

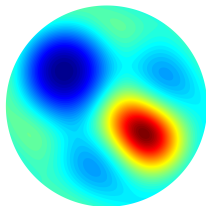
Mathematics of EIT

Lecture 7: The D-bar Method

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Outline

The D-bar method for EIT

Complex geometric optics solutions

Regularization theorem

Implementation of the D-bar method

Variants of the D-bar method

Virtual hybrid parallel-beam tomography

Radon transform and X-ray tomography

Electrical impedance tomography (EIT)

EIT and X-ray tomography are connected!

Complex geometric optics solutions

Noise, blur, and a new decomposition of real-world EIT

This part is a joint work with

David Isaacson, Rensselaer Polytechnic Institute, USA

Kim Knudsen, Technical University of Denmark

Matti Lassas, University of Helsinki, Finland

Jon Newell, Rensselaer Polytechnic Institute, USA

Jennifer Mueller, Colorado State University, USA

The mathematical model of EIT is the inverse conductivity problem introduced by Calderón

Let $\Omega \subset \mathbb{R}^2$ be the unit disc and let conductivity $\sigma : \Omega \rightarrow \mathbb{R}$ satisfy

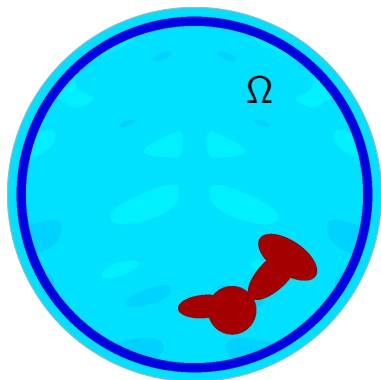
$$0 < M^{-1} \leq \sigma(z) \leq M.$$

Applying voltage f at the boundary $\partial\Omega$ leads to the elliptic PDE

$$\begin{cases} \nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= f. \end{cases}$$

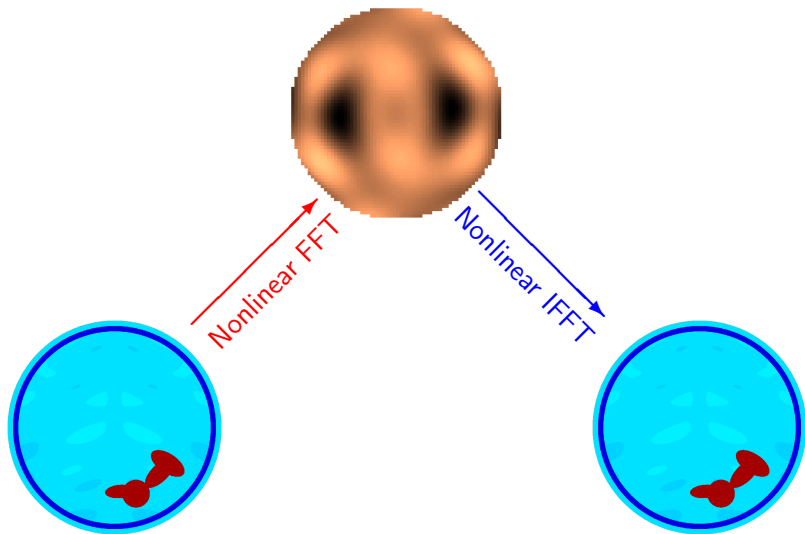
Boundary measurements are modelled by the Dirichlet-to-Neumann map

$$\Lambda_\sigma : f \mapsto \sigma \frac{\partial u}{\partial \vec{n}}|_{\partial\Omega}.$$

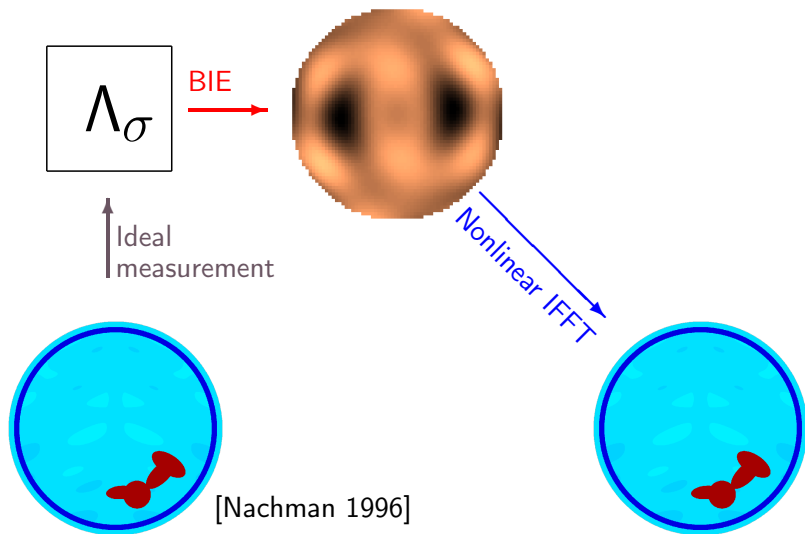


Calderón's problem is to recover σ from the knowledge of Λ_σ . It is a nonlinear and ill-posed inverse problem.

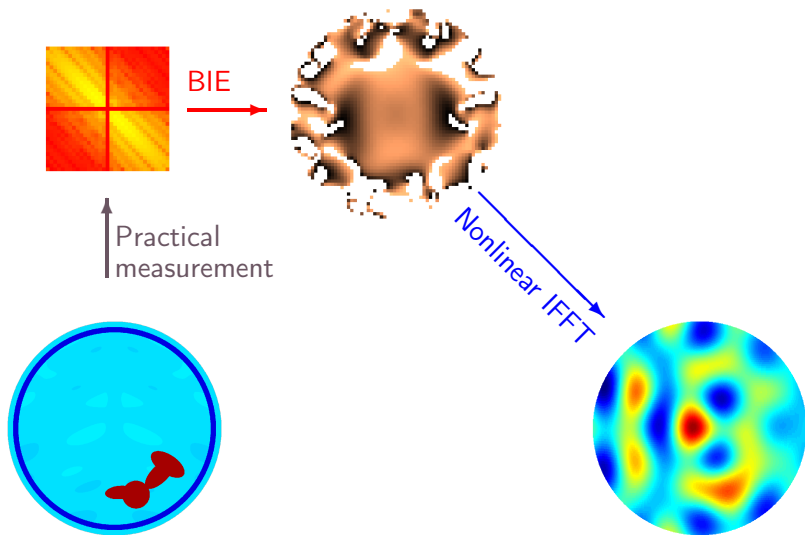
There exists a nonlinear Fourier transform adapted to electrical impedance tomography



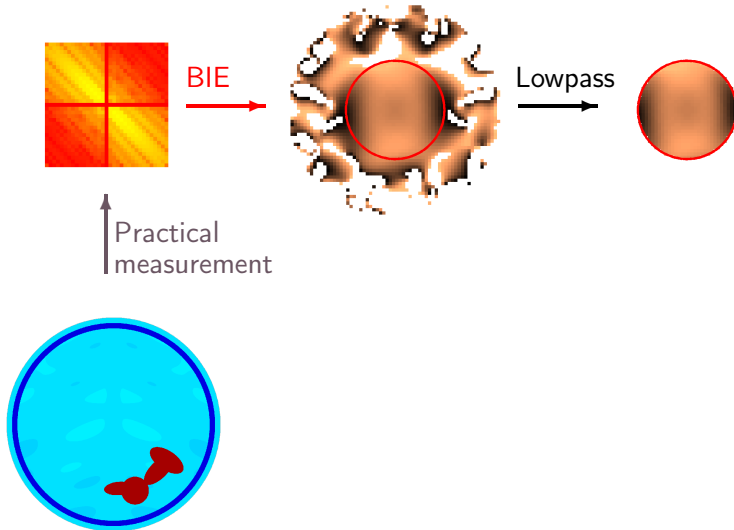
The nonlinear Fourier transform can be recovered from infinite-precision EIT measurements



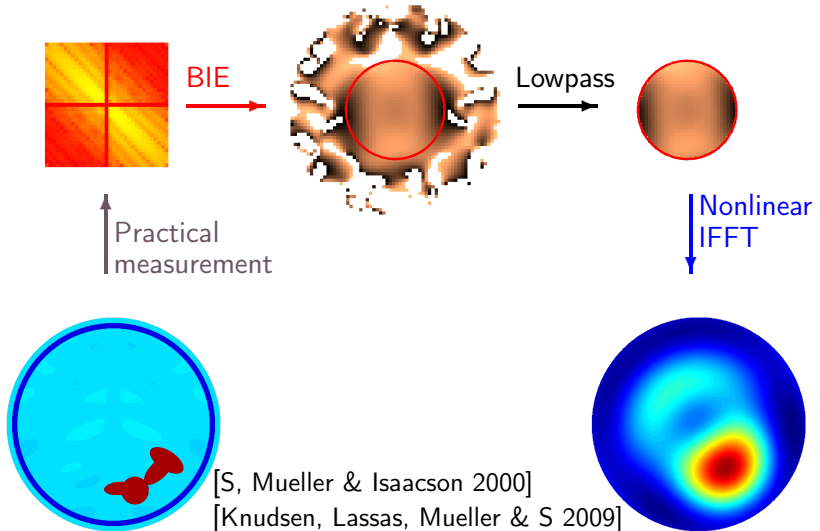
Measurement noise prevents the recovery of the nonlinear Fourier transform at high frequencies



We truncate away the bad part in the transform;
this is a nonlinear low-pass filter



The D-bar method is a regularization strategy for reconstructing the full conductivity distribution



This is a brief history of the two-dimensional regularized D-bar method for EIT

1966 Faddeev: Complex geometric optics (CGO) solutions

1987 Sylvester and Uhlmann: CGO solutions for inverse boundary-value problems; uniqueness for 3D EIT with smooth conductivities and infinite-precision data

1988 R. G. Novikov: Core ideas of the D-bar method

1988 Nachman: D-bar method for 3D EIT

1996 Nachman: Uniqueness and reconstruction for 2D EIT with C^2 conductivities and infinite-precision data

2000 S, Mueller and Isaacson: Numerical implementation of Nachman's proof using a Born approximation

2006 Isaacson, Mueller, Newell and S: Application of the D-bar method to EIT data measured from a human subject

2009 Knudsen, Lassas, Mueller and S: Regularization proof

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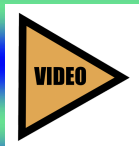
- Electrical impedance tomography (EIT)

- EIT and X-ray tomography are connected!

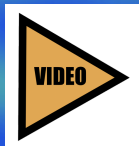
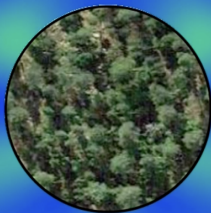
- Complex geometric optics solutions

- Noise, blur, and a new decomposition of real-world EIT

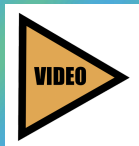
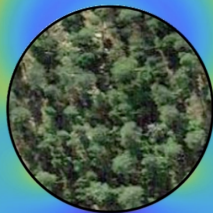
This is a simulation of a plane wave



If there is an island in the middle of the ocean,
the plane wave interacts with it and gets disturbed



Subtracting the plane wave from the disturbed wave reveals the scattered wave



Nachman (1996) transforms to the Schrödinger equation and uses CGO solutions

Define a potential q by setting $q(z) \equiv 0$ for z outside Ω and

$$q(z) = \frac{\Delta \sqrt{\sigma(z)}}{\sqrt{\sigma(z)}} \quad \text{for } z \in \Omega.$$

Then $q \in C_0(\Omega)$. We look for solutions of the Schrödinger equation

$$(-\Delta + q)\psi(\cdot, k) = 0 \quad \text{in } \mathbb{R}^2$$

parametrized by $k \in \mathbb{C} \setminus 0$ and satisfying the asymptotic condition

$$e^{-ikz}\psi(z, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2),$$

where $\tilde{p} > 2$ and $ikz = i(k_1 + ik_2)(x + iy)$.

The CGO solutions are constructed using a generalized Lippmann-Schwinger equation

Define $\mu(z, k) = e^{-ikz}\psi(z, k)$. Then $(-\Delta + q)\psi = 0$ implies

$$(-\Delta - 4ik\bar{\partial}_z + q)\mu(\cdot, k) = 0, \quad (1)$$

where the D-bar operator is defined by $\bar{\partial}_z = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$.

