = n - \underline{\text{dim}}_A(E)$, where $\underline{\text{dim}}_A$ is the lower dimension (or lower Assouad dimension) of $E \subset \mathbb{R}^n$.

We obtain the localized inequality (1.2) as a consequence of a pointwise fractional Hardy inequality, given in terms of a maximal operator. In the non-fractional case in \mathbb{R}^n , pointwise Hardy inequalities were introduced in [11] and [20]. Sufficient and necessary conditions for pointwise Hardy inequalities in metric spaces have been given in [21]; see also [22] for weighted variants. Fractional Hardy inequalities on open sets have been studied in the Euclidean space \mathbb{R}^n for instance in [4, 7, 9, 10, 18, 24] and in general metric spaces in [5, 8], but the present pointwise and localized versions of fractional Hardy inequalities, as well as the boundary Poincaré inequalities, are new even in the Euclidean case. Sobolev–Poincaré inequalities have also been considered in more general sets than balls, in particular in the so-called John domains, see [2, 16, 17] and the references therein.

The outline for the rest of the paper is as follows. In Section 2 we review the necessary definitions and notation on metric measure spaces and give in Lemma 2.2 the basic versions of fractional Poincaré inequalities for $q \leq p$; these are used as a starting point in the proofs of the stronger inequalities in the subsequent sections. Section 3 is devoted to extending the range in the fractional (Sobolev–)Poincaré inequalities to $q > p$, following the ideas in the proofs of the corresponding fractional results in the Euclidean case [6] as well in the non-fractional results in metric spaces [1]. In Section 4 we introduce a variant of the relative fractional capacity and prove a Maz'ya type capacity Poincaré inequality in Theorem 4.3. Boundary Poincaré inequalities are obtained in Theorem 4.7 and Corollary 4.8 under the dimensional condition $\overline{\text{codim}}_A(E) < sp$, which is connected to the relative capacity via suitable Hausdorff contents; see Definition 4.5 and Lemma 4.6. In Section 5, the localized Hardy inequality (1.2) is obtained in Theorem 5.2 as a consequence of a pointwise fractional Hardy inequality, see Theorem 5.1, which in turn is based on the boundary Poincaré inequality in Theorem 4.7. Theorem 5.3 then shows the necessity of the condition $\overline{\text{codim}}_A(E) \leq sp$ for the localized inequality (1.2). In Sections 4 and 5 our proofs often follow the main lines of the proofs from the non-fractional case, as for instance in [1, 15, 21], but due to the non-locality of the setting several modifications are needed in the proofs.

2. PRELIMINARIES

We assume throughout this paper that $X = (X, d, \mu)$ is a metric measure space (with at least two points), where μ is a Borel measure supported on X such that $\mu(\{x\}) = 0$ for all $x \in X$ and $0 < \mu(B) < \infty$ for all (open) balls

$$B = B(x, r) := \{y \in X : d(x, y) < r\}$$

with $x \in X$ and $r > 0$. We make the tacit assumption that each ball $B \subset X$ has a fixed center x_B and radius $\text{rad}(B)$, and thus notation such as $\lambda B = B(x_B, \lambda \text{rad}(B))$ is well-defined for all $\lambda > 0$. When $E, F \subset X$, we let $\text{diam}(E)$ denote the diameter of E and $\text{dist}(E, F)$ is the distance between the sets $E, F \subset X$. We use $d(x, E) = \text{dist}(x, E) = \text{dist}(\{x\}, E)$ to denote the distance from a point $x \in X$ to the set E . If $E \subset X$, then χ_E denotes the characteristic function of E ; that is, $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \in X \setminus E$.

We also assume throughout that μ is *doubling*, that is, there is a constant $c_D \geq 1$ such that whenever $x \in X$ and $r > 0$, we have

$$\mu(B(x, 2r)) \leq c_D \mu(B(x, r)). \quad (2.3)$$

Iteration of (2.3) shows that if μ is doubling, then there exist an exponent $Q > 0$ and a constant $c_Q > 0$, both only depending on c_D , such that the quantitative doubling condition

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq c_Q \left(\frac{r}{R}\right)^Q \quad (2.4)$$

holds whenever $y \in B(x, R) \subset X$ and $0 < r < R$. Condition (2.4) always holds for $Q \geq \log_2 c_D$, but it can hold for smaller values of Q as well. See [1, Lemma 3.3] for details.

In some of our results we also need to assume that μ is *reverse doubling*, in the sense that there are constants $0 < \kappa < 1$ and $0 < c_R < 1$ such that

$$\mu(B(x, \kappa r)) \leq c_R \mu(B(x, r)) \quad (2.5)$$

for every $x \in X$ and $0 < r < \text{diam}(X)/2$. If X is connected and $0 < \kappa < 1$, then inequality (2.5) follows from the doubling property (2.3) with $0 < c_R = c_R(c_D, \kappa) < 1$. See for instance [1, Lemma 3.7]. Iteration of (2.5) shows that if μ is reverse doubling, then there exist an exponent $\sigma > 0$ and a constant $c_\sigma > 0$, both only depending on κ and c_R , such that the quantitative reverse doubling condition

$$\frac{\mu(B(x, r))}{\mu(B(x, R))} \leq c_\sigma \left(\frac{r}{R}\right)^\sigma \quad (2.6)$$

holds for every $x \in X$ and $0 < r < R < 2 \text{diam}(X)$.

If the measure μ is Ahlfors Q -regular for some $Q > 0$, that is, there is a constant $C > 0$ such that

$$\frac{1}{C} r^Q \leq \mu(B(x, r)) \leq C r^Q$$

for every $x \in X$ and $0 < r < \text{diam}(X)$, then μ is both doubling and reverse doubling, and the quantitative estimates (2.4) and (2.6) hold with the exponent Q .

We abbreviate $d\mu(x) = dx$ and say that a function $u: X \rightarrow \mathbb{R}$ is integrable on balls, if u is μ -measurable and

$$\|u\|_{L^1(B)} = \int_B |u(x)| dx < \infty$$

for all balls $B \subset X$. In particular, for such functions the integral average

$$u_B = \int_B u(x) dx = \frac{1}{\mu(B)} \int_B u(x) dx$$

is well-defined whenever B is a ball in X . Observe that we do not always assume that the space X is complete, and hence continuous functions are not necessarily integrable on balls.

Definition 2.1. Let $1 \leq p, q, t < \infty$ and $0 < s < 1$. We say that X supports a (s, q, p, t) -Poincaré inequality, if there are constants $c_P > 0$ and $\lambda \geq 1$ such that inequality

$$\left(\int_B |u(x) - u_B|^q dx \right)^{1/q} \leq c_P r^s \left(\int_{\lambda B} \left(\int_{\lambda B} \frac{|u(x) - u(y)|^t}{d(x, y)^{st} \mu(B(x, d(x, y)))} dy \right)^{p/t} dx \right)^{1/p} \quad (2.7)$$

holds for every ball $B = B(x_0, r) \subset X$ and for all functions $u: X \rightarrow \mathbb{R}$ that are integrable on balls.

In particular the left-hand side of (2.7) is finite, if the right-hand side is finite.

If $u: X \rightarrow \mathbb{R}$ is a measurable function, $0 < s < 1$, $1 \leq t < \infty$, and $A \subset X$ is a measurable set, we write

$$g_{u,s,t,A}(x) = \left(\int_A \frac{|u(x) - u(y)|^t}{d(x, y)^{st} \mu(B(x, d(x, y)))} dy \right)^{1/t}, \quad \text{for every } x \in X.$$

Using this notation, the (s, q, p, t) -Poincaré inequality (2.7) can be written as

$$\left(\int_B |u(x) - u_B|^q dx \right)^{1/q} \leq c_P r^s \left(\int_{\lambda B} g_{u,s,t,\lambda B}(x)^p dx \right)^{1/p}.$$

We will repeatedly use the facts that $g_{|u|,s,t,A} \leq g_{u,s,t,A}$ and $g_{u,s,t,A} \leq g_{u,s,t,A'}$ when $A \subset A'$.

The following lemma shows that X supports a (s, q, p, t) -Poincaré inequality if $1 \leq q \leq \min\{p, t\}$. We emphasize that the doubling condition on μ is the only quantitative property of X that is needed in this case. This result is certainly known among experts, but we include

the short proof for the convenience of the reader. We refer to [16, Lemma 2.2] for a variant of this result in \mathbb{R}^n .

