Mathematics of Electrical Impedance Tomography
Lecture 6: DN map by the finite element method

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Outline

Principle of the finite element method

Computation of the ND matrix

Determining the DN matrix from the ND matrix

The starting point of FEM is the weak formulation

Multiply the conductivity equation by a test function $v \in H_0^1(\Omega)$ and integrate by parts:

$$\int_\Omega \sigma \nabla u \cdot \nabla v \, dx = 0.$$

Define the conductivity-weighted energy form by

$$a_\sigma(u, v) = \int_\Omega \sigma(x) \nabla u(x) \cdot \nabla v(x) \, dx.$$

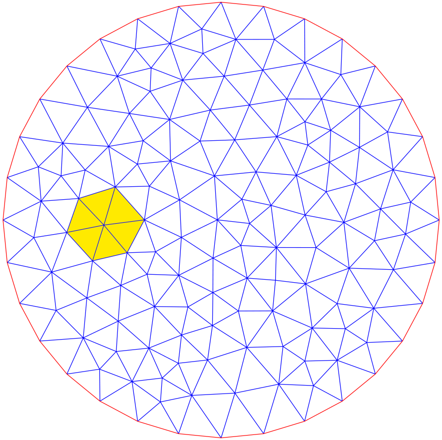
On $H_0^1(\Omega)$, this is an inner product equivalent to the usual H^1-inner product.
Weak formulation: find $u \in H^1(\Omega)$ such that

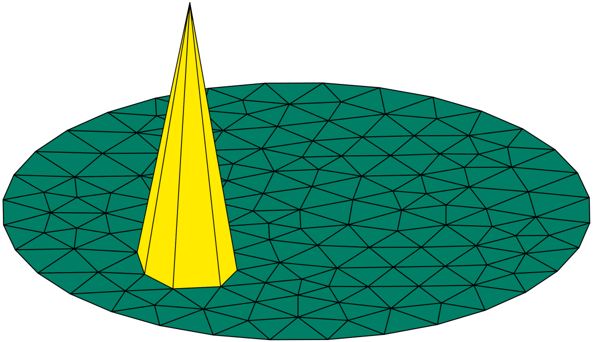
$$u|_{\partial\Omega} = f$$

and

$$a_\sigma(u, v) = 0 \qquad \text{for every } v \in H_0^1(\Omega).$$

Finite element mesh and piecewise linear basis function





Piecewise linear finite elements

Let $\mathcal{T}_h$ be a triangulation of Ω. We introduce the finite-dimensional space

$$V_h = \left\{ v_h \in C(\overline{\Omega}) : v_h|_T \text{ is affine for every } T \in \mathcal{T}_h \right\}.$$

Let $x_1, \ldots, x_N$ denote the vertices of the triangulation. The standard nodal basis functions $\varphi_j \in V_h$ are defined by

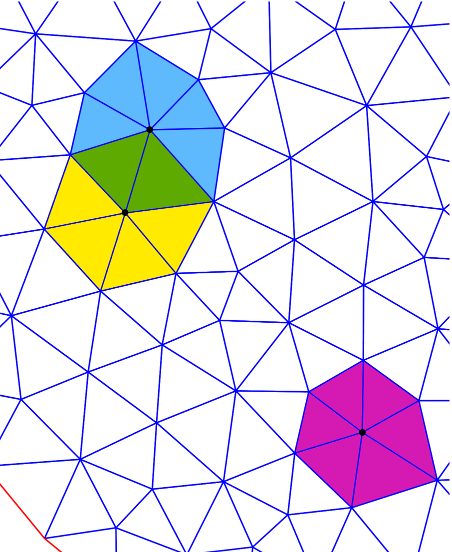
$$\varphi_j(x_i) = \delta_{ij}.$$

Thus, each φ_j is a continuous piecewise linear hat function supported on the triangles that meet at the vertex x_j.

Every finite element function $u_h \in V_h$ can be written as

$$u_h(x) = \sum_{j=1}^{N} U_j \varphi_j(x).$$

Inner products between FEM basis functions



The overlaps of basis function supports are crucial.