A solution of (1) satisfying $\mu(z, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2)$ can be constructed using the Lippmann-Schwinger type equation

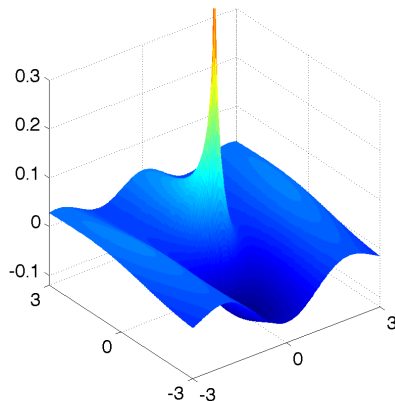
$$\mu = 1 - g_k * (q\mu),$$

where g_k satisfies $(-\Delta - 4ik\bar{\partial}_z)g_k = \delta$ and is defined by

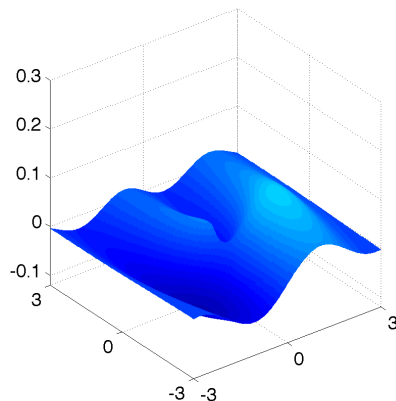
$$g_k(z) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{e^{iz \cdot \xi}}{|\xi|^2 + 2k(\xi_1 + i\xi_2)} d\xi_1 d\xi_2.$$

The Faddeev fundamental solution $g_1(z)$ has a logarithmic singularity at $z = 0$

Real part of $g_1(z)$



Imaginary part of $g_1(z)$



It is enough to know $g_1(z)$ because of the relation $g_k(z) = g_1(kz)$.
MATLAB: $g_1(z) = \exp(-i*z) .* \text{real}(\text{expint}(-i*z)) / (2*\text{pi})$

One of the breakthroughs in Nachman's 1996 article is showing uniqueness of μ

A solution of $(-\Delta - 4ik\bar{\partial}_z + q)\mu(\cdot, k) = 0$ satisfying $\mu(z, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2)$ can be constructed using the formula

$$\mu - 1 = [I + g_k * (q \cdot)]^{-1}(g_k * q),$$

provided that the inverse operator exists.

Now $q \in L^p(\mathbb{R}^2)$ with $1 < p < 2$ and $1/\tilde{p} = 1/p - 1/2$, and

$$\begin{aligned} q \cdot &: W^{1,\tilde{p}}(\mathbb{R}^2) \rightarrow L^p(\mathbb{R}^2) \quad \text{is bounded,} \\ g_k * &: L^p(\mathbb{R}^2) \rightarrow W^{1,\tilde{p}}(\mathbb{R}^2) \quad \text{is compact.} \end{aligned}$$

Thus $I + g_k * (q \cdot) : W^{1,\tilde{p}}(\mathbb{R}^2) \rightarrow W^{1,\tilde{p}}(\mathbb{R}^2)$ is Fredholm of index zero, and Nachman proved injectivity for all $k \neq 0$.

The non-physical scattering transform $t(k)$ can be seen as a nonlinear Fourier transform

We denote $z = x + iy \in \mathbb{C}$ and $z = (x, y) \in \mathbb{R}^2$. The scattering transform $t : \mathbb{C} \rightarrow \mathbb{C}$ is defined by

$$t(k) := \int_{\mathbb{R}^2} e^{i\bar{k}\bar{z}} q(z) \psi(z, k) \, dx dy.$$

Sometimes $t(k)$ is called the nonlinear Fourier transform of q . This is because asymptotically $\psi(z, k) \sim e^{ikz}$ as $|z| \rightarrow \infty$, and substituting e^{ikz} in place of $\psi(z, k)$ above gives

$$\begin{aligned} \int_{\mathbb{R}^2} e^{i(kz + \bar{k}\bar{z})} q(z) \, dx dy &= \int_{\mathbb{R}^2} e^{-i(-2k_1, 2k_2) \cdot (x, y)} q(z) \, dx dy \\ &= \hat{q}(-2k_1, 2k_2). \end{aligned}$$

Alessandrini's equation gives a way to write t in terms of Λ_σ and traces of the CGO solutions

The following boundary integral equation is a Fredholm equation of the second kind and uniquely solvable in the space $H^{1/2}(\partial\Omega)$:

$$\psi(\cdot, k)|_{\partial\Omega} = e^{ikz}|_{\partial\Omega} - \mathcal{S}_k(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k).$$

Here $\mathcal{S}_k$ is the single-layer operator with Faddeev Green's function:

$$(\mathcal{S}_k\phi)(z) := \int_{\partial\Omega} G_k(z - \zeta)\phi(\zeta) ds(\zeta),$$

where $G_k(z) := e^{ikz}g_k(z)$ satisfies $-\Delta G_k = \delta$.

The scattering transform can be evaluated by

$$t(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{z}}(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k) ds.$$

The functions μ can be recovered from the scattering transform t using a D-bar equation

It is natural to ask whether $\mu(z, k) := e^{-ikz}\psi(z, k)$ depends analytically on the parameter k . If it does, the D-bar operator

$$\frac{\partial}{\partial \bar{k}} = \frac{1}{2} \left(\frac{\partial}{\partial k_1} + i \frac{\partial}{\partial k_2} \right)$$

will give zero when applied to $\mu(z, k)$.

It turns out that

$$\frac{\partial}{\partial \bar{k}} \mu(z, k) = \frac{1}{4\pi \bar{k}} t(k) e^{-i(kz + \bar{k}\bar{z})} \overline{\mu(z, k)}.$$

Thus the dependence of $\mu(z, k)$ on k is not analytic. The above D-bar equation was discovered by Beals and Coifman in the 1980's.

History of computational approaches for the Schrödinger-type CGO solutions

Vainikko 1993,2000: fast solution method for the Lippmann-Schwinger equation.

S 1999: first-ever CGO computation, complexity N^6 .

S, Mueller & Isaacson 2000: first CGO-based EIT reconstruction, including a solution method for the D-bar equation.

Mueller & S 2003: CGO computation by Vainikko's method, complexity $N^4 \log(N)$.

Knudsen, Mueller & S 2004: solution of the D-bar equation by generalizing Vainikko's method.

Delbary, Hansen & Knudsen 2011: three-dimensional CGO computation using Vainikko's method.

Numerical solution of the D-bar equation is based on the periodization approach of G. Vainikko

The application of Vainikko's method for the CGO computation is described in [Mueller & S 2003,2012].

We wish to solve the Lippmann-Schwinger type equation

$$\mu = 1 - g_k * (q\mu),$$

where $\text{supp } (q) \subset B(0, \rho)$ with $\rho < 1$. More precisely,

$$\mu(z, k) = 1 - \int_{\mathbb{R}^2} g_k(z - w) q(w) \mu(w, k) dw_1 dw_2.$$

The trick is to solve a periodic equation whose solution coincides with $\mu(z, k)$ in the disc $|z| < 1$

Take $\epsilon > 0$ and $s = 2 + 3\epsilon$.

Let $\eta \in C_0^\infty(\mathbb{R}^2)$ satisfy

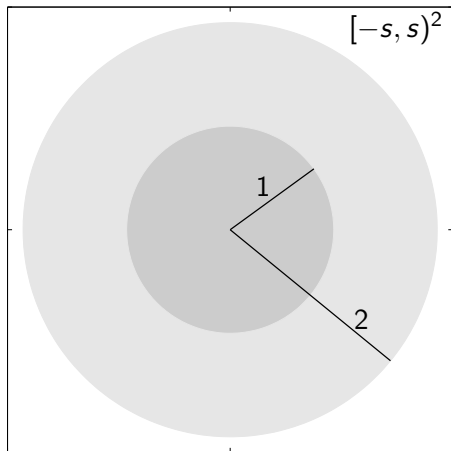
$$\eta(k) = \begin{cases} 1 & \text{for } |k| < 2 + \epsilon, \\ 0 & \text{for } |k| \geq 2 + 2\epsilon. \end{cases}$$

Define a $2s$ -periodic function $\tilde{g}_k$:

$$\tilde{g}_k(z + n2s + im2s) = g_k(z)\eta(z),$$

and periodic convolution:

$$\tilde{G}f(z) = \int_{\mathbb{T}^2} \tilde{g}_k(z-z')f(z') \, dx' dy'$$



We consider the periodic equation

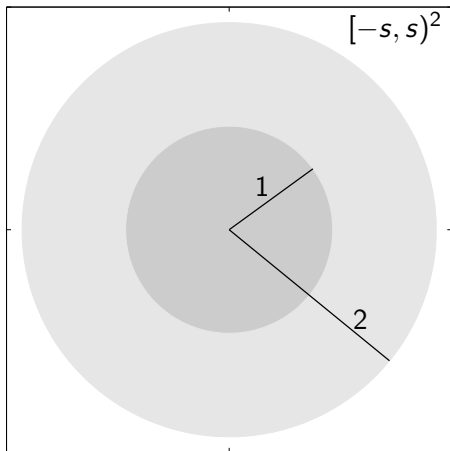
$$\phi = 1 - \tilde{G}q\phi.$$

Why does the periodic trick work? How to get the solution of the LS equation in the whole plane?

Claim 1: the periodic equation $\phi = 1 - \tilde{G}q\phi$ has a unique solution.

Claim 2: the two solutions coincide inside the smaller disc: $\phi(z, k) = \mu(z, k)$ for fixed $k \neq 0$ and $|z| < 1$.

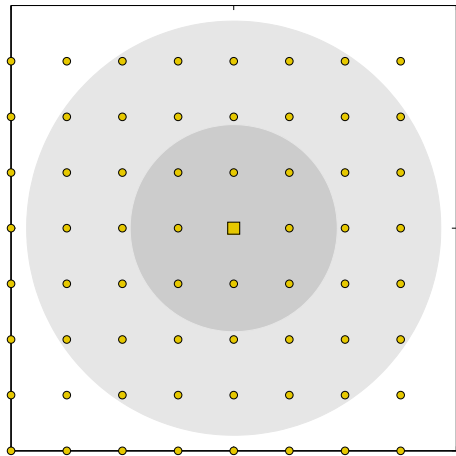
Claim 3: if we know $\mu(z, k)$ for all $|z| < 1$, then we know it for any $z \in \mathbb{C}$:



$$\mu(z, k) = 1 - \int_{|z|<1} g_k(z-w)q(w)\mu(w, k)dw_1dw_2.$$

We introduce a two-dimensional grid of points on the torus $\mathbb{T}^2$

For compatibility with the Fast Fourier Transform, we take the grid to have $2^M \times 2^M$ points.



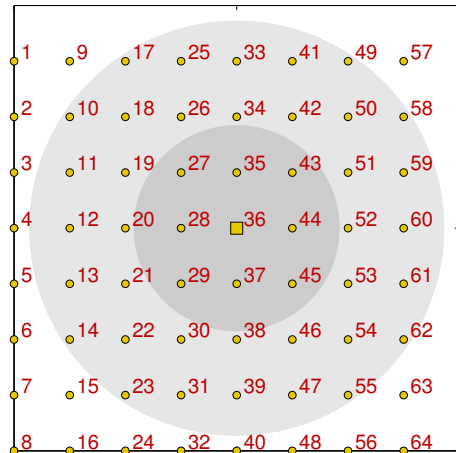
$$\phi = 1 - \tilde{G}q\phi$$

We represent the unknown as a vector of values at grid points

The grid points are numbered with one index as shown.

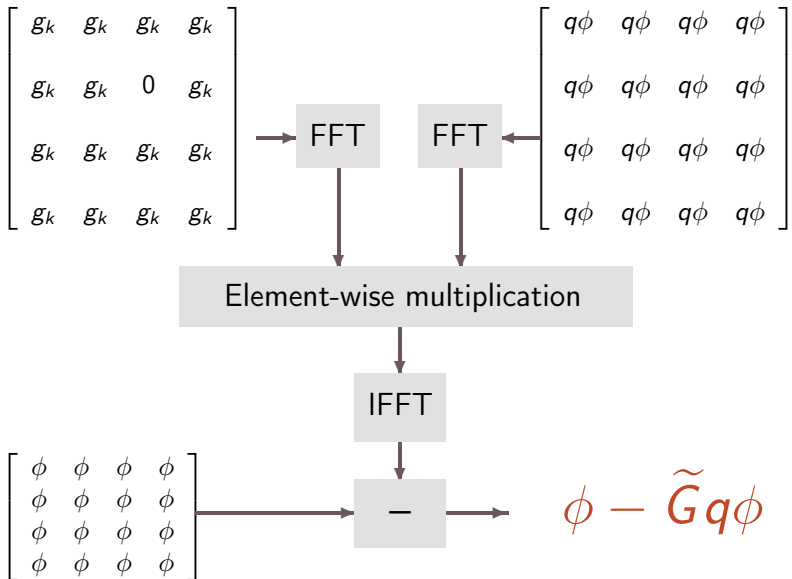
Function $\phi : [-s, s]^2 \rightarrow \mathbb{C}$ is represented by a vector of values at the grid points:

$$\begin{bmatrix} \phi(z_1) \\ \phi(z_2) \\ \vdots \\ \phi(z_{64}) \end{bmatrix} \in \mathbb{C}^{64}$$



$$[I + \tilde{G}q \cdot] \phi = 1$$

This is the linear operation given to GMRES



How does GMRES work?

