Mathematics of Electrical Impedance Tomography
Lecture 5: EIT on the disc

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Recall the definition of the Dirichlet-to-Neumann map

We start with the conductivity equation with Dirichlet boundary condition:

$$\begin{cases} \nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= f. \end{cases}$$

The domain Ω will be the unit disc throughout the discussion.

Infinite-precision continuum-model boundary measurements of EIT can be modelled by the Dirichlet-to-Neumann map

$$\Lambda_\sigma : f \mapsto \sigma \frac{\partial u}{\partial \vec{n}}|_{\partial\Omega}.$$

Is there a way to write Λ_σ down analytically for some examples so that we can implement theoretical ideas in a computational way? Yes, there is!

We use Fourier series to represent functions and operators defined on the unit circle $\partial\Omega$

Define Fourier basis functions $\varphi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}$ for all integers $n \in \mathbb{Z}$.

We can represent functions $f : \partial\Omega \to \mathbb{C}$ approximately like this:

$$f(\theta) \approx \sum_{n=-N}^{N} \langle f, \varphi_n \rangle \, \varphi_n(\theta), \qquad \langle f, \varphi_n \rangle := \int_0^{2\pi} f(\theta) \, \overline{\varphi_n(\theta)} \, d\theta.$$

Here the unit circle $\partial\Omega$ is parametrized by the angle θ running from 0 to 2π.
Furthermore, we can represent a linear operator $\mathcal{A}$ acting on the function f as a (possibly infinite) matrix A acting on the Fourier coefficients $\langle f, \varphi_n \rangle$:

$$A_{mn} := \int_0^{2\pi} (\mathcal{A}\varphi_n)(\theta) \, \overline{\varphi_m(\theta)} \, d\theta, \qquad m, n \in \mathbb{Z}.$$

Outline

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The important one-jump conductivity

The powerful concept of monotonicity

The operator Λ_1 is known analytically when Ω is the unit disc

$$L_1 = \begin{bmatrix} \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ & 3 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 2 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 3 & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots \end{bmatrix}$$

Next we show how to calculate this matrix.

Column zero in the Fourier matrix for Λ_1

Take the zeroth Fourier basis function $\varphi_0(\theta) = \frac{1}{\sqrt{2\pi}} e^{i0\theta} = \frac{1}{\sqrt{2\pi}}$ as Dirichlet data:

$$\begin{cases} \nabla \cdot \nabla u &= 0 \text{ in } \Omega, \qquad \text{(remember that } \sigma = 1\text{)} \\ u|_{\partial\Omega} &= \dfrac{1}{\sqrt{2\pi}}. \end{cases}$$

Now what is the unique harmonic function defined inside the unit disc having boundary value $(2\pi)^{-1/2}$? It is simply the constant function $u(z) \equiv (2\pi)^{-1/2}$ for all $z \in \Omega$.

By definition of the DN map and due to derivatives of constants being zero we get

$$\Lambda_1(\varphi_0) = \frac{\partial u}{\partial \vec{n}}|_{\partial\Omega} = 0.$$

So we get one column to the infinite matrix L_1 of the DN map Λ_1:

$$(L_1)_{m0} := \int_0^{2\pi} (\Lambda_1 \varphi_0)(\theta)\, \overline{\varphi_m(\theta)}\, d\theta = 0 \qquad \text{for all } m \in \mathbb{Z}.$$

Yay! We have constructed a part of the infinite L_1 matrix

$$L_1 = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ \color{red}{0} \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

Row zero in the Fourier matrix for Λ_1

Remember the definition of the DN map:

$$\begin{cases} \Delta u &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= f, \end{cases} \qquad \Lambda_\sigma(f) = \frac{\partial u}{\partial \vec{n}}|_{\partial\Omega}.$$

Now by integration by parts we have

$$0 = \int_\Omega \Delta u \, dx_1 dx_2 = \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}} \, dS = \int_{\partial\Omega} \Lambda_\sigma(f) \, dS.$$

So for any arbitrary Dirichlet data f we have that the integral of $\Lambda_\sigma(f)$ over the unit circle is zero. This is again conservation of charge appearing.