Lemma 2.2. *Assume that $1 \leq q, p, t < \infty$, $q \leq \min\{p, t\}$, and $0 < s < 1$. Then X supports the (s, q, p, t) -Poincaré inequality (2.7) with constants $\lambda = 1$ and $c_P = c_P(s, q, t, c_D)$.*

Proof. Fix a ball $B = B(x_0, r) \subset X$ and a function $u: X \rightarrow \mathbb{R}$ that is integrable on balls. Then

$$\begin{aligned} \int_B |u(x) - u_B|^q dx &\leq \int_B \int_B |u(x) - u(y)|^q dy dx \\ &\leq \int_B \left(\int_B |u(x) - u(y)|^t dy \right)^{q/t} dx \\ &\leq \left(\int_B \left(\int_B |u(x) - u(y)|^t dy \right)^{p/t} dx \right)^{q/p} \\ &\leq r^{sq} \left(\int_B \left(\int_B \frac{|u(x) - u(y)|^t}{r^{st} \mu(B)} dy \right)^{p/t} dx \right)^{q/p} \\ &\leq Cr^{sq} \left(\int_B \left(\int_B \frac{|u(x) - u(y)|^t}{d(x, y)^{st} \mu(4B)} dy \right)^{p/t} dx \right)^{q/p} \\ &\leq Cr^{sq} \left(\int_B \left(\int_B \frac{|u(x) - u(y)|^t}{d(x, y)^{st} \mu(B(x, d(x, y)))} dy \right)^{p/t} dx \right)^{q/p}. \end{aligned}$$

This yields the desired inequality (2.7) with $\lambda = 1$ and $c_P = c_P(s, q, t, c_D)$. \square

3. SOBOLEV–POINCARÉ INEQUALITIES

As with the usual Poincaré inequalities (see [1, 15]), also in the fractional case it is possible to improve inequalities from the case $q \leq p$ (in Lemma 2.2) to the case $q > p$, up to the ‘‘Sobolev exponent’’ $p^* = Qp/(Q - sp)$; see Theorem 3.4 below. For this purpose, we apply a metric measure space version of the fractional truncation method in [9, Proposition 5], [6, Theorem 4.1]; see also [3, Proposition 2.14]. In the proof we need the following auxiliary result, which is a special case of [12, Lemma 5].

Lemma 3.1. *Assume that $g \geq 0$ is a measurable function on a ball $B \subset X$ with*

$$\mu(\{x \in B : g(x) = 0\}) \geq \mu(B)/2.$$

Then inequality

$$\mu(\{x \in B : g(x) > t\}) \leq 2 \inf_{a \in \mathbb{R}} \mu(\{x \in B : |g(x) - a| > t/2\})$$

holds for every $t > 0$.

Theorem 3.2 below is metric measure space version of the Euclidean result in [6, Theorem 4.1]. We will later apply this theorem with the kernel

$$K(y, z) = \frac{1}{d(y, z)^{sp} \mu(B(y, d(y, z)))}, \quad y, z \in X,$$

but we formulate the result in terms of general kernels. The proof is a straightforward adaptation of the proof in [6], and it is based on a fractional Maz’ya truncation method.

Theorem 3.2. *Let $0 < s < 1$, $0 < p \leq q < \infty$, and $\lambda \geq 1$. Let $K: X \times X \rightarrow [0, \infty]$ be a measurable function and let $B = B(x_0, r) \subset X$ be a ball. Then the following conditions are equivalent:*

(A) *There is a constant $C_1 > 0$ such that inequality*

$$\inf_{a \in \mathbb{R}} \sup_{t > 0} \mu(\{x \in B : |u(x) - a| > t\}) t^q \leq C_1 \left(\int_{\lambda B} \int_{\lambda B} |u(y) - u(z)|^p K(y, z) dz dy \right)^{\frac{q}{p}}$$

holds for every $u \in L^\infty(\lambda B)$.

(B) *There is a constant $C_2 > 0$ such that inequality*

$$\inf_{a \in \mathbb{R}} \int_B |u(x) - a|^q dx \leq C_2 \left(\int_{\lambda B} \int_{\lambda B} |u(y) - u(z)|^p K(y, z) dz dy \right)^{\frac{q}{p}}$$

holds for every $u \in L^1(\lambda B)$, and the left-hand side is finite if the right-hand side is finite.

Moreover, in the implication from (A) to (B) the constant C_2 is of the form $C(p, q)C_1$, and in the implication from (B) to (A) we have $C_1 = C_2$.

Proof. The implication from (B) to (A) with $C_1 = C_2$ follows from Chebyshev's inequality. Let us then assume that condition (A) holds. Fix $u \in L^1(\lambda B)$ and let $b \in \mathbb{R}$ be such that

$$\mu(\{x \in B : u(x) \geq b\}) \geq \frac{\mu(B)}{2} \quad \text{and} \quad \mu(\{x \in B : u(x) \leq b\}) \geq \frac{\mu(B)}{2}. \quad (3.8)$$

We write $v_+ = \max\{u - b, 0\}$ and $v_- = -\min\{u - b, 0\}$. In the sequel v denotes either v_+ or v_- ; all the statements are valid in both cases. Moreover, without loss of generality, we may assume that $v \geq 0$ is defined and finite everywhere in λB .

For $0 < t_1 < t_2 < \infty$ and every $x \in \lambda B$, we define

$$v_{t_1}^{t_2}(x) = \begin{cases} t_2 - t_1, & \text{if } t_2 \leq v(x), \\ v(x) - t_1, & \text{if } t_1 < v(x) < t_2, \\ 0, & \text{if } v(x) \leq t_1. \end{cases}$$

Observe from (3.8) that

$$\mu(\{x \in B : v_{t_1}^{t_2}(x) = 0\}) \geq \mu(B)/2.$$

By Lemma 3.1 and condition (A), both applied to the non-negative function $v_{t_1}^{t_2} \in L^\infty(\lambda B)$,

$$\begin{aligned} \sup_{t > 0} \mu(\{x \in B : v_{t_1}^{t_2}(x) > t\}) t^q &\leq 2^{1+q} \inf_{a \in \mathbb{R}} \sup_{t > 0} \mu(\{x \in B : |v_{t_1}^{t_2}(x) - a| > t\}) t^q \\ &\leq 2^{1+q} C_1 \left(\int_{\lambda B} \int_{\lambda B} |v_{t_1}^{t_2}(y) - v_{t_1}^{t_2}(z)|^p K(y, z) dz dy \right)^{\frac{q}{p}}. \end{aligned} \quad (3.9)$$

We write $E_k = \{x \in \lambda B : v(x) > 2^k\}$ and $A_k = E_{k-1} \setminus E_k$, where $k \in \mathbb{Z}$. Since $v \geq 0$ is finite everywhere in B , we can write

$$B = \{x \in B : 0 \leq v(x) < \infty\} = \left(\bigcup_{i \in \mathbb{Z}} B \cap A_i \right) \cup \underbrace{\left(B \cap \{x \in \lambda B : v(x) = 0\} \right)}_{=: A_{-\infty}}. \quad (3.10)$$

Hence, by inequality (3.9) and the fact that $\sum_{k \in \mathbb{Z}} |a_k|^{q/p} \leq (\sum_{k \in \mathbb{Z}} |a_k|)^{q/p}$ for all real-valued sequences $(a_k)_{k \in \mathbb{Z}}$, we obtain

$$\begin{aligned} \int_B |v(x)|^q dx &\leq \sum_{k \in \mathbb{Z}} 2^{(k+1)q} \mu(B \cap A_{k+1}) \leq \sum_{k \in \mathbb{Z}} 2^{(k+1)q} \mu(\{x \in B : v_{2^{k-1}}^{2^k}(x) > 2^{k-2}\}) \\ &\leq 2^{1+4q} C_1 \left(\sum_{k \in \mathbb{Z}} \int_{\lambda B} \int_{\lambda B} |v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)|^p K(y, z) dz dy \right)^{\frac{q}{p}}. \end{aligned}$$

Using the definition of $v_{2^{k-1}}^{2^k}$, we can now estimate

$$\begin{aligned} & \sum_{k \in \mathbb{Z}} \int_{\lambda B} \int_{\lambda B} |v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)|^p K(y, z) dz dy \\ & \leq \left\{ \sum_{k \in \mathbb{Z}} \sum_{-\infty \leq i \leq k} \sum_{j \geq k} \int_{A_i} \int_{A_j} + \sum_{k \in \mathbb{Z}} \sum_{i \geq k} \sum_{-\infty \leq j \leq k} \int_{A_i} \int_{A_j} \right\} |v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)|^p K(y, z) dz dy. \end{aligned} \quad (3.11)$$

Let $y \in A_i$ and $z \in A_j$, where $j - 1 > i \geq -\infty$, and let $k \in \mathbb{Z}$. Then

$$|v(y) - v(z)| \geq |v(z)| - |v(y)| \geq 2^{j-2}$$

and $|v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)| \leq 2^k$, and so

$$|v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)| \leq 4 \cdot 2^{k-j} |v(y) - v(z)|. \quad (3.12)$$

On the other hand, the estimate

$$|v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)| \leq |v(y) - v(z)|$$

holds for every $k \in \mathbb{Z}$, and thus we conclude that inequality (3.12) holds whenever $-\infty \leq i \leq k \leq j$ and $(y, z) \in A_i \times A_j$.