Consider the equation $Ax = b$ with $x \in \mathbb{R}^m$ and A a $m \times m$ matrix.

The n th Krylov subspace is $K_n := \text{span}\{b, Ab, A^2b, \dots, A^{n-1}b\}$.

GMRES approximates the exact solution of $Ax = b$ by the vector $x_n \in K_n$ that minimizes the norm of the residual $r_n = Ax_n - b$.

Given n , calculate orthonormal basis $\{q_1, \dots, q_n\}$ for K_n with the Arnoldi method, and denote $x_n = Q_n y_n$. In the process we get an $(n+1) \times n$ upper Hessenberg matrix $\tilde{H}_n$ with $AQ_n = Q_{n+1}\tilde{H}_n$.

Since the columns of Q_n are orthogonal, we have

$$\|r_n\| = \|Ax_n - b\| = \left\| \tilde{H}_n y_n - (\|b - Ax_0\|) e_1 \right\|.$$

Find the vector y_n which minimizes $\|r_n\|$. If the residual is too large, repeat with $n \leftarrow n + 1$.

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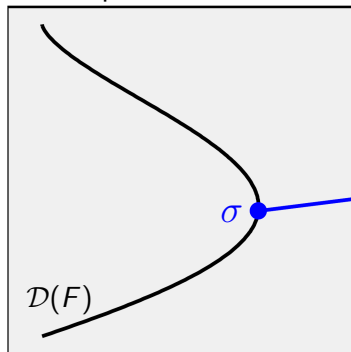
EIT and X-ray tomography are connected!

Complex geometric optics solutions

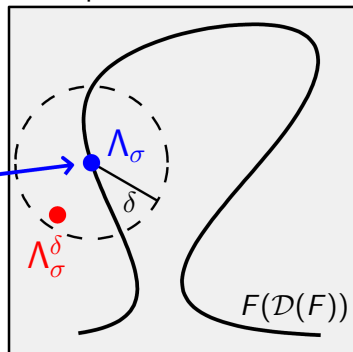
Noise, blur, and a new decomposition of real-world EIT

The forward map $F : X \supset \mathcal{D}(F) \rightarrow Y$ of an ill-posed problem does not have a continuous inverse

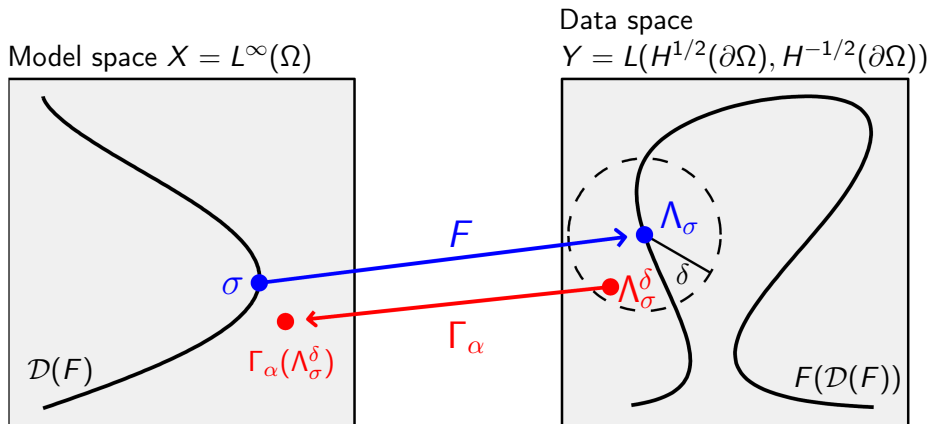
Model space X



Data space Y



Regularization means constructing a continuous map $\Gamma_\alpha : Y \rightarrow X$ that inverts F approximately



Regularization must be based on combining the incomplete measurement data with *a priori* information about the conductivity.

A regularization strategy needs to be constructed so that the assumptions below are satisfied

A family $\Gamma_\alpha : Y \rightarrow X$ of continuous mappings parameterized by $0 < \alpha < \infty$ is a *regularization strategy* for F if

$$\lim_{\alpha \rightarrow 0} \|\Gamma_\alpha(\Lambda_\sigma) - \sigma\|_X = 0$$

for each fixed $\sigma \in \mathcal{D}(F)$.

Further, a regularization strategy with a choice $\alpha = \alpha(\delta)$ of regularization parameter is called *admissible* if

$$\alpha(\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0,$$

and for any fixed $\sigma \in \mathcal{D}(F)$ the following holds:

$$\sup_{\Lambda_\sigma^\delta} \{ \|\Gamma_{\alpha(\delta)}(\Lambda_\sigma^\delta) - \sigma\|_X : \|\Lambda_\sigma^\delta - \Lambda_\sigma\|_Y \leq \delta \} \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

Infinite-precision data:

Solve boundary integral equation

$$\psi(\cdot, k)|_{\partial\Omega} = e^{ikz} - \mathcal{S}_k(\Lambda_\sigma - \Lambda_1)\psi$$

for every complex number $k \in \mathbb{C} \setminus 0$.

Evaluate the scattering transform:

$$t(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{z}} (\Lambda_\sigma - \Lambda_1) \psi(\cdot, k) ds.$$

Fix $z \in \Omega$. Solve D-bar equation

$$\frac{\partial}{\partial \bar{k}} \mu(z, k) = \frac{t(k)}{4\pi \bar{k}} e^{-i(kz + \bar{k}\bar{z})} \overline{\mu(z, k)}$$

with $\mu(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C})$.

Reconstruct: $\sigma(z) = (\mu(z, 0))^2$.

Practical data:

Solve boundary integral equation

$$\psi^\delta(\cdot, k)|_{\partial\Omega} = e^{ikz} - \mathcal{S}_k(\Lambda_\sigma^\delta - \Lambda_1)\psi^\delta$$

for all $0 < |k| < R = -\frac{1}{10} \log \delta$.

For $|k| \geq R$ set $t_R^\delta(k) = 0$. For $|k| < R$

$$t_R^\delta(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{z}} (\Lambda_\sigma^\delta - \Lambda_1) \psi^\delta(\cdot, k) ds.$$

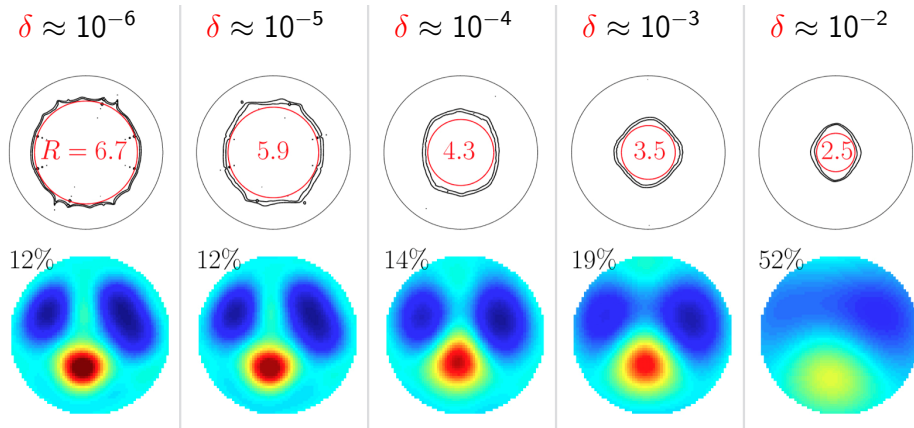
Fix $z \in \Omega$. Solve D-bar equation

$$\frac{\partial}{\partial \bar{k}} \mu_R^\delta(z, k) = \frac{t_R^\delta(k)}{4\pi \bar{k}} e^{-i(kz + \bar{k}\bar{z})} \overline{\mu_R^\delta(z, k)}$$

with $\mu_R^\delta(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C})$.

Set $\Gamma_{1/R(\delta)}(\Lambda_\sigma^\delta) := (\mu_R^\delta(z, 0))^2$.

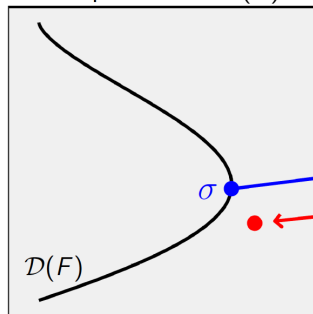
Regularized reconstructions from simulated data with noise amplitude $\delta = \|\Lambda_\sigma^\delta - \Lambda_\sigma\|_Y$



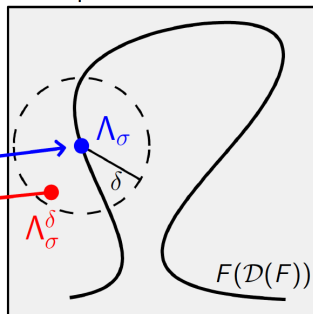
The percentages are the relative square norm errors in the reconstructions.

We define spaces for our regularization strategy

Model space $X = L^\infty(\Omega)$



Data space Y



Let $M > 0$ and $0 < \rho < 1$.
The domain $\mathcal{D}(F)$ consists of
functions $\sigma : \Omega \rightarrow \mathbb{R}$ with

- $\|\sigma\|_{C^2(\bar{\Omega})} \leq M$,
- $\sigma(z) \geq M^{-1}$,
- $\sigma(z) \equiv 1$ for $\rho < |z| < 1$.

Bounded linear operators
 $A : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)$
satisfying

- $A(1) = 0$,
- $\int_{\partial\Omega} A(f) ds = 0$.

Nonlinear low-pass filtering yields a regularization strategy with convergence speed

Theorem (Knudsen, Lassas, Mueller & S 2009)

Fix a conductivity $\sigma \in \mathcal{D}(F)$. Assume given noisy data Λ_σ^δ satisfying

$$\|\Lambda_\sigma^\delta - \Lambda_\sigma\|_Y \leq \delta.$$

Then Γ_α with the choice

$$R(\delta) = -\frac{1}{10} \log \delta, \quad \alpha(\delta) = \frac{1}{R(\delta)},$$

is well-defined, admissible and satisfies the estimate

$$\|\Gamma_{\alpha(\delta)}(\Lambda_\sigma^\delta) - \sigma\|_{L^\infty(\Omega)} \leq C(-\log \delta)^{-1/14}.$$

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We use Fourier series to represent functions and operators defined on the unit circle $\partial\Omega$

Define Fourier basis functions $\varphi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}$.

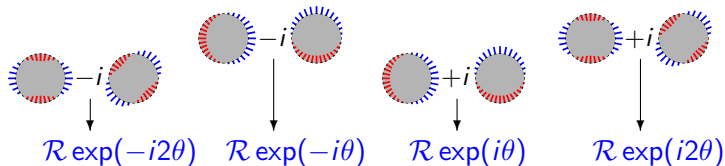
We can represent functions $f : \partial\Omega \rightarrow \mathbb{C}$ approximately like this:

$$f(\theta) \approx \sum_{n=-N}^N \langle f, \varphi_n \rangle \varphi_n(\theta), \quad \langle f, \varphi_n \rangle := \int_0^{2\pi} f(\theta) \overline{\varphi_n(\theta)} d\theta.$$

Furthermore, we can represent a linear operator $\mathcal{A}$ acting on the function f as a matrix A acting on the Fourier coefficients $\langle f, \varphi_n \rangle$:

$$A_{mn} := \int_0^{2\pi} (\mathcal{A}\varphi_n)(\theta) \overline{\varphi_m(\theta)} d\theta.$$

We use the current-to-voltage measurements to construct the Neumann-to-Dirichlet matrix



$$\begin{array}{l}
 \exp(-i2\theta) \\
 \exp(-i\theta) \\
 \exp(i\theta) \\
 \exp(i2\theta)
 \end{array}
 \begin{bmatrix}
 \langle \mathcal{R}\varphi_{-2}, \varphi_{-2} \rangle & \langle \mathcal{R}\varphi_{-1}, \varphi_{-2} \rangle & \langle \mathcal{R}\varphi_1, \varphi_{-2} \rangle & \langle \mathcal{R}\varphi_2, \varphi_{-2} \rangle \\
 \langle \mathcal{R}\varphi_{-2}, \varphi_{-1} \rangle & \langle \mathcal{R}\varphi_{-1}, \varphi_{-1} \rangle & \langle \mathcal{R}\varphi_1, \varphi_{-1} \rangle & \langle \mathcal{R}\varphi_2, \varphi_{-1} \rangle \\
 \langle \mathcal{R}\varphi_{-2}, \varphi_1 \rangle & \langle \mathcal{R}\varphi_{-1}, \varphi_1 \rangle & \langle \mathcal{R}\varphi_1, \varphi_1 \rangle & \langle \mathcal{R}\varphi_2, \varphi_1 \rangle \\
 \langle \mathcal{R}\varphi_{-2}, \varphi_2 \rangle & \langle \mathcal{R}\varphi_{-1}, \varphi_2 \rangle & \langle \mathcal{R}\varphi_1, \varphi_2 \rangle & \langle \mathcal{R}\varphi_2, \varphi_2 \rangle
 \end{bmatrix}$$

Inverting the above matrix gives the DN matrix up to constant functions.