This also implies that the zeroth Fourier coefficient $\langle \Lambda_\sigma(f), \varphi_0 \rangle$ must be zero.

Low and behold! We have some more of L_1 figured out!

$$L_1 = \begin{bmatrix} & & & & \vdots & & & & \\ & & & & 0 & & & & \\ & & & & 0 & & & & \\ & & & & 0 & & & & \\ \cdots & 0 & 0 & 0 & \color{red}{0} & 0 & 0 & 0 & \cdots \\ & & & & 0 & & & & \\ & & & & 0 & & & & \\ & & & & 0 & & & & \\ & & & & \vdots & & & & \end{bmatrix}$$

Homework: harmonic function z^n

Recall the Laplace operator for functions $f(x, x_2)$ in dimension two:

$$\nabla \cdot \nabla f = \Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}.$$

Functions that satisfy $\Delta f = 0$ are called *harmonic*. Consider the polar coordinates $x_1 = r\cos\theta, x_2 = r\sin\theta$, leading to

$$r = \sqrt{x_1^2 + x_2^2}, \qquad \theta = \arctan(\frac{x_2}{x}).$$

Show that the Laplacian in polar coordinates is given by

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial f^2}{\partial \theta^2}.$$

Prove that z^n is harmonic for all $z \in \mathbb{C}$, where $z = (x_1 + ix_2) = re^{i\theta}$ and $n > 0$. For which $z \in \mathbb{C}$ is z^n harmonic when $n < 0$?

The right half of the Fourier matrix for Λ_1

Consider the nth Fourier basis function $\varphi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}$ with $n > 0$ as Dirichlet data:

$$\begin{cases} \Delta u &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= \dfrac{1}{\sqrt{2\pi}} e^{in\theta}. \end{cases}$$

Now the unique solution is $u(z) = (2\pi)^{-1/2} z^n = (2\pi)^{-1/2} r^n e^{in\theta}$, and we can calculate

$$\Lambda_1(\varphi_n) = \frac{\partial u}{\partial \vec{n}}|_{\partial\Omega} = \frac{\partial u}{\partial r}|_{r=1} = (2\pi)^{-1/2} n r^{n-1} e^{in\theta}|_{r=1} = n \frac{e^{in\theta}}{\sqrt{2\pi}} = n\varphi_n.$$

Orthonormality of the Fourier basis gives us now the half $n > 0$ of the L_1 matrix:

$$(L_1)_{mn} := \int_0^{2\pi} (\Lambda_1 \varphi_n)(\theta) \overline{\varphi_m(\theta)} \, d\theta = \frac{n}{2\pi} \int_0^{2\pi} e^{i(n-m)\theta} d\theta \begin{cases} n & \text{when } m = n, \\ 0 & \text{when } m \neq n. \end{cases}$$

Super! We know more than half of L_1 now

$$L_1 = \begin{bmatrix} & & & & \vdots & \vdots & \vdots & \vdots & \\ & & & & 0 & 0 & 0 & 0 & \cdots \\ & & & & 0 & 0 & 0 & 0 & \cdots \\ & & & & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & & & & 0 & 1 & 0 & 0 & \cdots \\ & & & & 0 & 0 & 2 & 0 & \cdots \\ & & & & 0 & 0 & 0 & 3 & \\ & & & & \vdots & \vdots & \vdots & & \ddots \end{bmatrix}$$

Finding the missing parts will be left as an exercise.

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Notations for planar rotations

We denote planar points in several equivalent ways, depending on the context:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x = z = x_1 + i x_2.$$

We consider rotating the planar point by angle α, using equivalent notations:

$$z \mapsto e^{i\alpha} z, \qquad x \mapsto R_\alpha x = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Note that the rotation matrix is orthonormal: $R_\alpha^T R_\alpha = I = R_\alpha R_\alpha^T$.