By inequality (3.12), we have

$$\begin{aligned} & \sum_{k \in \mathbb{Z}} \sum_{-\infty \leq i \leq k} \sum_{j \geq k} \int_{A_i} \int_{A_j} |v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)|^p K(y, z) dz dy \\ & \leq 4^p \sum_{k \in \mathbb{Z}} \sum_{-\infty \leq i \leq k} \sum_{j \geq k} 2^{p(k-j)} \int_{A_i} \int_{A_j} |v(y) - v(z)|^p K(y, z) dz dy. \end{aligned} \quad (3.13)$$

Since $\sum_{k=i}^j 2^{p(k-j)} \leq (1 - 2^{-p})^{-1}$, changing the order of the summation shows that the right-hand side of inequality (3.13) is bounded by

$$\frac{4^p}{1 - 2^{-p}} \int_{\lambda B} \int_{\lambda B} |v(y) - v(z)|^p K(y, z) dz dy.$$

The second sum on the right-hand side of (3.11) can be estimated in the same way. To conclude that (B) holds with $C_2 = C(p, q)C_1$ it remains to recall that $|u - b| = v_+ + v_-$ and $q > 0$. Observe also that $|v_{\pm}(y) - v_{\pm}(z)| \leq |u(y) - u(z)|$ for all $y, z \in \lambda B$. \square

We also need certain maximal functions. If $B \subset X$ is an (open) ball and $u \in L^1(B)$, then the noncentred maximal function restricted to B is

$$M_B^* u(x) = \sup_{B'} \int_{B'} |u(y)| dy,$$

where the supremum is taken over all balls $B' \subset B$ containing $x \in B$. We will apply the following lemma from [1, Lemma 3.12].

Lemma 3.3. *Let $B \subset X$ be a ball and let $u \in L^1(B)$. Then $M_B^* u$ is lower semicontinuous in B and satisfies*

$$\mu(E_\tau) \leq \frac{c_D^3}{\tau} \int_{E_\tau} |u(x)| dx \quad \text{and} \quad \lim_{\tau \rightarrow \infty} \tau \mu(E_\tau) = 0,$$

where $E_\tau = \{x \in B : M_B^* u(x) > \tau\}$ and $\tau > 0$.

The next theorem gives a sufficient condition for the fractional (s, q, p, p) -Poincaré inequality with $q = p^* = Qp/(Q - sp)$. The proof is essentially the same as the argument in [1, pp. 95–97], but we present the details for the sake of completeness. In particular, the fractional Maz'ya truncation method is needed with sufficiently careful tracking of the constants. Recall that we

assume throughout that μ is doubling, with constant $c_D \geq 1$ in (2.3). Hence, for any fixed $1 \leq p < \infty$ and $0 < s < 1$ there exists an exponent $Q > sp$ such that (2.4) holds, and then $p^* = Qp/(Q - sp) > p$. The exponent Q in (2.4) is not uniquely determined, and a smaller value of $Q > sp$ gives in Theorem 3.4 a larger exponent p^* , which in turn yields a stronger version of the Sobolev–Poincaré inequality.

Theorem 3.4. *Assume that μ is reverse doubling, with constants $\sigma > 0$ and $c_\sigma > 0$ in (2.6), and let $Q > 0$ and $c_Q > 0$ be the constants in (2.4). Let $1 \leq p < \infty$ and $0 < s < 1$ be such that $sp < Q$, and let $p^* = Qp/(Q - sp)$. Then X supports a (s, p^*, p, p) -Poincaré inequality, with constants $\lambda = 2$ and $c_P = c_P(Q, p, s, \sigma, c_D, c_Q, c_\sigma)$.*

Proof. Let $B = B(x_0, r)$ be a ball in X and let $u \in L^\infty(2B)$. It suffices to prove that there exists a constant $C = C(Q, p, s, \sigma, c_D, c_Q, c_\sigma)$ such that

$$\mu(\{x \in B : |u(x) - u_{2B}| > t\}) t^{p^*} \leq Cr^{sp^*} \mu(B)^{1-p^*/p} \left(\int_{2B} g_{u,s,p,2B}(y)^p dy \right)^{\frac{p^*}{p}} \quad (3.14)$$

whenever $t > 0$. Then the (s, p^*, p, p) -Poincaré inequality follows from Theorem 3.2, applied with the kernel

$$K(y, z) = \frac{1}{d(y, z)^{sp} \mu(B(y, d(y, z)))}, \quad y, z \in X,$$

together with the doubling property of μ and the inequality

$$\int_B |u(x) - u_B|^p dx \leq 2^p \inf_{a \in \mathbb{R}} \int_B |u(x) - a|^p dx,$$

which in turn follows from Hölder's inequality.

We prove (3.14) for a fixed $t > 0$. We may assume that $r < 2 \operatorname{diam}(X)$ and

$$0 < \int_{2B} g_{u,s,p,2B}(y)^p dy < \infty. \quad (3.15)$$

Indeed, if the integral in (3.15) vanishes, then u is a constant almost everywhere in the ball B by the (s, p, p, p) -Poincaré inequality given in Lemma 2.2. Write $B_0 = 2B$, $r_0 = 2r$ and $M = M_{B_0}^*((g_{u,s,p,2B})^p)$. By [13, Lemma 1.8], μ -almost every point $x \in B$ is a Lebesgue point of u . Lemma 3.3 implies that the function M is finite μ -almost everywhere in B .

Let $x \in B$ be a Lebesgue point of u , with $M(x) < \infty$, and write $r_j = 2^{-j}r$ and $B_j = B(x, r_j)$, for $j = 1, 2, \dots$. By the doubling property of μ and the $(s, 1, p, p)$ -Poincaré inequality in Lemma 2.2,

$$\begin{aligned} |u(x) - u_{B_0}| &= \lim_{k \rightarrow \infty} |u_{B_k} - u_{B_0}| = \lim_{k \rightarrow \infty} \left| \sum_{j=0}^{k-1} (u_{B_{j+1}} - u_{B_j}) \right| \\ &\leq \sum_{j=0}^{\infty} \int_{B_{j+1}} |u(y) - u_{B_j}| dy \leq c_D^3 \sum_{j=0}^{\infty} \int_{B_j} |u(y) - u_{B_j}| dy \\ &\leq C(p, s, c_D) \sum_{j=0}^{\infty} r_j^s \left(\int_{B_j} g_{u,s,p,B_j}(y)^p dy \right)^{\frac{1}{p}} \\ &\leq C(p, s, c_D) \sum_{j=0}^{\infty} r_j^s \left(\int_{B_j} g_{u,s,p,2B}(y)^p dy \right)^{\frac{1}{p}}. \end{aligned}$$

Condition (2.4), applied to the balls $B_j \subset B_0$ on the right-hand side, gives

$$|u(x) - u_{B_0}| \leq C(Q, p, s, c_D, c_Q) \frac{r^s}{\mu(B_0)^{s/Q}} \underbrace{\sum_{j=0}^{\infty} \mu(B_j)^{s/Q-1/p} \left(\int_{B_j} g_{u,s,p,2B}(y)^p dy \right)^{\frac{1}{p}}}_{\Sigma' + \Sigma''}. \quad (3.16)$$

We write the sum in (3.16) as $\Sigma' + \Sigma''$, where the summations are over $0 \leq j < j_0$ and $j \geq j_0$, respectively, and the cut-off number $j_0 \in \mathbb{N}$ is chosen as follows (depending on x). Since $B_0 \subset 8B_1$ and

$$0 < \int_{B_0} g_{u,s,p,2B}(y)^p dy \leq M(x) < \infty,$$

there exists $j_0 \geq 1$ such that

$$c_D^2 \mu(B_{j_0}) \leq \frac{1}{M(x)} \int_{B_0} g_{u,s,p,2B}(y)^p dy \leq c_D^3 \mu(B_{j_0}). \quad (3.17)$$

More precisely, by (2.6) $\mu(B_j) \rightarrow 0$ as $j \rightarrow \infty$, and hence we can choose the largest integer j_0 for which the right inequality holds. The left inequality then follows from the doubling property of μ .

In the first sum Σ' we have $\mu(B_j) \geq c_\sigma^{-1} 2^{\sigma(j_0-j)} \mu(B_{j_0})$ for every $0 \leq j < j_0$, by (2.6). Since $s/Q - 1/p < 0$, we obtain

$$\begin{aligned} \Sigma' &= \sum_{j=0}^{j_0-1} \mu(B_j)^{s/Q-1/p} \left(\int_{B_j} g_{u,s,p,2B}(y)^p dy \right)^{\frac{1}{p}} \\ &\leq C(Q, p, s, c_\sigma) \mu(B_{j_0})^{s/Q-1/p} \left(\int_{B_0} g_{u,s,p,2B}(y)^p dy \right)^{\frac{1}{p}} \sum_{j=0}^{j_0-1} 2^{\sigma(j_0-j)(s/Q-1/p)}, \\ &\leq C(Q, p, s, \sigma, c_D, c_\sigma) \left(\int_{B_0} g_{u,s,p,2B}(y)^p dy \right)^{\frac{s}{Q}} M(x)^{1/p-s/Q}, \end{aligned}$$

where the sum on the second line is bounded from above by a constant $0 < C(Q, p, s, \sigma) < \infty$ that can be chosen to be independent of j_0 , and the last step follows from the right-hand inequality in (3.17).