Numerical solution of traces of CGO solutions from the boundary integral equation

We invert the linear operator appearing in the equation

$$\psi^\delta(\cdot, k)|_{\partial\Omega} = [I + \mathcal{S}_k(\Lambda_\sigma^\delta - \Lambda_1)]^{-1} e^{ikz}|_{\partial\Omega}$$

as a matrix in $\text{span}(\{\varphi_n\}_{n=-N}^N)$.

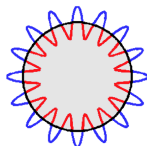
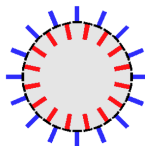
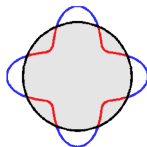
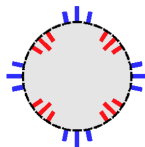
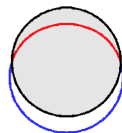
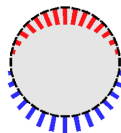
The single-layer operator

$$(\mathcal{S}_k\phi)(z) = \int_{\partial\Omega} G_k(z-w)\phi(w) ds(w)$$

uses Faddeev's Green's function.

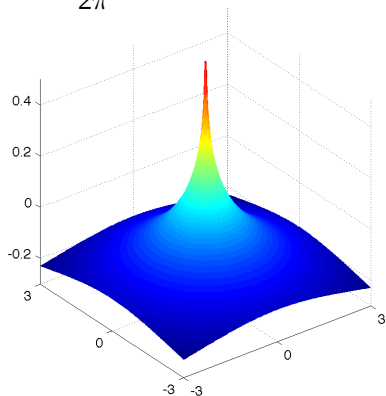
Electrodes

Theory

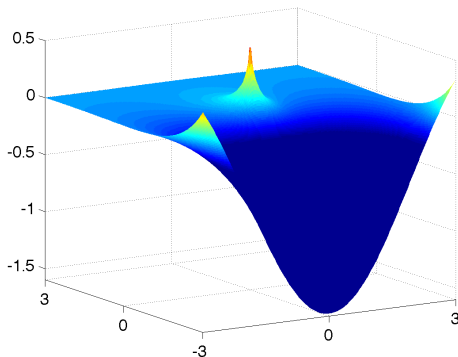


The difference between the usual Green's function and Faddeev Green's function is exponential

$$-\frac{1}{2\pi} \log |z|$$



$$G_1(z) = \frac{e^{iz}}{2\pi} \int_{\mathbb{R}^2} \frac{e^{iz \cdot \xi}}{\xi(\bar{\xi} + 2k)} dz$$



MATLAB: $G_1(z) = \text{real}(\text{expint}(-i*z))/(2*\pi)$

We approximate the operator $\mathcal{S}_k$ as a matrix in the truncated Fourier basis

$$\begin{bmatrix} \langle \mathcal{S}_k \varphi_{-2}, \varphi_{-2} \rangle & \langle \mathcal{S}_k \varphi_{-1}, \varphi_{-2} \rangle & \langle \mathcal{S}_k \varphi_0, \varphi_{-2} \rangle & \langle \mathcal{S}_k \varphi_1, \varphi_{-2} \rangle & \langle \mathcal{S}_k \varphi_2, \varphi_{-2} \rangle \\ \langle \mathcal{S}_k \varphi_{-2}, \varphi_{-1} \rangle & \langle \mathcal{S}_k \varphi_{-1}, \varphi_{-1} \rangle & \langle \mathcal{S}_k \varphi_0, \varphi_{-1} \rangle & \langle \mathcal{S}_k \varphi_1, \varphi_{-1} \rangle & \langle \mathcal{S}_k \varphi_2, \varphi_{-1} \rangle \\ \langle \mathcal{S}_k \varphi_{-2}, \varphi_0 \rangle & \langle \mathcal{S}_k \varphi_{-1}, \varphi_0 \rangle & \langle \mathcal{S}_k \varphi_0, \varphi_0 \rangle & \langle \mathcal{S}_k \varphi_1, \varphi_0 \rangle & \langle \mathcal{S}_k \varphi_2, \varphi_0 \rangle \\ \langle \mathcal{S}_k \varphi_{-2}, \varphi_1 \rangle & \langle \mathcal{S}_k \varphi_{-1}, \varphi_1 \rangle & \langle \mathcal{S}_k \varphi_0, \varphi_1 \rangle & \langle \mathcal{S}_k \varphi_1, \varphi_1 \rangle & \langle \mathcal{S}_k \varphi_2, \varphi_1 \rangle \\ \langle \mathcal{S}_k \varphi_{-2}, \varphi_2 \rangle & \langle \mathcal{S}_k \varphi_{-1}, \varphi_2 \rangle & \langle \mathcal{S}_k \varphi_0, \varphi_2 \rangle & \langle \mathcal{S}_k \varphi_1, \varphi_2 \rangle & \langle \mathcal{S}_k \varphi_2, \varphi_2 \rangle \end{bmatrix}$$

The operator Λ_1 is known analytically when Ω is the unit disc

$$\Lambda_1 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Numerical solution of the D-bar equation is based on the periodization approach of G. Vainikko

The generalization of Vainikko's method for the D-bar equation is described in [Knudsen, Mueller & S 2004].

The D-bar equation

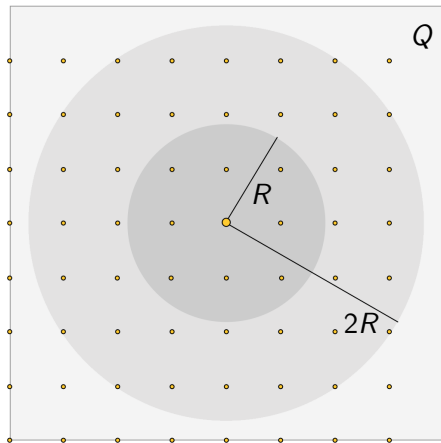
$$\frac{\partial}{\partial \bar{k}} \mu_R^\delta = \frac{1}{4\pi \bar{k}} t_R^\delta(k) e_{-z}(k) \overline{\mu_R^\delta}$$

together with the asymptotics

$$\mu_R^\delta(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C})$$

can be combined in a generalized Lippmann-Schwinger equation:

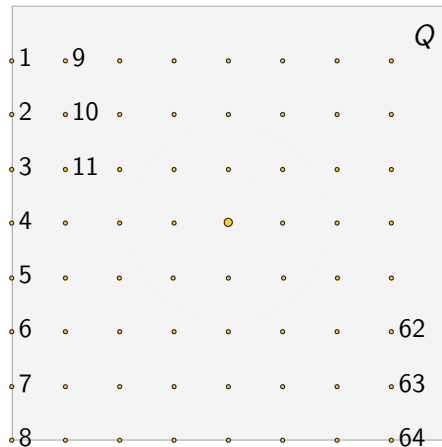
$$\mu_R^\delta(z, k) = 1 - \frac{1}{4\pi^2} \int_{\mathbb{C}} \frac{t_R^\delta(k')}{(k - k')k'} e_{-z}(k') \overline{\mu_R^\delta(z, k')} dk'_1 dk'_2.$$



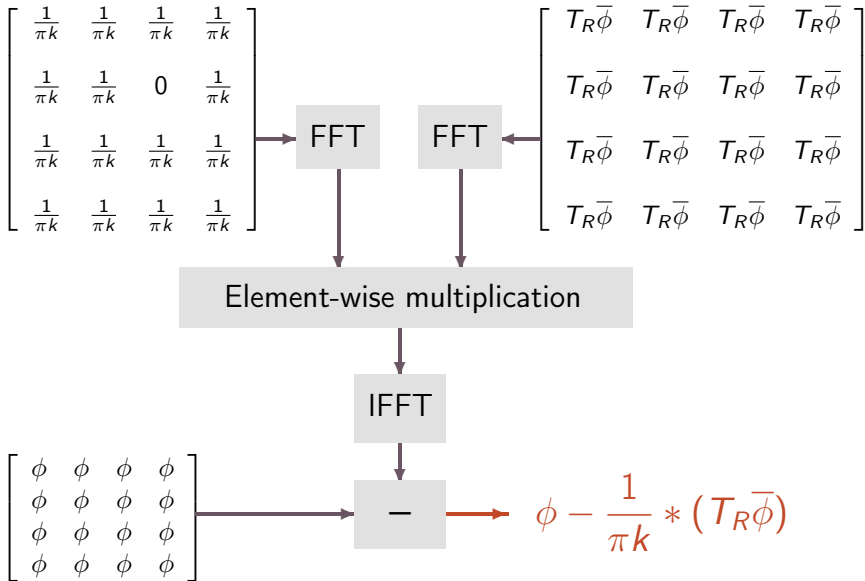
The D-bar equation is not complex-linear, so real and imaginary parts must be written separately

The grid points are numbered with one index as shown. Any function $\phi : Q \rightarrow \mathbb{C}$ is represented by a vector of its values at the grid points, organized as follows:

$$\begin{bmatrix} \operatorname{Re}\phi(z_1) \\ \operatorname{Re}\phi(z_2) \\ \vdots \\ \operatorname{Re}\phi(z_{64}) \\ \operatorname{Im}\phi(z_1) \\ \operatorname{Im}\phi(z_2) \\ \vdots \\ \operatorname{Im}\phi(z_{64}) \end{bmatrix} \in \mathbb{R}^{128}$$



This is the real-linear operation given to GMRES



Literature review: computational solution methods for the D-bar equation

2000 S, Mueller JL and Isaacson: Nyström method

2004 Knudsen, Mueller and S: Vainikko's method

2004 Eirola, Huhtanen and von Pfaler: real-linear GMRES

2014 Mueller P: finite difference method

2017 Klein and McLaughlin: spectral method

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Noise, blur, and a new decomposition of real-world EIT

Various D-bar method works for EIT

Partial-boundary data: Alsaker, Hamilton, and Hauptmann (2017), Hauptmann, Santacesaria and S (2017), Hamilton and S (2014)

A priori information: Alsaker and Mueller (in review), Hamilton, Mueller and Alsaker (2017), Alsaker and Mueller (2016)

Real-time computation: Dodd and Mueller (2014)

Complex-valued coefficient: Hamilton (2017), Herrera, Vallejo, Mueller and Lima (2015), Hamilton and Mueller (2013), Hamilton, Herrera, Mueller and von Herrmann (2012)

Anisotropic coefficient: Hamilton, Lassas and S (2014), Henkin and Santacesaria (2010)

Sharpening: Hamilton, Reyes, S and Zhang (2016), Hamilton, Hauptmann and S (2014)

The D-bar method based on the Schrödinger equation ($\sigma \in C^2(\Omega)$) is currently most developed

Theoretical studies

Uniqueness, no-noise reconstruction

1996 Nachman

Conditional stability

1997 Liu

Regularization strategy

2009 Knudsen, Lassas, Mueller, S

Computational methods

Feasibility study

2000 S, Mueller, Isaacson

2003 Mueller, S

2004 Knudsen, Mueller, S

Reconstruction from measured data

2004 Isaacson, Mueller, Newell, S

2006 Isaacson, Mueller, Newell, S

2007 Murphy, Mueller, Newell

2010 deAngelo, Mueller

... many many more papers

Real-time algorithm

2014 Dodd, Mueller

The D-bar method based on a 2×2 system ($\sigma \in C^1(\Omega)$) extends to complex impedances

Theoretical studies

Uniqueness, no-noise reconstruction

1997 Brown, Uhlmann

2000 Francini

Conditional stability

2001 Barceló, Barceló, Ruiz

2010 Beretta, Francini

Computational methods

Feasibility study

2001 Knudsen, Tamasan

2003 Knudsen

2012 Hamilton, Herrera, Mueller,
von Herrmann

2013 Hamilton, Mueller

Reconstruction from measured data

2015 Herrera, Vallejo, Mueller, Lima

2017 Hamilton

The D-bar method based on Beltrami equation can deal with discontinuities ($\sigma \in L^\infty(\Omega)$)