Inner products and the stiffness matrix

The usual $H^1(\Omega)$-inner product between two basis functions is

$$\langle \varphi_i, \varphi_j \rangle_{H^1(\Omega)} = \int_\Omega \varphi_i \varphi_j \, dx + \int_\Omega \nabla \varphi_i \cdot \nabla \varphi_j \, dx.$$

For the conductivity equation, however, the most important inner products are the energy inner products

$$\boxed{a_\sigma(\varphi_i, \varphi_j) = \int_\Omega \sigma \nabla \varphi_i \cdot \nabla \varphi_j \, dx.}$$

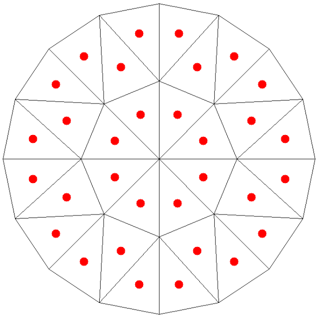
These inner products form the entries of the stiffness matrix:

$$K_{ij} = a_\sigma(\varphi_j, \varphi_i) = \int_\Omega \sigma \nabla \varphi_j \cdot \nabla \varphi_i \, dx.$$

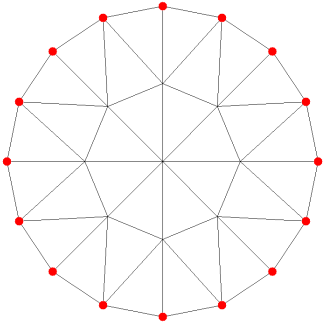
Because the basis functions have local support, $K_{ij} = 0$ whenever the supports of φ_i and φ_j do not overlap. Therefore, the stiffness matrix is sparse.

Specifying model information for the FEM solver

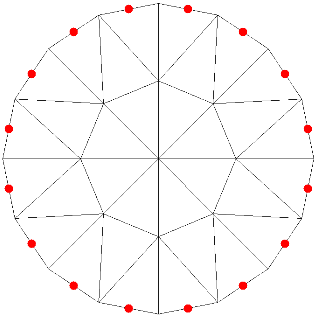
Conductivity



Dirichlet data



Neumann data



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Principle of the finite element method

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Recall the definition of the Neumann-to-Dirichlet map

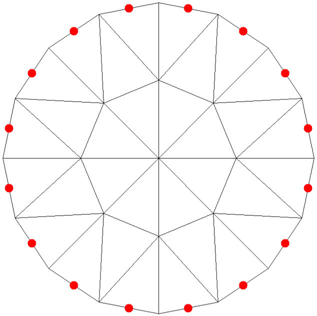
Consider the Neumann-to-Dirichlet map $\mathcal{R}_\sigma : \widetilde{H}^{-1/2}(\partial\Omega) \to \widetilde{H}^{1/2}(\partial\Omega)$, where $\widetilde{H}^s$ spaces consist of H^s functions with mean value zero. We wish to compute the matrix approximation R_σ to $\mathcal{R}_\sigma$. In the Neumann problem

$$\nabla \cdot \sigma \nabla u_n = 0 \text{ in } \Omega, \qquad \sigma \frac{\partial u_n}{\partial \nu}|_{\partial\Omega} = \varphi_n,$$

the solution u_n is only determined up to an additive constant. We make the solution unique by the requirement $\int_{\partial\Omega} u_n \, ds = 0$.

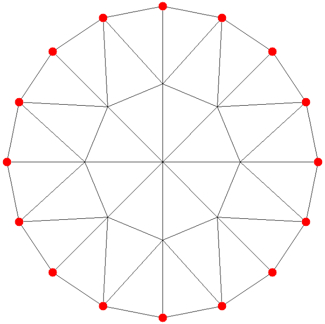
Note that the Neumann problem is well-defined only for $n \neq 0$ because the Neumann data must satisfy $\int_{\partial\Omega} \sigma \frac{\partial u_n}{\partial \nu} \, ds = 0$ due to conservation of charge. Consequently, we define R_σ as a $2N \times 2N$ matrix and use the basis $\{\varphi_{-N}, \ldots, \varphi_{-1}, \varphi_1, \ldots, \varphi_N\}$ that does not include the constant function $\varphi_0(\theta) = (2\pi)^{-1/2}$.

Solving the Neumann problem using FEM



The first step is to solve the Neumann problem for $n = -N, -N+1, \ldots, -1, 1, \ldots, N$ using fem. Neumann data needs to be specified at the centers of edge segments.

Picking out the Dirichlet data



One can then evaluate the trace $u_n|_{\partial\Omega}$ by picking out the values of the solution u_n at the boundary nodes. Note that this step is stable, as opposed to the numerical differentiation needed if you would compute the DN matrix instead of the ND matrix.