Outline

Rotating solutions u when conductivity is symmetric

Assume $\sigma(x) = \sigma(|x|)$. Let u be the solution of

$$\begin{cases} \nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= f. \end{cases}$$

Define the rotated function

$$u_\alpha(x) = u(R_\alpha x).$$

We verify explicitly that u_α also satisfies the conductivity equation and check the appropriate Dirichlet condition for it.

Rotating solution u when conductivity is symmetric: gradient

Introduce the new variable $y = R_\alpha x$. By the chain rule we have

$$\frac{\partial u_\alpha}{\partial x_i}(x) = \sum_{j=1}^{2} \frac{\partial u}{\partial y_j}(y) \frac{\partial y_j}{\partial x_i}.$$

Since

$$y_j = \sum_{k=1}^{2} [R_\alpha]_{jk} \, x_k,$$

we have $\frac{\partial y_j}{\partial x_i} = [R_\alpha]_{ji}$. Consequently,

$$\frac{\partial u_\alpha}{\partial x_i}(x) = \sum_{j=1}^{2} [R_\alpha]_{ji} \frac{\partial u}{\partial y_j}(R_\alpha x).$$

In vector notation we have

$$\nabla_x u_\alpha(x) = R_\alpha^T \nabla_y u(R_\alpha x).$$

Preparation for applying the divergence

It follows from $y = R_\alpha x$ and $\sigma(x) = \sigma(R_\alpha x)$ that

$$\sigma(x)\nabla_x u_\alpha(x) = \sigma(R_\alpha x) R_\alpha^T \nabla_y u(R_\alpha x) = R_\alpha^T \left[\sigma(y)\nabla_y u(y)\right].$$

Set $F(y) = \sigma(y)\nabla_y u(y)$. Then

$$\sigma(x)\nabla_x u_\alpha(x) = R_\alpha^T F(R_\alpha x).$$

We next calculate the divergence of this vector field. Its ith component is

$$\left(R_\alpha^T F(R_\alpha x)\right)_i = \sum_{j=1}^{2} [R_\alpha]_{ji} F_j(R_\alpha x).$$

Hence

$$\nabla_x \cdot \left(R_\alpha^T F(R_\alpha x)\right) = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{2} [R_\alpha]_{ji} F_j(R_\alpha x) \right) = \sum_{i,j=1}^{2} [R_\alpha]_{ji} \frac{\partial}{\partial x_i} F_j(R_\alpha x).$$

Divergence calculation

Now we get

$$\nabla_x \cdot \left(R_\alpha^T F(R_\alpha x)\right) = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{2} [R_\alpha]_{ji} F_j(R_\alpha x) \right) = \sum_{i,j=1}^{2} [R_\alpha]_{ji} \frac{\partial}{\partial x_i} F_j(R_\alpha x).$$

Applying the chain rule once more gives

$$\frac{\partial}{\partial x_i} F_j(R_\alpha x) = \sum_{k=1}^{2} [R_\alpha]_{ki} \frac{\partial F_j}{\partial y_k}(R_\alpha x).$$

Therefore,

$$\nabla_x \cdot \left(R_\alpha^T F(R_\alpha x)\right) = \sum_{i,j,k=1}^{2} [R_\alpha]_{ji} [R_\alpha]_{ki} \frac{\partial F_j}{\partial y_k}(R_\alpha x) = \sum_{j,k=1}^{2} \left(\sum_{i=1}^{2} [R_\alpha]_{ji} [R_\alpha]_{ki} \right) \frac{\partial F_j}{\partial y_k}(R_\alpha x).$$

Conductivity equation for the rotated potential

Since $R_\alpha R_\alpha^T = I$, we get $\sum_{i=1}^2 [R_\alpha]_{ji}[R_\alpha]_{ki} = \delta_{jk}$. It follows that

$$\nabla_x \cdot \big(R_\alpha^T F(R_\alpha x)\big) = \sum_{j,k=1}^{2} \delta_{jk} \frac{\partial F_j}{\partial y_k}(R_\alpha x) = \sum_{j=1}^{2} \frac{\partial F_j}{\partial y_j}(R_\alpha x) = (\nabla_y \cdot F)(R_\alpha x).$$