Theoretical studies

Uniqueness, no-noise reconstruction

2003 Astala, Päivärinta

2005 Astala, Lassas, Päivärinta

Conditional stability

2007 Barceló, Faraco, Ruiz

2008 Clop, Faraco, Ruiz

Virtual hybrid extension

2018 Greenleaf, Lassas, Santacesaria, S and Uhlmann

Computational methods

Feasibility study

2010 Astala, Mueller, Päivärinta, S

2011 Astala, Mueller, Päivärinta, Perämäki, S

2014 Astala, Päivärinta, Reyes, S

Reconstruction from measured data

2025 Alsaker, Rautio, Moura, Agnelli, Murthy, Lassas, Mueller and S

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- Complex geometric optics solutions

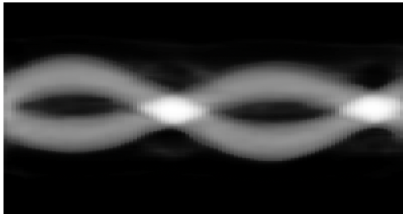
- Noise, blur, and a new decomposition of real-world EIT

Measurements from Jennifer Mueller's lab

Phantom



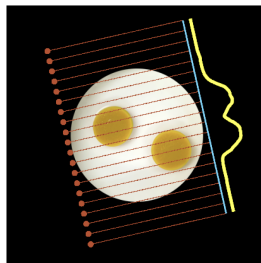
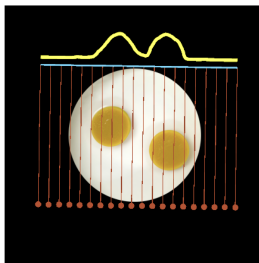
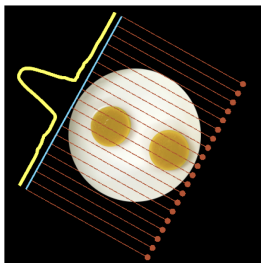
VHPT sinogram



TV reconstruction



One-dimensional projection data along three directions of virtual rays:



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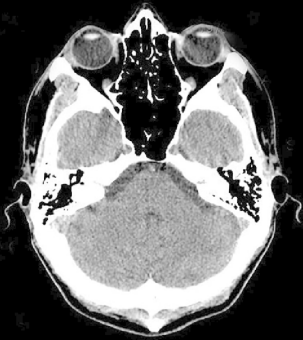
Couch unit for EMI brain scanner.



Image: Science Museum Group.

This sweeping movement is the data collection mode of first-generation CT scanners

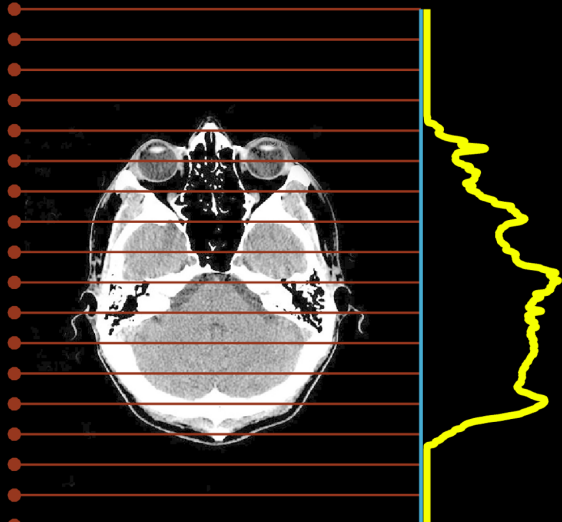
Slice image source:
Wikimedia commons,
Andrew Ciscel



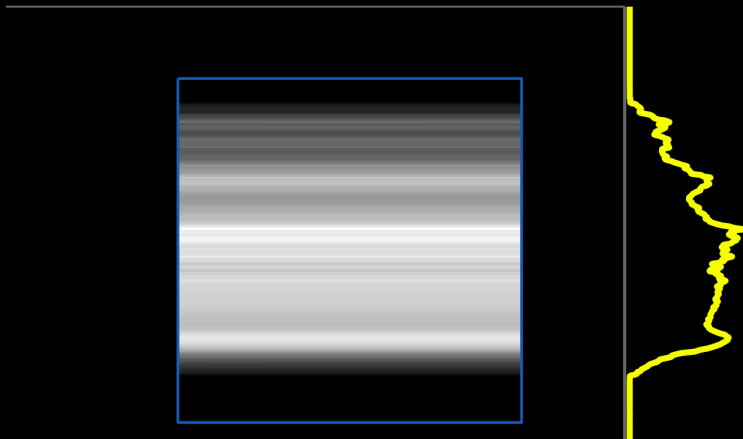




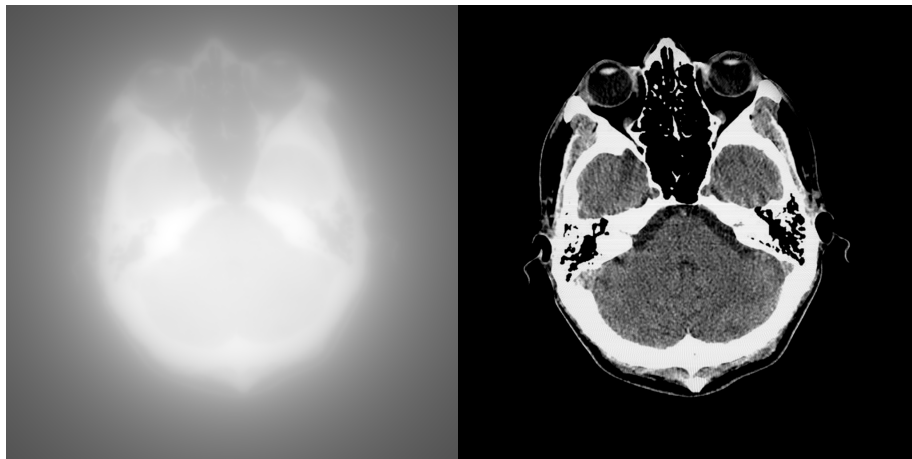
Data is collected by rotating the system around the patient



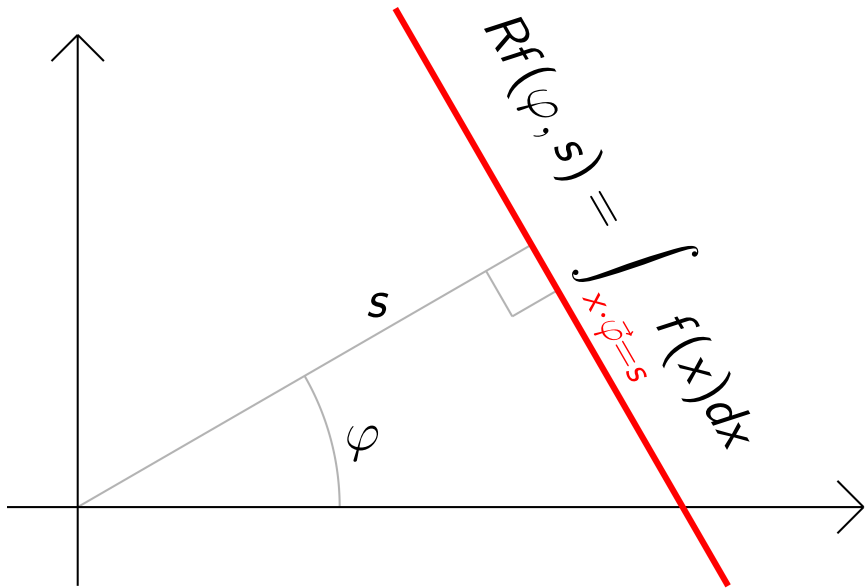
Back-projection operation produces a blurry initial reconstruction



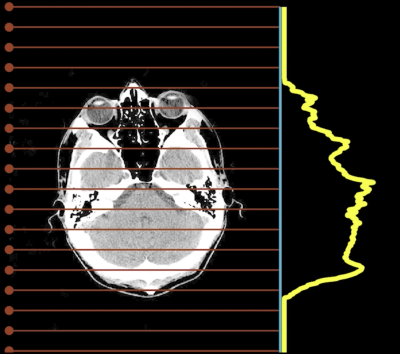
Remove blur by filtering in the frequency domain



Radon transform organizes sets of line integrals



Radon transform as sinogram



We arrive at the modern way of writing the
filtered back-projection reconstruction formula

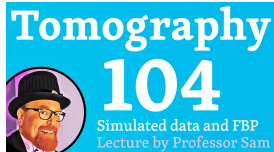
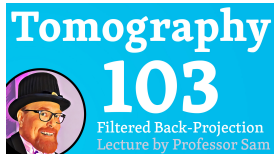
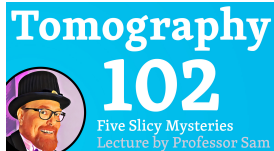
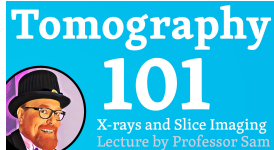


$$f = \frac{1}{2} \Lambda R^* R f$$

$$\Lambda g(x) = \mathcal{F}^{-1} |\xi| \hat{g}(\xi)$$

Johann Radon (1887-1956)
published FBP formula in 1917

Short lectures available in the
YouTube channel Professor Sam



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Calderón's inverse conductivity problem (1980) is an ideal mathematical model for EIT

Let $\Omega \subset \mathbb{R}^2$ be the unit disc and let conductivity $\sigma : \Omega \rightarrow \mathbb{R}$ satisfy

$$0 < M^{-1} \leq \sigma(z) \leq M.$$

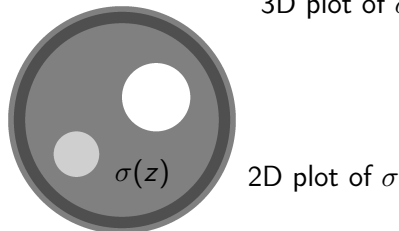
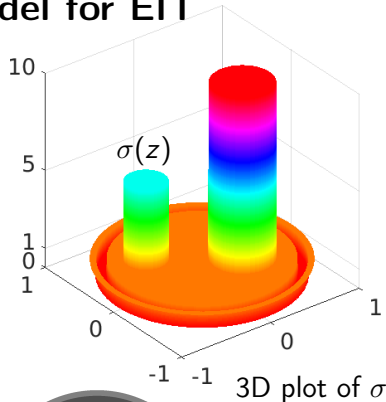
Voltage u solves conductivity equation

$$\begin{cases} \nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= f. \end{cases}$$

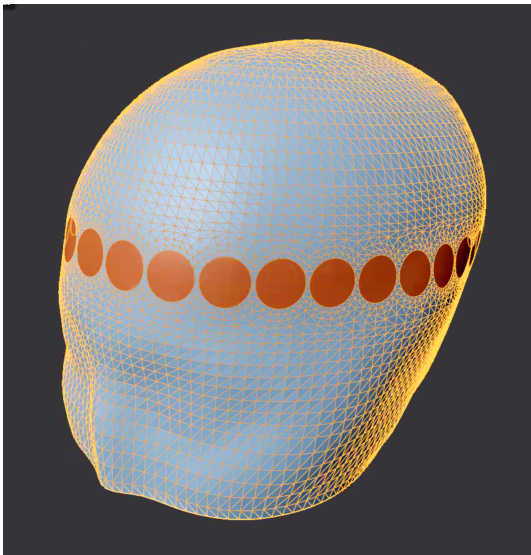
Boundary measurements are modelled by the Dirichlet-to-Neumann map

$$\Lambda_\sigma : f \mapsto \sigma \frac{\partial u}{\partial \bar{n}}|_{\partial\Omega}.$$

The task is to recover σ from Λ_σ .



Brain EIT imaging is based on covering the head partly by electrodes



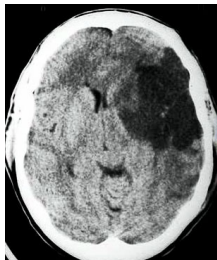
Tissue	Ωcm
Cortex	229
White matter	344
Blood	125
CS fluid	69
Scalp	490
Skull	6500



Imaging stroke with EIT. Same symptoms in both c

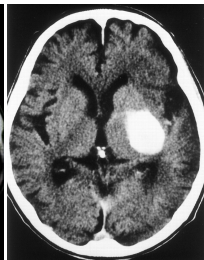
Ischemic stroke:
low conductivity.

CT image from
Jansen 2008



Hemorrhagic stroke:
high conductivity.

CT image from
Nakano *et al.* 2001



Difficulties: resistive skull layer and unknown background. However, see

- Romsauerova, McEwan, Horesh, Yerworth, Bayford & Holder 2006
- Shi, You, Xu, Wang, Fu, Liu & Dong 2009
- Bayford & Tizzard 2012
- Malone, Jehl, Arridge, Betcke & Holder 2014
- Candiani, Hyvönen, Kaipio and Kolehmainen 2021
- Agnelli, Moura, Rautio, Alsaker, Murthy, Lassas and S 2026

Monitoring a patient in an intensive care unit is an important application of stroke-EIT



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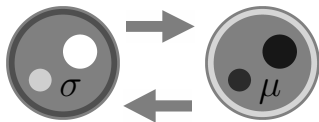
EIT and X-ray tomography are connected!