Correspondingly, in the second sum Σ'' we have $\mu(B_j) \leq c_\sigma 2^{\sigma(j_0-j)} \mu(B_{j_0})$ for every $j \geq j_0$, by (2.6). Using also the maximal function $M = M_{B_0}^*((g_{u,s,p,2B})^p)$, we obtain

$$\begin{aligned} \Sigma'' &= \sum_{j=j_0}^{\infty} \mu(B_j)^{s/Q} \left(\int_{B_j} g_{u,s,p,2B}(y)^p dy \right)^{\frac{1}{p}} \\ &\leq C(Q, s, c_\sigma) \mu(B_{j_0})^{s/Q} M(x)^{1/p} \sum_{j=j_0}^{\infty} 2^{\sigma(j_0-j)s/Q}, \\ &\leq C(Q, p, s, \sigma, c_D, c_\sigma) \left(\int_{B_0} g_{u,s,p,2B}(y)^p dy \right)^{\frac{s}{Q}} M(x)^{1/p-s/Q}, \end{aligned}$$

where the last sum is bounded from above by a constant $0 < C(Q, s, \sigma) < \infty$ and the final step follows from the left-hand inequality in (3.17).

Substituting the above estimates for Σ' and Σ'' to (3.16) gives

$$\begin{aligned} |u(x) - u_{B_0}| &\leq C(Q, p, s, c_D, c_Q) \frac{r^s}{\mu(B_0)^{s/Q}} (\Sigma' + \Sigma'') \\ &\leq C(Q, p, s, \sigma, c_D, c_Q, c_\sigma) r^s \left(\int_{B_0} g_{u,s,p,2B}(y)^p dy \right)^{\frac{s}{Q}} M(x)^{\frac{1}{p^*}}, \end{aligned}$$

for $p^* = Qp/(Q - sp)$. In particular, if $|u(x) - u_{B_0}| > t > 0$, then

$$M(x) > C(Q, p, s, \sigma, c_D, c_Q, c_\sigma) \frac{t^{p^*}}{r^{sp^*}} \left(\int_{B_0} g_{u,s,p,2B}(y)^p dy \right)^{-\frac{sp^*}{Q}} = \tau(t) > 0.$$

From this estimate, which is valid for μ -almost every $x \in B$, and Lemma 3.3, we obtain

$$\begin{aligned} \mu(\{x \in B : |u(x) - u_{B_0}| > t\}) t^{p^*} &\leq \mu(\{x \in B_0 : M(x) > \tau(t)\}) t^{p^*} \leq \frac{c_D^3 t^{p^*}}{\tau(t)} \int_{B_0} g_{u,s,p,2B}(x)^p dx \\ &\leq C(Q, p, s, \sigma, c_D, c_Q, c_\sigma) r^{sp^*} \mu(B)^{1-p^*/p} \left(\int_{B_0} g_{u,s,p,2B}(x)^p dx \right)^{\frac{p^*}{p}} \end{aligned}$$

for every $t > 0$. Inequality (3.14) follows, and the proof is complete. \square

4. CAPACITARY AND BOUNDARY POINCARÉ INEQUALITIES

Next we study versions of fractional Poincaré inequalities, in which the zero sets of functions are taken into account. As a tool we will apply a variant of the fractional relative capacity, compare to [25, Definition 7.1] and see also [9] and [26, §11].

Definition 4.1. Let $0 < s < 1$, $1 \leq t, p < \infty$, and $\Lambda \geq 2$. Let $B \subset X$ be a ball and let $E \subset \bar{B}$ be a closed set. Then we write

$$\text{cap}_{s,p,t}(E, 2B, \Lambda B) = \inf_{\varphi} \int_{\Lambda B} \left(\int_{\Lambda B} \frac{|\varphi(x) - \varphi(y)|^t}{d(x,y)^{st} \mu(B(x, d(x,y)))} dy \right)^{p/t} dx$$

where the infimum is taken over all continuous functions $\varphi: X \rightarrow \mathbb{R}$ that are integrable on balls, such that $\varphi(x) \geq 1$ for every $x \in E$ and $\varphi(x) = 0$ for every $x \in X \setminus 2B$.

The following simple lemma is needed in the proof of Theorem 4.3.

Lemma 4.2. Let $\alpha > 0$. There is a constant $C(\alpha, c_D) > 0$ such that

$$r^{-\alpha} \int_{B(x,r)} \frac{d(x,y)^\alpha}{\mu(B(x, d(x,y)))} dy \leq C(\alpha, c_D)$$

for every $x \in X$ and $r > 0$.

Proof. Let $x \in X$ and $r > 0$. For each $j \in \{0, 1, \dots\}$ we write

$$A_j(x, r) = \{y \in X : 2^{-j-1}r \leq d(x, y) < 2^{-j}r\}.$$

By the doubling condition (2.3) of the measure μ and the standing assumption that $\mu(\{x\}) = 0$, we obtain

$$\begin{aligned} \int_{B(x,r)} \frac{d(x,y)^\alpha}{\mu(B(x, d(x,y)))} dy &= \sum_{j=0}^{\infty} \int_{A_j(x,r)} \frac{d(x,y)^\alpha}{\mu(B(x, d(x,y)))} dy \\ &\leq \sum_{j=0}^{\infty} (2^{-j}r)^\alpha \frac{\mu(A_j(x, r))}{\mu(B(x, 2^{-j-1}r))} \\ &\leq \sum_{j=0}^{\infty} (2^{-j}r)^\alpha \frac{\mu(B(x, 2^{-j}r))}{\mu(B(x, 2^{-j-1}r))} \leq C(\alpha, c_D) r^\alpha. \end{aligned} \quad \square$$

The next result is a fractional version of Maz'ya's capacitary Poincaré inequality, compare to [1, Theorem 6.21]. The argument is similar to that in [1], but there are several technical differences due to the present non-local setting.

Theorem 4.3. Let $q \geq p \geq 1$, $0 < s < 1$, $1 \leq t < \infty$ and $\Lambda \geq 2$. Assume that X supports a (s, q, p, t) -Poincaré inequality with constants $c_P > 0$ and $\lambda \geq 1$. Let $u: X \rightarrow \mathbb{R}$ be a continuous function and let

$$Z = \{x \in X : u(x) = 0\}.$$

Then, for all balls $B = B(x_0, r) \subset X$,

$$\left(\int_{\Lambda B} |u(x)|^q dx \right)^{p/q} \leq \frac{C(s, t, p, c_D, c_P, \Lambda)}{\text{cap}_{s,p,t}(\bar{B} \cap Z, 2B, \Lambda B)} \int_{\Lambda B} g_{u,s,t,\Lambda B}(x)^p dx. \quad (4.18)$$

Proof. By replacing u with $u_k = \min\{|u|, k\}$, for $k \in \mathbb{N}$, applying Fatou's lemma, and using inequalities $g_{u_k,s,t,\Lambda B} \leq g_{u,s,t,\Lambda B}$, we may assume that $u \geq 0$ and that u is bounded. Fix a ball $B = B(x_0, r)$ in X . Without loss of generality we may assume that the right-hand side of inequality (4.18) is finite. Let

$$\bar{u} = \left(\int_{\Lambda B} |u(x)|^q dx \right)^{1/q} < \infty.$$

We may assume that $\bar{u} > 0$, as otherwise there is nothing to prove.