Recalling that $F = \sigma \nabla u$, we have proved the identity

$$\nabla_x \cdot \big(\sigma(x)\nabla_x u_\alpha(x)\big) = \big[\nabla_y \cdot \big(\sigma(y)\nabla_y u(y)\big)\big] = 0.$$

Consequently,

$$\nabla \cdot (\sigma \nabla u_\alpha) = 0 \qquad \text{in } \Omega.$$

Outline

Boundary condition for the rotated potential

Denote $(T_\alpha f)(x) = f(R_\alpha x)$. Then $u_\alpha|_{\partial\Omega} = T_\alpha f$. Since Ω is the unit disc, its outward unit normal is $\nu(x) = x$. Therefore,

$$\nu(R_\alpha x) = R_\alpha \nu(x).$$

Using the transformation formula for the gradient, we obtain

$$\begin{aligned}
\frac{\partial u_\alpha}{\partial \nu}(x) &= \nabla u_\alpha(x) \cdot \nu(x) \\
&= R_\alpha^T \nabla u(R_\alpha x) \cdot \nu(x) \\
&= \nabla u(R_\alpha x) \cdot R_\alpha \nu(x) \\
&= \nabla u(R_\alpha x) \cdot \nu(R_\alpha x) \\
&= \frac{\partial u}{\partial \nu}(R_\alpha x).
\end{aligned}$$

DN map commutes with rotations and angular differentiation

Since $\sigma(x) = \sigma(R_\alpha x)$, it follows that

$$\sigma(x)\frac{\partial u_\alpha}{\partial \nu}(x) = \left(\sigma \frac{\partial u}{\partial \nu}\right)(R_\alpha x).$$

Thus, rotating the boundary voltage rotates the corresponding boundary current in exactly the same way. By uniqueness, u_α is the solution corresponding to the rotated boundary voltage $T_\alpha f$.

We get the commutation relation

$$\Lambda_\sigma T_\alpha = T_\alpha \Lambda_\sigma.$$

The angular derivative argument

Compute the derivative

$$\frac{d}{d\alpha} T_\alpha f \bigg|_{\alpha=0} = \lim_{\alpha \to 0} \frac{T_\alpha f(x) - f(x)}{\alpha} = \lim_{\alpha \to 0} \frac{f(\theta + \alpha) - f(\theta)}{\alpha} = \frac{\partial f}{\partial \theta}.$$

We can use $\Lambda_\sigma T_\alpha = T_\alpha \Lambda_\sigma$ to obtain

$$\Lambda_\sigma \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta} \Lambda_\sigma.$$

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Fourier basis functions are eigenfunctions of the DN map

Differentiating the basis function gives $\frac{\partial \varphi_n}{\partial \theta} = in\,\varphi_n$. The commutation relation yields

$$\frac{\partial}{\partial \theta}\big(\Lambda_\sigma \varphi_n\big) = \Lambda_\sigma \frac{\partial \varphi_n}{\partial \theta} = in\,\Lambda_\sigma \varphi_n.$$

By basic ordinary differential equation theory we can conclude that

$$\Lambda_\sigma e^{in\theta} = \lambda_n e^{in\theta}$$

for some complex number $\lambda_n \in \mathbb{C}$. Using $\varphi_n(\theta) = (2\pi)^{-1/2} e^{in\theta}$ we can also write

$$\Lambda_\sigma \varphi_n = \lambda_n \varphi_n.$$

Eigenvalues are real, nonnegative and symmetric: $\lambda_n = \lambda_{-n}$

Take $n \neq 0$ and let u be the solution of

$$\begin{cases} \nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= \varphi_n. \end{cases}$$

Since the conductivity is real-valued, the eigenvalues of Λ_σ satisfy

$$\lambda_n = \lambda_n \|\varphi_n\|^2_{L^2(\partial\Omega)} = \langle \Lambda_\sigma \varphi_n, \varphi_n \rangle = \int_\Omega \sigma |\nabla u|^2 \geq 0.$$