Complex geometric optics solutions

Noise, blur, and a new decomposition of real-world EIT

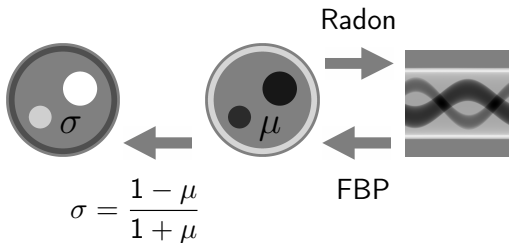
Define a simple nonlinear transform between σ and Beltrami coefficient μ

$$\frac{1 - \sigma}{1 + \sigma} =: \mu$$

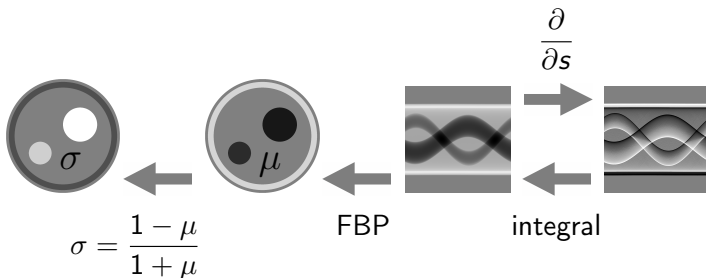


$$\sigma = \frac{1 - \mu}{1 + \mu}$$

We can apply parallel-beam Radon transform and invert it with filtered backprojection

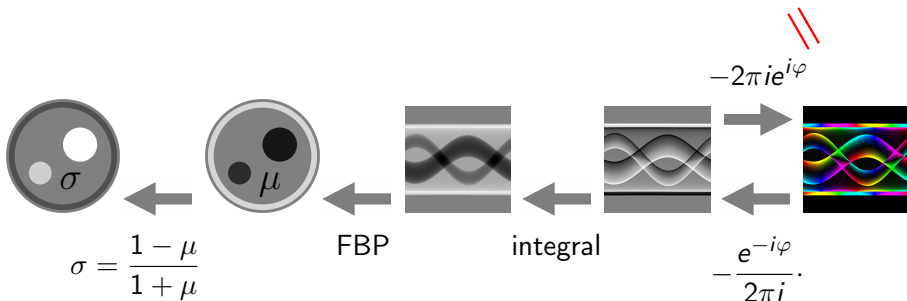


Differentiation of the columns of the sinogram,
reversed by the fundamental theorem of calculus

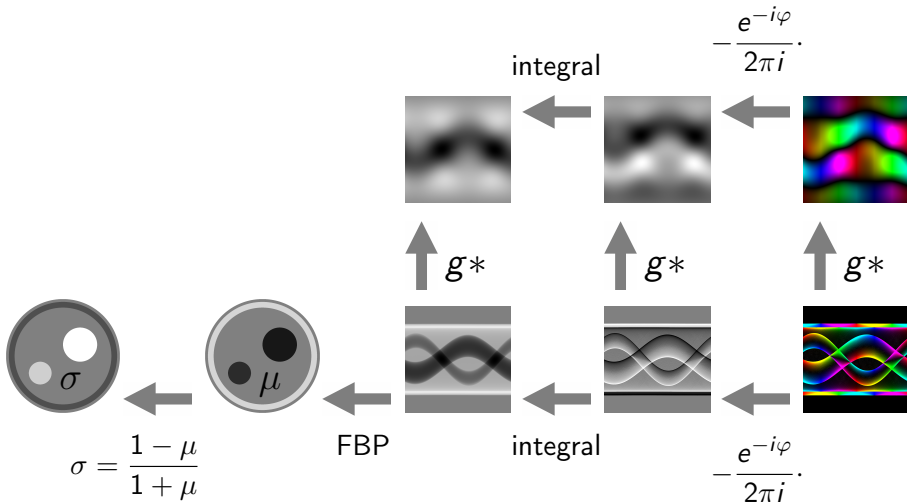


Multiplication by a complex-valued phase function
can be inverted by division

$$-2\pi i e^{i\varphi} \left(\frac{\partial}{\partial s} R\mu \right) \left(\frac{1}{2}t, e^{i\varphi} \right) = \underbrace{T_1^+ \mu(t, e^{i\varphi})}_{\text{red bracket}}$$



Sneak peek: we will need blurred versions of the sinograms in the novel EIT diagram



Take-home message: ideal EIT allows solution by complex “filtered back-projection”

Theorem (Greenleaf, Lassas, Santacesaria, S and Uhlmann 2018).
Define the operator $T_1^\pm$ like this:

$$T_1^\pm \mu(t, e^{i\varphi}) = \frac{1}{2\pi i} \int_{\partial\Omega} \hat{\omega}_1^\pm(z, t, e^{i\varphi}) dz.$$

Then we have a filtered back-projection (FBP) formula

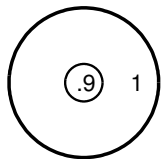
$$(-\Delta)^{-1/2} (T_1^\pm)^* T_1^\pm \mu = \mu.$$

(Compare to the X-ray FBP formula $\frac{1}{2}(-\Delta)^{1/2} R^* R \mu = \mu$.)

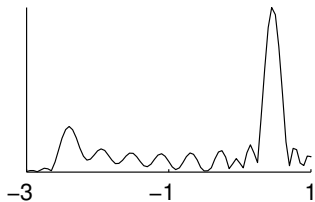
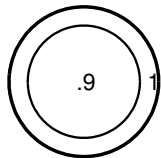
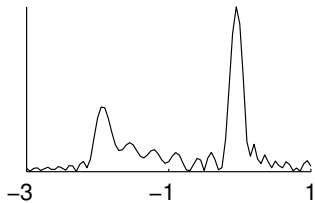
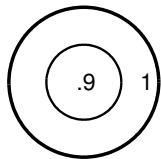
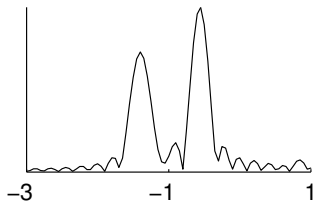
This project started in 2000



Original conductivity



$|F(\psi(1,k)-e^{ik})(t)|$





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Complex geometric optics solutions

Noise, blur, and a new decomposition of real-world EIT

We use the Beltrami-type complex geometric optics (CGO) solutions

Set $\mu := (1 - \sigma)/(1 + \sigma)$. Write $f = u + iv$ and note that

$$\overline{\partial}_z f_\mu = \mu \overline{\partial}_z \overline{f_\mu} \quad \Leftrightarrow \quad \nabla \cdot \sigma \nabla u = 0 \quad \text{and} \quad \nabla \cdot \sigma^{-1} \nabla v = 0.$$

The CGO solutions of [Astala-Päivärinta 2006] have the form

$$\begin{aligned} f_\mu(z, k) &= e^{ikz}(1 + \omega^+(z, k)), \\ f_{-\mu}(z, k) &= e^{ikz}(1 + \omega^-(z, k)), \end{aligned}$$

with the asymptotic condition

$$\omega^\pm(z, k) = \mathcal{O}\left(\frac{1}{|z|}\right) \text{ as } |z| \rightarrow \infty.$$

Here $ikz = i(k_1 + ik_2)(x + iy)$ and $\overline{\partial}_z = \frac{1}{2}(\partial_x + i\partial_y)$.

This is a brief history of computational solution methods for the Beltrami CGO solutions

2006 Astala and Päivärinta : Construction

2010 Astala, Mueller, Päivärinta and S: First numerical solution method

2011 Astala, Mueller, Päivärinta, Perämäki and S: Novel EIT reconstruction method

2012 Huhtanen and Perämäki: Preconditioned Krylov subspace method for solving real-linear systems

2014 Astala, Päivärinta, Reyes and S: Computational high-frequency experiments

Cauchy and Beurling transforms

Define the Cauchy and Beurling transforms by

$$Pf(z) = \bar{\partial}^{-1}f(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\lambda)}{\lambda - z} d\lambda_1 d\lambda_2,$$

$$Sg(z) = \partial \bar{\partial}^{-1}g(z) = -\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{|\lambda - z| > \varepsilon} \frac{g(\lambda)}{(\lambda - z)^2} d\lambda_1 d\lambda_2.$$

Also, set

$$\begin{aligned}e_k(z) &= \exp(i(kz + \bar{k}\bar{z})), \\ \alpha(z, k) &= -i\bar{k}e_{-k}(z)\mu(z), \\ \nu(z, k) &= e_{-k}(z)\mu(z),\end{aligned}$$

and define the operator A by

$$A := (-\bar{\nu}S - \bar{\alpha}P).$$

We introduced a non-physical scattering series

Huhtanen and Perämäki (2012) modified the original construction of Astala and Päiväranta (2006) for computational purposes. We use the 2012 technique for the construction of a novel scattering series

$$\omega^+ = \sum_{n=1}^{\infty} \omega_n^+,$$

where

$$\omega_n^+ = -\bar{\partial}_z^{-1} \overline{u_n}, \quad u_n = -A \overline{u_{n-1}}, \quad u_1 = -\bar{\alpha}$$

and $A := (-\bar{\nu}S - \bar{\alpha}P)$. The **single scattering** term $\omega_1^+ = \bar{\partial}_z^{-1} \alpha$ determines singularities of μ .

[Greenleaf, Lassas, Santacesaria, S and Uhlmann 2018]

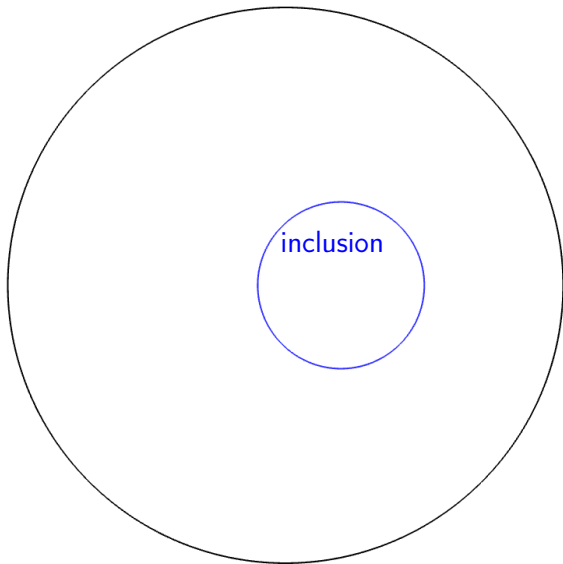
We apply one-dimensional Fourier transform to the complex spectral parameter

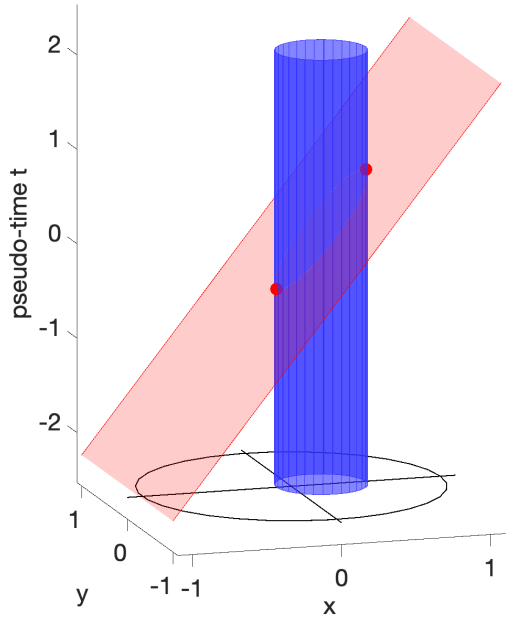
For $z \in \mathbb{C}$, $t \in \mathbb{R}$ and $\varphi \in [0, 2\pi]$ define

$$\begin{aligned}\widehat{\omega}_{\textcolor{red}{n}}^{\pm}(z, t, \varphi) &:= \mathcal{F}_{\tau \rightarrow t}(\omega_{\textcolor{red}{n}}^{\pm}(z, \tau e^{i\varphi})) \\ &= \int_{-\infty}^{\infty} e^{-it\tau} \omega_{\textcolor{red}{n}}^{\pm}(z, \tau e^{i\varphi}) d\tau,\end{aligned}$$