Let $\eta(x) = \max\{0, 1 - \text{dist}(x, B)/r\}$ for every $x \in X$. Then

$$|\eta(x) - \eta(y)| \leq d(x, y)/r, \quad \text{for every } x, y \in X,$$

$0 \leq \eta \leq 1$ in X , $\eta = 1$ in \bar{B} and $\eta = 0$ outside $2B$. The function $\varphi = (1 - u/\bar{u})\eta$ is bounded and continuous, $\varphi = 1$ in $\bar{B} \cap Z$, and $\varphi = 0$ outside $2B$. By Definition 4.1 of the capacity, we have

$$\begin{aligned} \text{cap}_{s,p,t}(\bar{B} \cap Z, 2B, \Lambda B) &\leq \int_{\Lambda B} \left(\int_{\Lambda B} \frac{|\varphi(x) - \varphi(y)|^t}{d(x, y)^{st} \mu(B(x, d(x, y)))} dy \right)^{p/t} dx \\ &= \frac{1}{\bar{u}^p} \int_{\Lambda B} \left(\int_{\Lambda B} \frac{|\eta(x)(\bar{u} - u(x)) - \eta(y)(\bar{u} - u(y))|^t}{d(x, y)^{st} \mu(B(x, d(x, y)))} dy \right)^{p/t} dx \\ &= \frac{1}{\bar{u}^p} I. \end{aligned}$$

To estimate I , we write

$$\begin{aligned} I &= \int_{\Lambda B} \left(\int_{\Lambda B} \frac{|\eta(x)(\bar{u} - u(x)) - \eta(y)(\bar{u} - u(x)) + \eta(y)(\bar{u} - u(x)) - \eta(y)(\bar{u} - u(y))|^t}{d(x, y)^{st} \mu(B(x, d(x, y)))} dy \right)^{p/t} dx \\ &\leq C(t, p) \int_{\Lambda B} |\bar{u} - u(x)|^p \left(\int_{\Lambda B} \frac{|\eta(x) - \eta(y)|^t}{d(x, y)^{st} \mu(B(x, d(x, y)))} dy \right)^{p/t} dx \\ &\quad + C(t, p) \int_{\Lambda B} \left(\int_{\Lambda B} \eta(y)^t \frac{|u(y) - u(x)|^t}{d(x, y)^{st} \mu(B(x, d(x, y)))} dy \right)^{p/t} dx. \end{aligned}$$

Fix $x \in \Lambda B$. Since $|\eta(x) - \eta(y)| \leq d(x, y)/r$ for each $y \in \Lambda B$, by Lemma 4.2 we have

$$\int_{\Lambda B} \frac{|\eta(x) - \eta(y)|^t}{d(x, y)^{st} \mu(B(x, d(x, y)))} dy \leq r^{-t} \int_{B(x, 2\Lambda r)} \frac{d(x, y)^{t(1-s)}}{\mu(B(x, d(x, y)))} dy \leq C(s, t, c_D, \Lambda) r^{-st}.$$

Taking also into account that $0 \leq \eta^t \leq 1$ in ΛB , we obtain

$$I \leq C(s, t, p, c_D, \Lambda) r^{-sp} \int_{\Lambda B} |\bar{u} - u(x)|^p dx + C(t, p) \int_{\Lambda B} g_{u,s,t,\Lambda B}(x)^p dx.$$

Hence, we are left with estimating the following integral, with $a = (q - p)/(pq)$,

$$\begin{aligned} \left(\int_{\Lambda B} |\bar{u} - u(x)|^p dx \right)^{1/p} &\leq \mu(\Lambda B)^a \left(\int_{\Lambda B} |\bar{u} - u(x)|^q dx \right)^{1/q} \\ &\leq \mu(\Lambda B)^a \left(\int_{\Lambda B} |u(x) - u_{\Lambda B}|^q dx \right)^{1/q} + |\bar{u} - u_{\Lambda B}| \mu(\Lambda B)^{a+1/q}. \end{aligned}$$

The first step above relies on the assumption $q \geq p$. The right-hand side can be estimated exactly as in [1, pp. 144–145]. Indeed, the second term may be estimated by the first one, since

$$\begin{aligned} |\bar{u} - u_{\Lambda B}| \mu(\Lambda B)^{a+1/q} &= \mu(\Lambda B)^a \left| \|u\|_{L^q(\Lambda B)} - \|u_{\Lambda B}\|_{L^q(\Lambda B)} \right| \\ &\leq \mu(\Lambda B)^a \|u - u_{\Lambda B}\|_{L^q(\Lambda B)} = \mu(\Lambda B)^a \left(\int_{\Lambda B} |u(x) - u_{\Lambda B}|^q dx \right)^{1/q}. \end{aligned}$$

The first term is in turn estimated by the assumed (s, q, p, t) -Poincaré inequality,

$$\begin{aligned} \mu(\Lambda B)^a \left(\int_{\Lambda B} |u(x) - u_{\Lambda B}|^q dx \right)^{1/q} &= \mu(\Lambda B)^{1/p} \left(\int_{\Lambda B} |u(x) - u_{\Lambda B}|^q dx \right)^{1/q} \\ &\leq c_P r^{-s} \left(\int_{\lambda \Lambda B} g_{u,s,t,\lambda \Lambda B}(x)^p dx \right)^{1/p}. \end{aligned}$$

This results in

$$I \leq C(s, t, p, c_D, c_P, \Lambda) \int_{\lambda \Lambda B} g_{u,s,t,\lambda \Lambda B}(x)^p dx,$$

and it follows that

$$\left(\int_{\Lambda B} |u(x)|^q dx \right)^{p/q} = \bar{u}^p \leq \frac{C(s, t, p, c_D, c_P, \Lambda)}{\text{cap}_{s,p,t}(\bar{B} \cap Z, 2B, \Lambda B)} \int_{\lambda \Lambda B} g_{u,s,t,\lambda \Lambda B}(x)^p dx,$$

as required. \square

Next we consider two notions that are closely related to the relative capacity but have a more geometric flavor. The following concept of (co)dimension was introduced in [19].

Definition 4.4. Let $E \subset X$. For $r > 0$, the open r -neighborhood of E is the set

$$E_r = \{x \in X : \text{dist}(x, E) < r\}.$$

The upper Assouad codimension of E , denoted by $\overline{\text{codim}}_A(E)$, is the infimum of all $Q \geq 0$ for which there is a constant $c > 0$ such that

$$\frac{\mu(E_r \cap B(x, R))}{\mu(B(x, R))} \geq c \left(\frac{r}{R} \right)^Q$$

for every $x \in E$ and all $0 < r < R < \text{diam}(E)$. If E consists of one point, then the restriction $R < \text{diam}(E)$ is removed.

If the measure μ is Q -regular, then $\overline{\text{codim}}_A(E) = Q - \underline{\text{dim}}_A(E)$ for all $E \subset X$, where $\underline{\text{dim}}_A(E)$ is the lower (Assouad) dimension of E ; see [19, (3.11)]. In the Euclidean space \mathbb{R}^n , which is regular with $Q = n$, the connection between fractional Hardy inequalities and the lower Assouad dimension (as well as its dual, the upper Assouad dimension) has been considered in [7, 8]; see also [23].

We also need suitable versions of Hausdorff contents, which give lower bounds for capacities, as in Lemma 4.6 below. In the case of non-fractional capacities, similar ideas can be found for instance in [14, Theorem 5.9] and in several subsequent papers.

Definition 4.5. The $(\rho$ -restricted) Hausdorff content of codimension $\eta \geq 0$ is defined for sets $E \subset X$ by setting

$$\mathcal{H}_\rho^{\mu,\eta}(E) = \inf \left\{ \sum_k \mu(B(x_k, r_k)) r_k^{-\eta} : E \subset \bigcup_k B(x_k, r_k) \text{ and } 0 < r_k \leq \rho \right\}.$$

Lemma 4.6. *Let $0 < s < 1$, $1 \leq p, t < \infty$, $0 \leq \eta < p$ and $\Lambda > 2$. Assume that μ is reverse doubling, with constants $\kappa = 2/\Lambda$ and $0 < c_R < 1$ in (2.5). Let $B = B(x_0, r) \subset X$ be a ball with $r < \text{diam}(X)/(2\Lambda)$, and assume that $E \subset \bar{B}$ is a closed set. Then*

$$\mathcal{H}_{5\Lambda r}^{\mu, s\eta}(E) \leq C(s, t, p, \eta, c_R, c_D, \Lambda) r^{s(p-\eta)} \text{cap}_{s,p,t}(E, 2B, \Lambda B).$$

Proof. Fix $x \in E$ and write $B_0 = \Lambda B = B(x_0, \Lambda r)$, $r_0 = \Lambda r$, $r_j = 2^{-j+1}r$ and $B_j = B(x, r_j)$, $j = 1, 2, \dots$. Observe that there are test functions for $\text{cap}_{s,p,t}(E, 2B, \Lambda B)$; let φ be any one of them. By replacing φ with $\max\{0, \min\{\varphi, 1\}\}$, if necessary, we may assume that $0 \leq \varphi \leq 1$. Thus φ is continuous on X , $\varphi = 1$ on E , and $\varphi = 0$ on $B_0 \setminus 2B$. By inequality (2.5), we have

$$0 \leq \varphi_{B_0} = \int_{B_0} \varphi(y) dy \leq \frac{\mu(2B)}{\mu(\Lambda B)} \leq c_R < 1.$$

As a consequence, since $x \in E$, we find that

$$|\varphi(x) - \varphi_{B_0}| \geq 1 - c_R > 0.$$

Let $\delta = s(p - \eta)/p > 0$. Proceeding as in the proof of Theorem 3.4 with the $(s, 1, p, t)$ -Poincaré inequality given by Lemma 2.2, we obtain

$$\begin{aligned} \sum_{j=0}^{\infty} 2^{-j\delta} &= C(s, p, \eta, c_R)(1 - c_R) \leq C(s, p, \eta, c_R) |\varphi(x) - \varphi_{B_0}| \\ &\leq C(s, t, p, \eta, c_R, c_D, \Lambda) \sum_{j=0}^{\infty} r_j^s \left(\int_{B_j} g_{\varphi, s, t, B_0}(y)^p dy \right)^{\frac{1}{p}}. \end{aligned}$$