Forming the ND matrix element by element

Set $\mathrm{R}_\sigma = [(\mathrm{R}_\sigma)_{m,n}]$ with

$$(\mathrm{R}_\sigma)_{m,n} = \langle u_n|_{\partial\Omega}, \varphi_m \rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u_n|_{\partial\Omega}(\theta) e^{-im\theta} d\theta.$$

Here m is row index and n is column index. The integration can be implemented with a simple quadrature using the lengths of boundary segments as weights.

Outline

Recall the definition of the Dirichlet-to-Neumann map

We start with the conductivity equation with Dirichlet boundary condition:

$$\begin{cases} \nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= f. \end{cases}$$

The domain Ω will be the unit disc throughout the discussion.

Infinite-precision continuum-model boundary measurements of EIT can be modelled by the Dirichlet-to-Neumann map

$$\Lambda_\sigma : f \mapsto \sigma \frac{\partial u}{\partial \vec{n}}|_{\partial\Omega}.$$

Is there a way to write Λ_σ down analytically for some examples so that we can implement theoretical ideas in a computational way? Yes, there is!

Are DN and ND maps inverses of each other?

The electric potential $u \in H^1(\Omega)$ is the same in both Dirichlet and Neumann problems, so the DN and ND maps are inverses of each other in some sense. But how, exactly?

The ND map $\mathcal{R}_\sigma : \widetilde{H}^{-1/2}(\partial\Omega) \to \widetilde{H}^{1/2}(\partial\Omega)$ is bounded, where $\widetilde{H}^s$ spaces consist of H^s functions with mean value zero. Also, $\Lambda_\sigma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$.

Define a projection operator $P\phi := |\partial\Omega|^{-1} \int_{\partial\Omega} \phi$. Then for any $f \in H^{1/2}(\partial\Omega)$ we have

$$P\Lambda_\sigma f = |\partial\Omega|^{-1} \int_{\partial\Omega} \sigma \frac{\partial u}{\partial \nu}\, ds = |\partial\Omega|^{-1} \int_{\Omega} \nabla \cdot \sigma \nabla u \, dz = 0,$$

so actually $\Lambda_\sigma : H^{1/2}(\partial\Omega) \to \widetilde{H}^{-1/2}(\partial\Omega)$. We now have

$$\begin{aligned} \Lambda_\sigma \mathcal{R}_\sigma &= I & &: \widetilde{H}^{-1/2}(\partial\Omega) \to \widetilde{H}^{-1/2}(\partial\Omega), \\ \mathcal{R}_\sigma \Lambda_\sigma &= I - P & &: H^{1/2}(\partial\Omega) \to \widetilde{H}^{1/2}(\partial\Omega). \end{aligned}$$

Computing the DN matrix using the ND matrix

Define $\mathrm{L}'_\sigma := \mathrm{R}_\sigma^{-1}$. Then L'_σ is a matrix of size $2N \times 2N$.

However, the DN matrix L_σ acts on the $\{\varphi_n\}$ basis with $-N \leq n \leq N$, so it needs to have size $(2N+1) \times (2N+1)$ and map constant functions appropriately.

the DN operator satisfies

$$\begin{aligned} \Lambda_\sigma 1 &= 0, \\ \int_{\partial\Omega} \Lambda_\sigma f \, ds &= 0 \qquad \text{for all } f \in H^{1/2}(\partial\Omega). \end{aligned}$$

So, we can enforce the above conditions in L_σ by adding one zero row and one zero column in the middle of the matrix L'_σ

Adding the zero row and column to L'_σ

Divide the matrix L'_σ into four $N \times N$ blocks named like this:

$$\mathsf{L}'_\sigma = \left[\begin{array}{c|c} \mathsf{L}'_\sigma\{1,1\} & \mathsf{L}'_\sigma\{1,2\} \\ \hline \mathsf{L}'_\sigma\{2,1\} & \mathsf{L}'_\sigma\{2,2\} \end{array}\right].$$

Now construct L_σ by adding a zero row and a zero column:

$$\mathsf{L}_\sigma = \begin{bmatrix} \mathsf{L}'_\sigma\{1,1\} & \underset{N\times 1}{0} & \mathsf{L}'_\sigma\{1,2\} \\ \underset{1\times N}{0} & \underset{1\times 1}{0} & \underset{1\times N}{0} \\ \mathsf{L}'_\sigma\{2,1\} & \underset{N\times 1}{0} & \mathsf{L}'_\sigma\{2,2\} \end{bmatrix}.$$