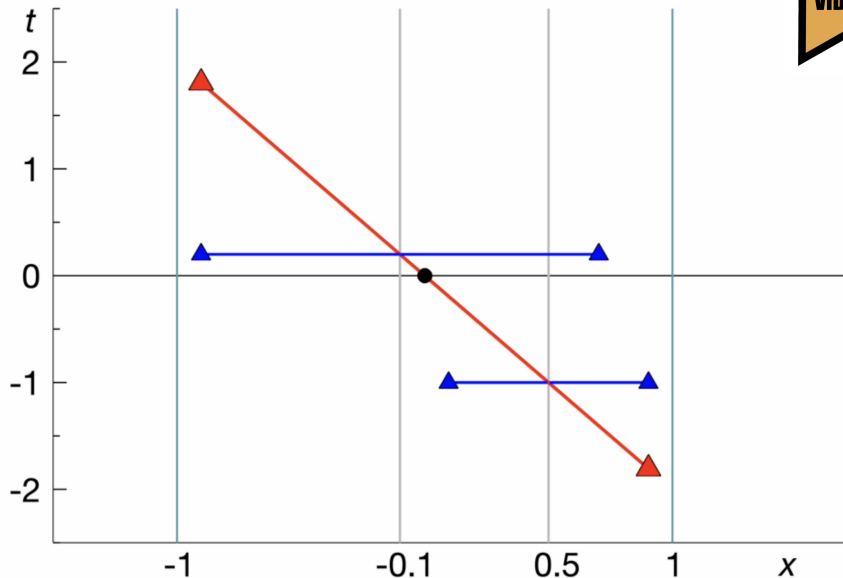
where $n = 1, 2, 3, \dots$

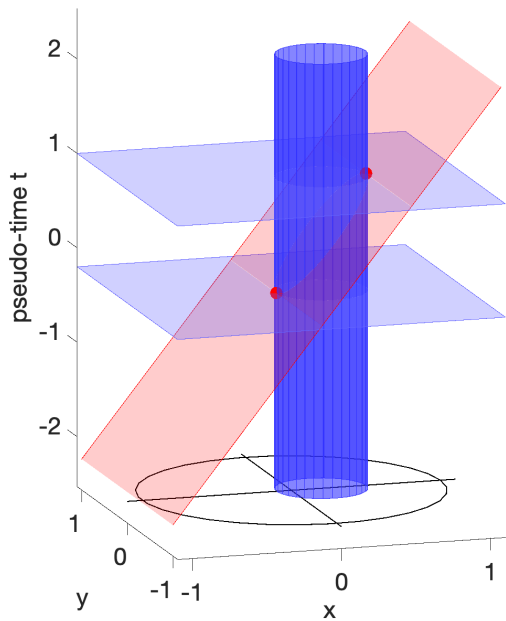
Consider a disc inclusion with center coordinates $x = 0.2$, $y = 0$, and radius 0.3

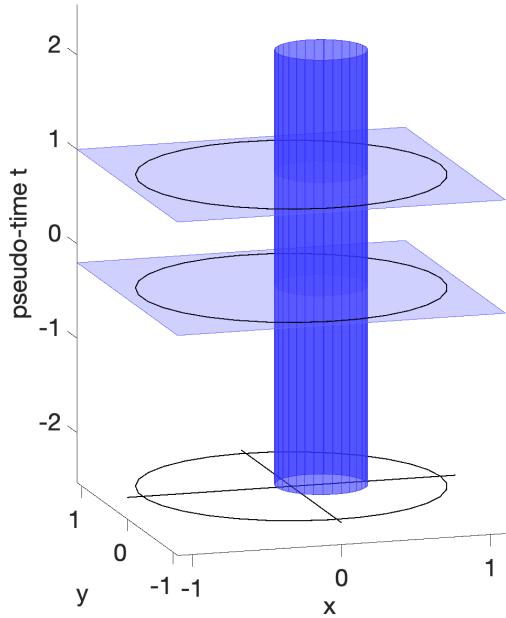




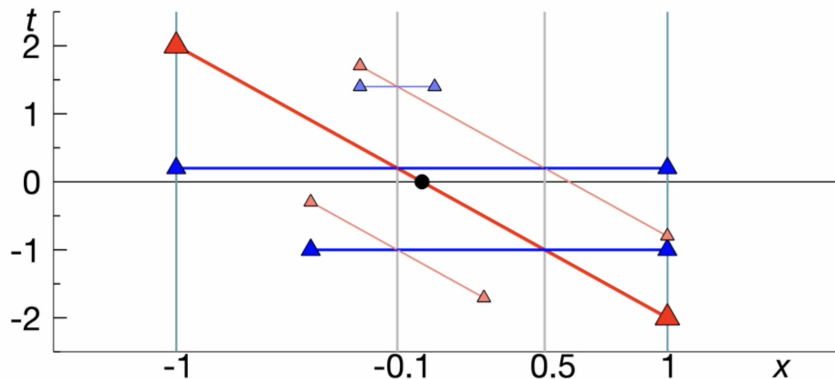
First-order singularity propagation

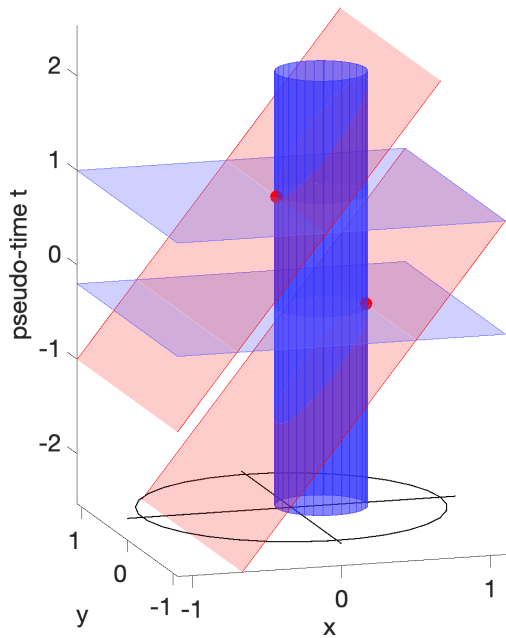


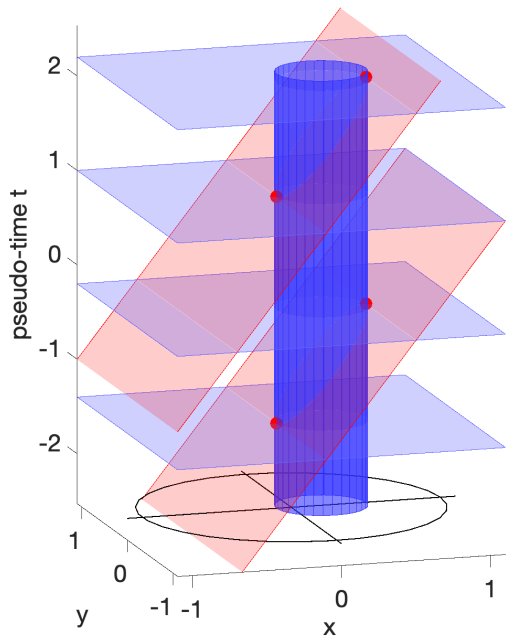


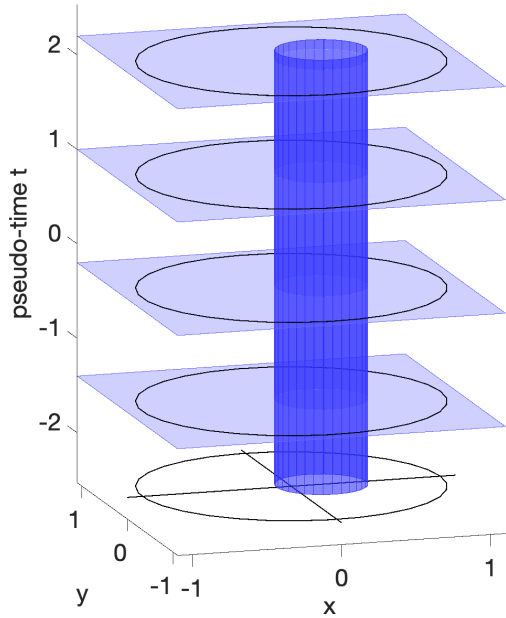


Higher-order singularity propagation

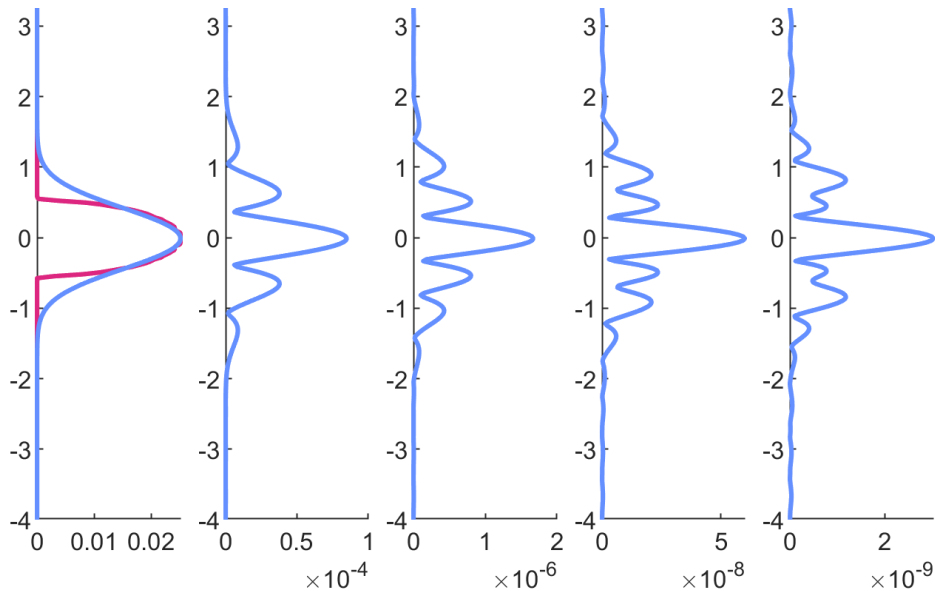








Calculations by Sara Sippola



Now we have a better understanding of the complex “filtered back-projection”

Theorem (Greenleaf, Lassas, Santacesaria, S and Uhlmann 2018).
Define the operator $T_1^\pm$ like this:

$$T_1^\pm \mu(t, e^{i\varphi}) = \frac{1}{2\pi i} \int_{\partial\Omega} \widehat{\omega}_1^\pm(z, t, e^{i\varphi}) dz.$$

Then we have a filtered back-projection (FBP) formula

$$(-\Delta)^{-1/2} (T_1^\pm)^* T_1^\pm \mu = \mu.$$

Outline

The D-bar method for EIT

- Complex geometric optics solutions

- Regularization theorem

- Implementation of the D-bar method

- Variants of the D-bar method

Virtual hybrid parallel-beam tomography

- Radon transform and X-ray tomography

- Electrical impedance tomography (EIT)

- EIT and X-ray tomography are connected!

- Complex geometric optics solutions

- Noise, blur, and a new decomposition of real-world EIT

The results in this section are joint with

Juan Pablo Agnelli

Melody Alsaker

Matti Lassas

Fernando Moura

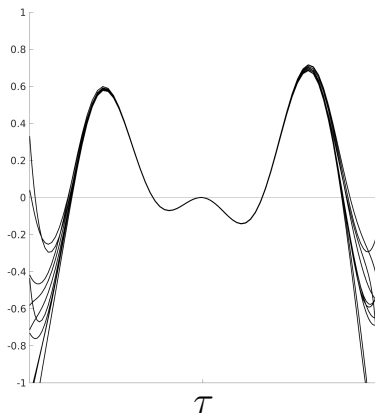
Jennifer Mueller

Rashmi Murthy

Siiri Rautio

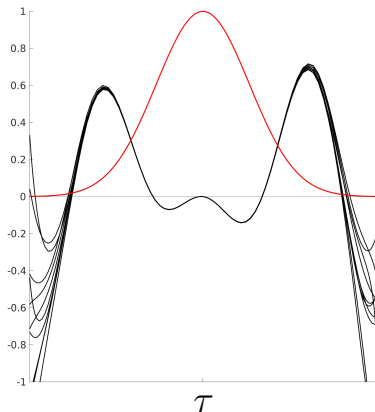
Note the improved gender balance in this new team!