In particular, there exists $j \in \{0, 1, 2, \dots\}$, depending on x , such that

$$2^{-j\delta p} \leq C(s, t, p, \eta, c_R, c_D, \Lambda) r_j^{sp} \int_{B_j} g_{\varphi, s, t, B_0}(y)^p dy.$$

Write $r_x = r_j$ and $B_x = B(x, r_x) = B_j$. Then the previous estimate gives

$$\mu(B_x) r_x^{-s\eta} \leq C(s, t, p, \eta, c_R, c_D, \Lambda) r^{s(p-\eta)} \int_{B_x} g_{\varphi, s, t, B_0}(y)^p dy.$$

By the $5r$ -covering lemma [1, Lemma 1.7], we obtain points $x_k \in E$, $k = 1, 2, \dots$, such that the balls $B_{x_k} \subset B_0 = \Lambda B$ with radii $r_{x_k} \leq \Lambda r$ are pairwise disjoint and $E \subset \bigcup_{k=1}^{\infty} 5B_{x_k}$. Hence,

$$\begin{aligned} \mathcal{H}_{5\Lambda r}^{\mu, s\eta}(E) &\leq \sum_{k=1}^{\infty} \mu(5B_{x_k}) (5r_{x_k})^{-s\eta} \leq C \sum_{k=1}^{\infty} r^{s(p-\eta)} \int_{B_{x_k}} g_{\varphi, s, t, B_0}(x)^p dx \\ &\leq C r^{s(p-\eta)} \int_{\Lambda B} g_{\varphi, s, t, \Lambda B}(x)^p dx, \end{aligned}$$

where $C = C(s, t, p, \eta, c_R, c_D, \Lambda)$. The desired inequality follows by taking infimum over all functions φ as above. \square

The main result of this section is the following version of the fractional (Sobolev-)Poincaré inequality, where the mean value u_B can be omitted from the integral on the left-hand side. Due to the zero values on the set E , this kind of inequalities are often called boundary Poincaré inequalities. The proof below requires completeness of X via [23, Lemma 5.1], which gives uniform lower bounds for Hausdorff contents under the assumption that $\overline{\text{codim}}_A(E) < sp$. Hence, in the forthcoming applications of Theorem 4.7 we also make the assumption that the space X is complete. Alternatively, in the following results the condition $\overline{\text{codim}}_A(E) < sp$ could be replaced by an explicit condition in terms of the relative capacity or a suitable Hausdorff content, and then the completeness assumption would not be needed. However, in non-complete spaces one

then has to add to Theorems 5.1 and 5.2 also the assumption that the continuous function u is integrable on balls.

Theorem 4.7. *Let $q \geq p \geq 1$, $0 < s < 1$ and $1 \leq t < \infty$. Assume that the space X is complete and supports a (s, q, p, t) -Poincaré inequality, with constants c_P and $\lambda \geq 1$, and that μ is reverse doubling, with constants $0 < \kappa < 1$ and $0 < c_R < 1$ in (2.5). Let E be a closed set with $\overline{\text{codim}}_A(E) < sp$. Then there is a constant $C > 0$ such that*

$$\left(\int_B |u(x)|^q dx \right)^{p/q} \leq CR^{sp} \int_{\lambda B} \left(\int_{\lambda B} \frac{|u(x) - u(y)|^t}{d(x, y)^{st} \mu(B(x, d(x, y)))} dy \right)^{p/t} dx \quad (4.19)$$

whenever $u: X \rightarrow \mathbb{R}$ is a continuous function such that $u = 0$ on E and $B = B(w, R)$ is a ball with $w \in E$ and $0 < R < \text{diam}(E)/2$.

Note that (4.19) can be written as

$$\left(\int_B |u(x)|^q dx \right)^{p/q} \leq CR^{sp} \int_{\lambda B} g_{u, s, t, \lambda B}(x)^p dx.$$

Proof. Fix a number $0 \leq \eta < p$ such that $\overline{\text{codim}}_A(E) < s\eta$, and let $w \in E$ and $0 < R < \text{diam}(E)/2$. Write $\Lambda = 2/\kappa > 2$ and $r = R/\Lambda < \text{diam}(E)/(2\Lambda) \leq \text{diam}(X)/(2\Lambda)$. We prove the claim (4.19) for the ball $B(w, R)$, but for simplicity we write during the proof that $B = B(w, r) = B(w, R/\Lambda)$.

By a covering argument using the doubling condition and completeness of X , see [23, Lemma 5.1], we obtain

$$\begin{aligned} r^{-s\eta} \mu(B) &\leq C\mathcal{H}_r^{\mu, s\eta}(\overline{B} \cap E) \leq C\mathcal{H}_{5\Lambda r}^{\mu, s\eta}(\overline{B} \cap E) \\ &\leq Cr^{s(p-\eta)} \text{cap}_{s, p, t}(\overline{B} \cap E, 2B, \Lambda B). \end{aligned}$$

Write $Z = \{y \in X : u(y) = 0\} \supset E$. By the monotonicity of capacity and the doubling condition we have

$$\frac{1}{\text{cap}_{s, p, t}(\overline{B} \cap Z, 2B, \Lambda B)} \leq \frac{1}{\text{cap}_{s, p, t}(\overline{B} \cap E, 2B, \Lambda B)} \leq \frac{Cr^{sp}}{\mu(B)} \leq \frac{CR^{sp}}{\mu(\lambda\Lambda B)}.$$

The desired inequality, for the ball $B(w, R) = B(w, \Lambda r)$, follows from Theorem 4.3. \square

Corollary 4.8. *Assume that X is complete. Let $Q > 0$ and $c_Q > 0$ be the constants in (2.4), and let $q, p \geq 1$, $0 < s < 1$ and $1 \leq t < \infty$ be such that either $q \leq p \leq t$, or $q \leq p^* = \frac{Qp}{(Q-sp)} < \infty$ and $t = p$. Assume that μ is reverse doubling, and let E be a closed set with $\overline{\text{codim}}_A(E) < sp$. Then there is a constant $C > 0$ such that the boundary Poincaré inequality (4.19) holds whenever $u: X \rightarrow \mathbb{R}$ is a continuous function such that $u = 0$ on E and $B = B(w, R)$ is a ball with $w \in E$ and $0 < R < \text{diam}(E)/2$.*

Proof. By Lemma 2.2, X supports a (s, q, p, t) -Poincaré inequality whenever $q \leq \min\{p, t\}$. In particular X supports a (s, p, p, t) -Poincaré inequality whenever $p \leq t$, and for $q \leq p \leq t$ the claim then follows from Theorem 4.7 and Hölder's inequality on the left-hand side.

On the other hand, X supports a (s, p^*, p, p) -Poincaré inequality by Theorem 3.4. Theorem 4.7 gives the desired inequality for $q = p^*$ and $t = p$, and for $q \leq p^*$ and $t = p$ the claim follows again from Hölder's inequality on the left-hand side. \square

5. POINTWISE AND INTEGRAL HARDY INEQUALITIES

In this section we apply the Sobolev–Poincaré and boundary Poincaré inequalities from the previous sections to fractional Hardy-type inequalities involving distance weights. We begin with

a pointwise version of the fractional Hardy inequality, given in terms of the fractional maximal function. For $\alpha \in \mathbb{R}$ and a measurable function u on X , this is defined as

$$M_\alpha u(x) = \sup_{r>0} r^\alpha \int_{B(x,r)} |u(y)| dy, \quad \text{for every } x \in X.$$

In particular, if $\alpha = 0$, then $M_\alpha = M_0 = M$ is the centered Hardy–Littlewood maximal function.

Theorem 5.1. *Let $\alpha \in \mathbb{R}$, $q \geq p \geq 1$, $\alpha < s < 1$ and $1 \leq t < \infty$. Assume that X is complete and supports a (s, q, p, t) -Poincaré inequality and that μ is reverse doubling. Let $E \subset X$ be a closed set with $\overline{\text{codim}}_\Lambda(E) < sp$, and assume that $u: X \rightarrow \mathbb{R}$ is a continuous function such that $u = 0$ on E . Then there is a constant $C > 0$, independent of u , such that*

$$|u(x)| \leq Cd(x, E)^{s-\alpha} (M_{\alpha p}(\chi_B(g_{u,s,t,B})^p)(x))^{1/p}$$

whenever $0 < d(x, E) < \text{diam}(E)$ and $B = B(x, 2d(x, E))$.

Proof. Observe that the continuous function u is integrable on balls since X is complete, see [1, Proposition 3.1]. Fix $x \in X$ with $0 < d(x, E) < \text{diam}(E)$ and let $B = B(x, 2d(x, E))$. Write $r = 2d(x, E) > 0$ and choose $w \in E$ such that $d(x, w) < (3/2)d(x, E)$. Then

$$\tilde{B} = B(w, r/(4\lambda)) \subset B,$$

where $\lambda \geq 1$ is the constant in the assumed (s, q, p, t) -Poincaré inequality, and

$$|u(x)| = |u(x) - u_B + u_B - u_{\tilde{B}} + u_{\tilde{B}}| \leq |u(x) - u_B| + |u_B - u_{\tilde{B}}| + |u_{\tilde{B}}|. \quad (5.20)$$

We estimate each of the terms on the right-hand side separately.

First observe that $\lambda\tilde{B} \subset B$ and that the measures of these two balls are comparable, with constants only depending on c_D . Hence, by applying Theorem 4.7 for the ball \tilde{B} , whose radius is $r/(4\lambda) < \text{diam}(E)/2$, we obtain

$$\begin{aligned} |u_{\tilde{B}}| &\leq \left(\int_{\tilde{B}} |u(y)|^q dy \right)^{1/q} \leq Cr^{s-\alpha} \left(r^{\alpha p} \int_{\lambda\tilde{B}} \chi_B(y) g_{u,s,t,\lambda\tilde{B}}(y)^p dy \right)^{1/p} \\ &\leq Cr^{s-\alpha} \left(r^{\alpha p} \int_B \chi_B(y) g_{u,s,t,B}(y)^p dy \right)^{1/p} \\ &\leq Cd(x, E)^{s-\alpha} (M_{\alpha p}(\chi_B(g_{u,s,t,B})^p)(x))^{1/p}. \end{aligned}$$

Recall from Lemma 2.2 that X supports a $(s, 1, p, t)$ -Poincaré inequality, with constants $\lambda = 1$ and $C(s, t, c_D)$. By the doubling condition, followed by the $(s, 1, p, t)$ -Poincaré inequality, we obtain

$$\begin{aligned} |u_B - u_{\tilde{B}}| &\leq C \int_B |u(y) - u_B| dy \leq Cr^{s-\alpha} \left(r^{\alpha p} \int_B \chi_B(y) g_{u,s,t,B}(y)^p dy \right)^{1/p} \\ &\leq Cd(x, E)^{s-\alpha} (M_{\alpha p}(\chi_B(g_{u,s,t,B})^p)(x))^{1/p}. \end{aligned}$$

In order to estimate the term $|u(x) - u_B|$, we write $B_j = 2^{-j}B = B(x, 2^{-j}r)$ for $j = 0, 1, 2, \dots$. Since $\lim_{j \rightarrow \infty} u_{B_j} = u(x)$, we find that

$$\begin{aligned} |u(x) - u_B| &\leq \sum_{j=0}^{\infty} |u_{B_j} - u_{B_{j+1}}| \leq C \sum_{j=0}^{\infty} \int_{B_j} |u(y) - u_{B_j}| dy \\ &\leq C \sum_{j=0}^{\infty} (2^{-j}r)^s \left(\int_{B_j} g_{u,s,t,B_j}(y)^p dy \right)^{1/p} \\ &\leq C r^{s-\alpha} \sum_{j=0}^{\infty} 2^{-j(s-\alpha)} \left((2^{-j}r)^{\alpha p} \int_{B_j} \chi_B(y) g_{u,s,t,B}(y)^p dy \right)^{1/p} \\ &\leq C r^{s-\alpha} \sum_{j=0}^{\infty} 2^{-j(s-\alpha)} (M_{\alpha p}(\chi_B(g_{u,s,t,B})^p)(x))^{1/p} \\ &= Cd(x, E)^{s-\alpha} (M_{\alpha p}(\chi_B(g_{u,s,t,B})^p)(x))^{1/p}. \end{aligned}$$

The claim follows from (5.20) and the estimates above. \square

Pointwise Hardy inequalities imply localized Hardy inequalities for balls centered at E . Here we restrict ourselves to the case $q = p$.

Theorem 5.2. *Let $0 < s < 1$ and $1 < t < \infty$. Assume that X is complete and that μ is reverse doubling. Let $E \subset X$ be a closed set with $\text{co dim}_A(E) < st$, and let $u: X \rightarrow \mathbb{R}$ be a continuous function such that $u = 0$ on E . Then there is a constant $C > 0$, independent of u , such that*

$$\int_{B \setminus E} \frac{|u(x)|^t}{d(x, E)^{st}} dx \leq C \int_{3B} \int_{3B} \frac{|u(x) - u(y)|^t}{d(x, y)^{st} \mu(B(x, d(x, y)))} dy dx$$

whenever $B = B(w, r)$ with $w \in E$ and $0 < r < \text{diam}(E)$.

Proof. Fix an exponent $1 \leq p < t$ in such a way that $\text{co dim}_A(E) < sp$. By Lemma 2.2 we find that X supports the (s, p, p, t) -Poincaré inequality with constants $\lambda = 1$ and $c_P = c_P(s, p, t, c_D)$. Fix $x \in B \setminus E$. Then $0 < d(x, E) < r < \text{diam}(E)$ and $B(x, 2d(x, E)) \subset 3B$. Hence, by Theorem 5.1 with $\alpha = 0$ and $q = p$,

$$\begin{aligned} \frac{|u(x)|^t}{d(x, E)^{st}} &\leq C (M(\chi_{B(x, 2d(x, E))}(g_{u,s,t,B(x, 2d(x, E)))^p)(x))^{t/p} \\ &\leq C (M(\chi_{3B}(g_{u,s,t,3B})^p)(x))^{t/p}. \end{aligned}$$

Integrating this inequality over the set $B \setminus E$ we obtain

$$\int_{B \setminus E} \frac{|u(x)|^t}{d(x, E)^{st}} dx \leq C \int_X (M(\chi_{3B}(g_{u,s,t,3B})^p)(x))^{t/p} dx.$$

Since $t > p$, the Hardy–Littlewood maximal theorem [1, Theorem 3.13] implies that

$$\int_{B \setminus E} \frac{|u(x)|^t}{d(x, E)^{st}} dx \leq C \int_{3B} g_{u,s,t,3B}(x)^t dx.$$

This concludes the proof. \square

Next, we obtain a (partial) converse of Theorem 5.2. This shows that the dimensional condition $\text{co dim}_A(E) < st$ in Theorems 5.1 and 5.2 is essentially sharp, up to the end point. The idea behind the proof goes back to [4, Section 2], where the impossibility of a fractional Hardy inequality was shown in certain open sets Ω of the Euclidean space, for instance if Ω is a Lipschitz domain and $st \leq 1$ (note that in this case $\text{co dim}_A(\partial\Omega) = 1$). On the other hand, a necessary condition for

non-fractional pointwise Hardy inequalities in metric spaces has been given in [21, Lemma 3] in terms of a Hausdorff content density condition.

Theorem 5.3. *Let $0 < s < 1$, $1 < t < \infty$, and $\lambda \geq 1$. Assume that $E \subset X$ is a (nonempty) closed set such that*

$$\int_{B \setminus E} \frac{|u(x)|^t}{d(x, E)^{st}} dx \leq C \int_{\lambda B} \int_{\lambda B} \frac{|u(x) - u(y)|^t}{d(x, y)^{st} \mu(B(x, d(x, y)))} dy dx$$

whenever $u: X \rightarrow \mathbb{R}$ is a bounded continuous function such that $u = 0$ on E , and $B = B(w, r)$ with $w \in E$ and $0 < r < \text{diam}(E)$. Then $\text{co dim}_A(E) \leq st$.

Proof. Let $w \in E$ and $0 < r < R_0 < \text{diam}(E)$. It suffices to show that

$$\frac{\mu(E_r \cap B(w, R_0))}{\mu(B(w, R_0))} \geq c \left(\frac{r}{R_0} \right)^{st}, \quad (5.21)$$

where the constant c is independent of w , r and R_0 . For convenience, write $R = R_0/\lambda$ and $B = B(w, R)$, so that $\lambda B = B(w, R_0)$.

If $\mu(E_r \cap B(w, R)) \geq \frac{1}{2} \mu(B(w, R))$, the claim is clear since then, by doubling,

$$\mu(E_r \cap B(w, R_0)) \geq \mu(E_r \cap B(w, R)) \geq \frac{1}{2} \mu(B(w, R)) \geq c \mu(B(w, R_0)),$$

and on the other hand $\left(\frac{r}{R_0}\right)^{st} \leq 1$. Thus we may assume that $\mu(E_r \cap B(w, R)) < \frac{1}{2} \mu(B(w, R))$, whence

$$\mu(B(w, R) \setminus E_r) \geq \frac{1}{2} \mu(B(w, R)) > 0. \quad (5.22)$$

Notice that then in particular $r < R = R_0/\lambda$ since otherwise $B(w, R) \setminus E_r = \emptyset$.