Noise in the data Λ_σ causes inaccuracy in the solution of the BIE. The error grows with $|\tau|$

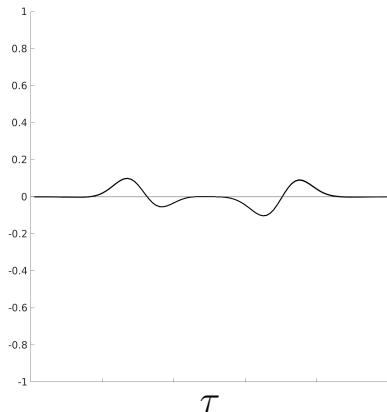


$\text{Re}(\omega^+(1, \tau))$ from many
realizations of **low-noise** Λ_σ

Noise in the data Λ_σ causes inaccuracy in the solution of the BIE. The error grows with $|\tau|$

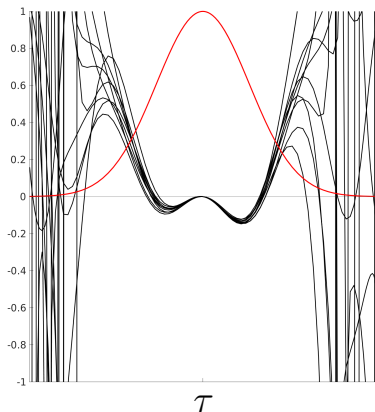


$\text{Re}(\omega^+(1, \tau))$ from many realizations of **low-noise** Λ_σ

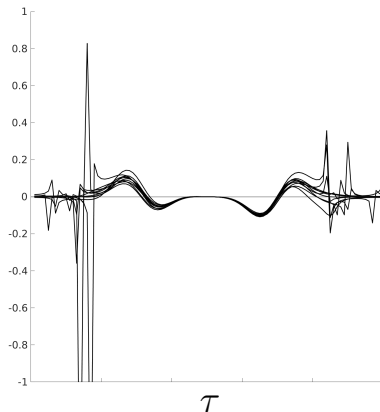


$\text{Re}(\omega^+(1, \tau))$ after multiplication with Gaussian window

Noise in the data Λ_σ causes inaccuracy in the solution of the BIE. The error grows with $|\tau|$

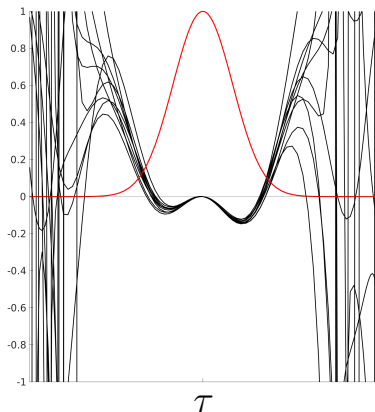


$\text{Re}(\omega^+(1, \tau))$ from many realizations of **high-noise** Λ_σ

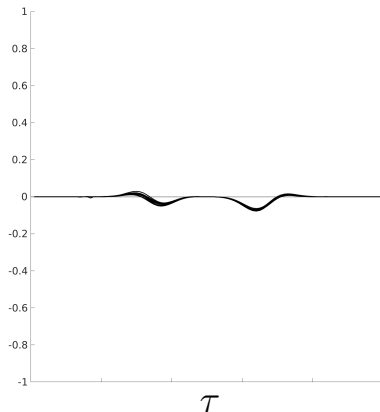


$\text{Re}(\omega^+(1, \tau))$ after multiplication with Gaussian window
(Window width too wide)

Noise in the data Λ_σ causes inaccuracy in the solution of the BIE. The error grows with $|\tau|$



$\text{Re}(\omega^+(1, \tau))$ from many realizations of **high-noise** Λ_σ



$\text{Re}(\omega^+(1, \tau))$ after multiplication with Gaussian window

For studying the noise issue, let's change the order of z integration and τ integration

Define

$$\tilde{T}^{\pm}\mu(\tau, \varphi) = \frac{1}{2\pi i} \int_{\partial\Omega} \omega^{\pm}(z, \tau, \varphi) dz.$$

We can compute $\tilde{T}^{\pm}\mu(\tau, \varphi)$ robustly from the DN map Λ_{σ} for

$$|\tau| \leq R,$$

where $R > 0$ depends on the noise level. Note that

$$T^{\pm}\mu(t, \varphi) = \mathcal{F}_{\tau \rightarrow t}(\tilde{T}^{\pm}\mu(\tau, \varphi)).$$

The above definitions also work for $\omega_n^{\pm}$, T_n , $\tilde{T}_n$.

We need to window the 1D Fourier transform, with window width depending on noise amplitude

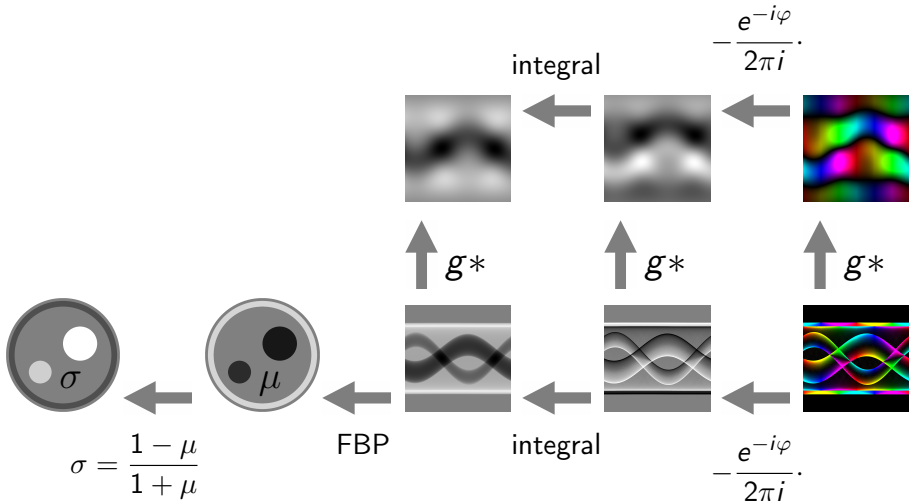
Restrict the integration to interval $[-R, R]$ and insert a Gaussian function with suitable parameter $a > 0$ into the Fourier integral:

$$\check{\omega}^{\pm}(z, t, \varphi) := \int_{-R}^R e^{-a\tau^2} e^{-it\tau} \omega^{\pm}(z, \tau e^{i\varphi}) d\tau.$$

Note that $\check{\omega}^{\pm}$ can be computed from Λ_{σ} in a noise-robust way. Denote the inverse Fourier transform of $e^{-a\tau^2}$ by $g(t)$. Then

$$\begin{aligned} (g * T_1^{\pm} \mu)(t, \varphi) &= \int_{-\infty}^{\infty} g(t-s) \left(\frac{1}{2\pi i} \int_{\partial\Omega} \hat{\omega}_1^{\pm}(z, s, \varphi) dz \right) ds \\ &= \frac{1}{2\pi i} \int_{\partial\Omega} \left(\int_{-\infty}^{\infty} e^{-a\tau^2} e^{-it\tau} \omega_1^{\pm}(z, \tau e^{i\varphi}) d\tau \right) dz \\ &\approx \frac{1}{2\pi i} \int_{\partial\Omega} \check{\omega}^{\pm}(z, t, \varphi) dz. \end{aligned}$$

Take-home message 2: in CGO&EIT, dealing with measurement noise causes blurring

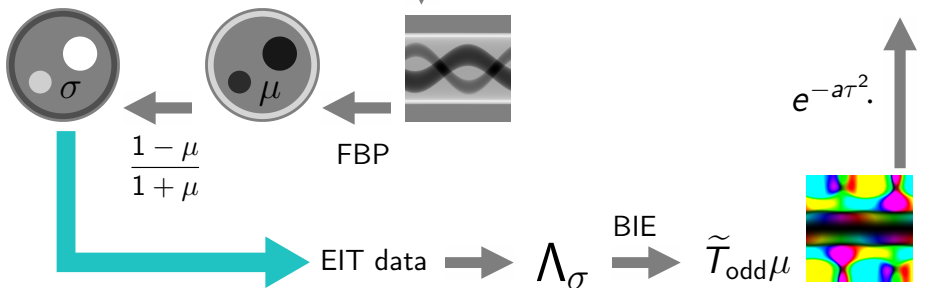


Decomposition of real-world EIT

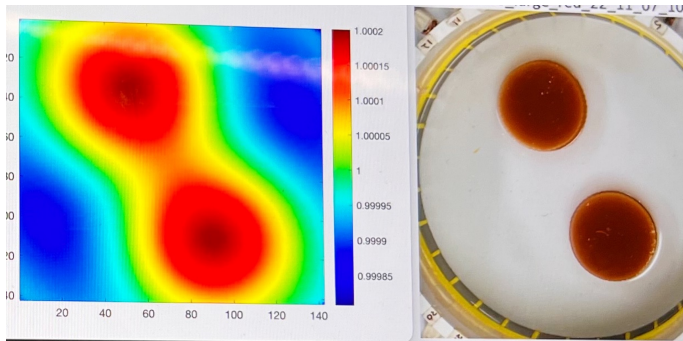
Ill-posedness is restricted to two steps only:

- ▶ 1D deconvolution,
- ▶ 2D FBP.

Both of them are linear.



Virtual Hybrid Parallel-beam Tomography VHPT



Breakthrough
in June 2023

Virtual Hybrid Parallel-beam Tomography VHPT: details of machine learning

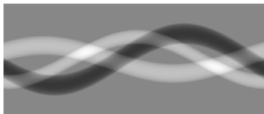
Input R_{ω_1}



Prediction $\hat{R}_\mu$



Ground truth R_μ



- ▶ 7000 training data pairs: (R_{ω_1}, R_μ)

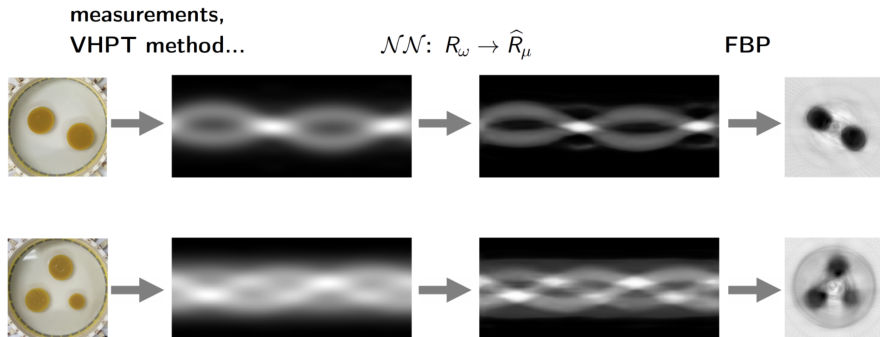
- ▶ Loss function:
MSE with sinogram penalty term:

$$\underbrace{\frac{1}{N} \|R_\mu - \hat{R}_\mu\|^2}_{\text{MSE}} + \underbrace{\alpha \left\| \sum_{k=1}^n \hat{R}_{\mu_{k,i}} - \sum_{k=1}^n \hat{R}_{\mu_{k,j}} \right\|_1}_{\text{penalty term}},$$

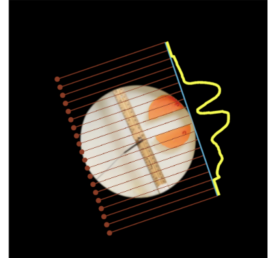
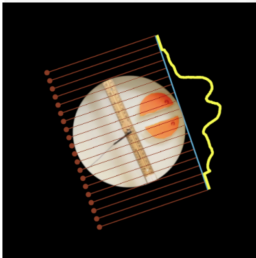
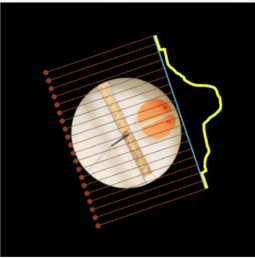
where $j = 1, \dots, s$.

- ▶ Optimizer: Adam
- ▶ Learning rate: 0.001

Virtual Hybrid Parallel-beam Tomography VHPT: experimental results

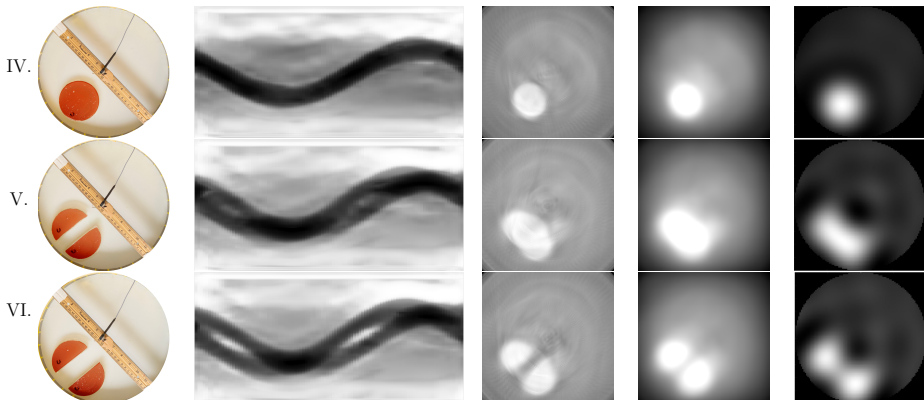


Virtual Hybrid Parallel-beam Tomography VHPT: slit experiment



[Alsaker, Rautio, Moura, Agnelli, Murthy, Lassas, Mueller and S 2025]

Virtual Hybrid Parallel-beam Tomography VHPT: slit experiment



[Alsaker, Rautio, Moura, Agnelli, Murthy, Lassas, Mueller and S 2025]

Thank you for your attention!



← Slime mold called *Lycogala conicum*