Let us now consider the continuous and bounded function $u: X \rightarrow \mathbb{R}$,

$$u(x) = \min\{1, 4r^{-1}d(x, E)\}, \quad x \in X.$$

Then $u = 0$ on E , $u = 1$ in $X \setminus E_{r/4}$, and

$$|u(x) - u(y)| \leq \min\{1, 4r^{-1}d(x, y)\} \quad \text{for all } x, y \in X.$$

Since $d(x, E)^{-st} \geq R^{-st}$ for $x \in B(w, R) \setminus E_r$, we obtain

$$\begin{aligned} \int_{B \setminus E} \frac{|u(x)|^t}{d(x, E)^{st}} dx &\geq \int_{B \setminus E_r} d(x, E)^{-st} dx \geq R^{-st} \mu(B(w, R) \setminus E_r) \\ &\geq \frac{1}{2} R^{-st} \mu(B(w, R)) \geq c R_0^{-st} \mu(B(w, R_0)), \end{aligned} \quad (5.23)$$

where the penultimate step follows from (5.22) and the final inequality holds by doubling.

To prove the claim (5.21), it hence suffices to show that

$$\int_{\lambda B} \int_{\lambda B} \frac{|u(x) - u(y)|^t}{d(x, y)^{st} \mu(B(x, d(x, y)))} dy dx \leq C r^{-st} \mu(E_r \cap B(w, R_0)). \quad (5.24)$$

Then (5.21) follows directly from estimates (5.23) and (5.24) and the assumed local fractional Hardy inequality.

Write

$$K(x, y) = \frac{|u(x) - u(y)|^t}{d(x, y)^{st} \mu(B(x, d(x, y)))}$$

whenever $x, y \in \lambda B$, $x \neq y$. Since $u(x) = 1$ for $x \in \lambda B \setminus E_{r/4}$, and $K(x, y) \leq c_D K(y, x)$ by doubling for $x, y \in \lambda B$, $x \neq y$, we have

$$\begin{aligned} \int_{\lambda B} \int_{\lambda B} K(x, y) dy dx &\leq \int_{E_r \cap \lambda B} \int_{E_r \cap \lambda B} K(x, y) dy dx + (1 + c_D) \int_{\lambda B \setminus E_r} \int_{E_{r/4} \cap \lambda B} K(x, y) dy dx \\ &=: I_1 + (1 + c_D) I_2. \end{aligned}$$

Define

$$F_k = \{x \in \lambda B : 2^{-k} \leq d(x, E) < 2^{-k+1}\}$$

and

$$A_j(x) = \{y \in \lambda B : 2^{-j-1} \leq d(y, x) < 2^{-j}\},$$

for $k, j \in \mathbb{Z}$ and $x \in \lambda B$. Let also $k_1, k_2 \in \mathbb{Z}$ be such that

$$2^{-k_1-1} \leq \lambda R < 2^{-k_1} \quad \text{and} \quad 2^{-k_2} \leq r < 2^{-k_2+1}.$$

When $k \leq k_2$ and $x \in F_k$, it holds that $E_{r/4} \cap A_j(x) = \emptyset$ for all $j \geq k+1$. Using the estimate $|u(x) - u(y)| \leq 1$ and changing also the orders of summation and integration, we thus obtain

$$\begin{aligned} I_2 &\leq \sum_{k=k_1}^{k_2} \int_{F_k} \sum_{j=k_1-1}^k \int_{E_{r/4} \cap A_j(x)} \frac{1}{2^{-(j+1)st} \mu(B(x, d(x, y)))} dy dx \\ &\leq C \sum_{j=k_1-1}^{k_2} 2^{jst} \int_{E_r \cap \lambda B} \sum_{k=j}^{k_2} \int_{\{x \in F_k : y \in A_j(x)\}} \frac{1}{\mu(B(x, d(x, y)))} dx dy. \end{aligned}$$

But if $y \in A_j(x)$, then $d(x, y) < 2^{-j}$ and so

$$\bigcup_{k=j}^{k_2} \{x \in F_k : y \in A_j(x)\} \subset B(y, 2^{-j}).$$

Since $B(y, d(x, y)) \subset B(x, 2d(x, y))$, we obtain by doubling that (still for $y \in A_j(x)$)

$$\begin{aligned} \mu(B(y, 2^{-j})) &\leq c_D \mu(B(y, 2^{-j-1})) \leq c_D \mu(B(y, d(x, y))) \\ &\leq c_D \mu(B(x, 2d(x, y))) \leq c_D^2 \mu(B(x, d(x, y))). \end{aligned}$$

We conclude that

$$\begin{aligned} I_2 &\leq C \sum_{j=k_1-1}^{k_2} 2^{jst} \int_{E_r \cap \lambda B} \int_{B(y, 2^{-j})} \frac{1}{\mu(B(y, 2^{-j}))} dx dy \\ &\leq C \sum_{j=k_1-1}^{k_2} 2^{jst} \mu(E_r \cap \lambda B) \leq C 2^{k_2 st} \mu(E_r \cap \lambda B) \leq C r^{-st} \mu(E_r \cap \lambda B). \end{aligned}$$

On the other hand, since $|u(x) - u(y)|^t \leq \min\{1, 4^t r^{-t} d(x, y)^t\}$ for all $x, y \in \lambda B$, integral I_1 can be estimated as follows:

$$\begin{aligned} I_1 &\leq \int_{E_r \cap \lambda B} \sum_{j=k_1-1}^{\infty} \int_{E_r \cap A_j(x)} \frac{|u(x) - u(y)|^t}{d(x, y)^{st} \mu(B(x, d(x, y)))} dy dx \\ &\leq \int_{E_r \cap \lambda B} \sum_{j=k_1-1}^{k_2} \int_{E_r \cap A_j(x)} \frac{1}{2^{-(j+1)st} \mu(B(x, d(x, y)))} dy dx \\ &\quad + C \int_{E_r \cap \lambda B} \sum_{j=k_2}^{\infty} \int_{E_r \cap A_j(x)} \frac{r^{-t} d(x, y)^t}{d(x, y)^{st} \mu(B(x, d(x, y)))} dy dx. \end{aligned} \tag{5.25}$$

In the first integral on the right-hand side of (5.25)

$$\int_{E_r \cap A_j(x)} \frac{1}{2^{-(j+1)st} \mu(B(x, d(x, y)))} dy \leq \frac{1}{2^{-(j+1)st}} \int_{A_j(x)} \frac{1}{\mu(B(x, 2^{-j-1}))} dy \leq C 2^{jst}$$

since the measures of the balls $B(x, 2^{-j})$ and $B(x, 2^{-j-1})$ are comparable, while in the second integral on the right-hand side of (5.25)

$$\begin{aligned} \sum_{j=k_2}^{\infty} \int_{E_r \cap A_j(x)} \frac{r^{-t} d(x, y)^t}{d(x, y)^{st} \mu(B(x, d(x, y)))} dy &\leq \sum_{j=k_2}^{\infty} r^{-t} \int_{A_j(x)} \frac{d(x, y)^{t(1-s)}}{\mu(B(x, d(x, y)))} dy \\ &\leq r^{-t} \sum_{j=k_2}^{\infty} 2^{-jt(1-s)} \int_{B(x, 2^{-j})} \frac{1}{\mu(B(x, 2^{-j-1}))} dy \\ &\leq Cr^{-t} 2^{-k_2 t(1-s)} \leq Cr^{-t} r^{t(1-s)} = Cr^{-st}. \end{aligned}$$

Here we had again a converging geometric series since $t(1-s) > 0$.

Substituting the above two estimates to (5.25), we obtain

$$\begin{aligned} I_1 &\leq C \int_{E_r \cap \lambda B} \sum_{j=k_1-1}^{k_2} 2^{jst} dx + C \int_{E_r \cap \lambda B} r^{-st} dx \\ &\leq C \int_{E_r \cap \lambda B} (2^{k_2 st} + r^{-st}) dx \leq Cr^{-st} \mu(E_r \cap \lambda B). \end{aligned}$$

As $\lambda B = B(w, R_0)$, we conclude that $I_1 + (1 + c_D)I_2 \leq Cr^{-st} \mu(E_r \cap B(w, R_0))$, and this proves the claim. \square

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(B.D.) FACULTY OF PURE AND APPLIED MATHEMATICS, WROCLAW UNIVERSITY OF SCIENCE AND TECHNOLOGY, WYBRZEŻE WYSPIAŃSKIEGO 27, 50-370 WROCLAW, POLAND
Email address: bdyda@pwr.edu.pl dyda@math.uni-bielefeld.de

(J.L.) UNIVERSITY OF JYVASKYLA, DEPARTMENT OF MATHEMATICS AND STATISTICS, P.O. BOX 35, FI-40014 UNIVERSITY OF JYVASKYLA, FINLAND
Email address: juha.lehrback@jyu.fi

(A.V.V.) UNIVERSITY OF JYVASKYLA, DEPARTMENT OF MATHEMATICS AND STATISTICS, P.O. BOX 35, FI-40014 UNIVERSITY OF JYVASKYLA, FINLAND
Email address: antti.vahakangas@iki.fi