In particular, the eigenvalue is real, so $\lambda_n = \overline{\lambda_n}$. Further, since $\overline{\varphi_n} = \varphi_{-n}$,

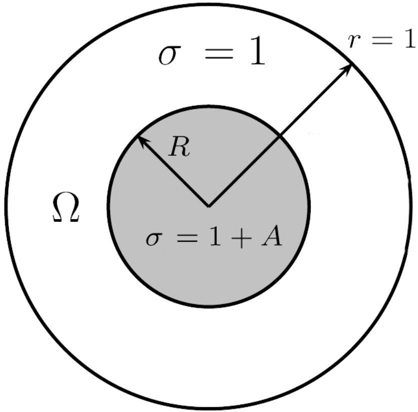
$$\Lambda_\sigma \varphi_{-n} = \Lambda_\sigma \overline{\varphi_n} = \overline{\Lambda_\sigma \varphi_n} = \overline{\lambda_n \varphi_n} = \overline{\lambda_n} \varphi_{-n},$$

implying the symmetry relation

$$\lambda_n = \overline{\lambda_n} = \lambda_{-n}.$$

Outline

Next we want to compute the matrix L_σ for a nontrivial conductivity distribution $\sigma \neq 1$



We choose a discontinuous conductivity with a jump along the circle with radius R and center at the origin. Here either $A > 0$ or $-1 < A < 0$.

Then we have rotational symmetry $\sigma(z) = \sigma(|z|)$, and Fourier basis functions will be eigenfunctions for the DN map:

$$\Lambda_\sigma \varphi_n = \lambda_n \varphi_n.$$

Consequently L_σ will be a diagonal matrix, and we only need to calculate the eigenvalues λ_n for all $n \in \mathbb{Z}$.

We again have that $\nabla \cdot \sigma \nabla u = 0$ for constant functions, so we get the central zero. To do: other diagonal elements

$$L_\sigma = \begin{bmatrix} \color{red}{\ddots} & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ & & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \color{red}{0} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \color{red}{\ddots} \end{bmatrix}$$

Calculation of eigenvalues λ_n for nonzero n

Our conductivity is

$$\sigma(r,\theta) = \begin{cases} 1+A, & 0 \le r \le R, \\ 1, & R < r \le 1, \end{cases}$$

and we are looking for the unique $H^1(\Omega)$ solution of the Dirichlet problem

$$\begin{cases} \nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= \varphi_n, \end{cases}$$

with some $n \neq 0$.

Since the conductivity jumps on the circle $r = R$, we need to carefully consider continuity conditions for u there.

Denote $u = u^{\text{int}}(r,\theta)$ for $r < R$ and $u = u^{\text{ext}}(r,\theta)$ for $R < r < 1$.

Now u^{ext} is harmonic in the annulus and

$$u^{\text{ext}}(1,\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}.$$

Therefore, we make the educated guess

$$u^{\text{ext}}(r,\theta) = a_n r^{|n|} e^{in\theta} + b_n r^{-|n|} e^{in\theta},$$

with constants a_n and b_n that satisfy

$$a_n + b_n = \frac{1}{\sqrt{2\pi}}.$$

Matching u^{int} and u^{ext} at $r = R$ in two ways

We need continuity of the voltage potential function u over $r = R$:

$$u^{\text{int}}(R, \theta) = u^{\text{ext}}(R, \theta).$$

Since u^{int} is harmonic in the disc $r < R$ and needs to match a factor $e^{in\theta}$ at $r = R$, we make the ansatz

$$u^{\text{int}}(r, \theta) = c_n r^{|n|} e^{in\theta}.$$

This implies

$$c_n R^{|n|} = a_n R^{|n|} + b_n R^{-|n|}.$$

Next we match the electric currents through the circle $r = R$:

$$(1 + A)\frac{\partial u^{\text{int}}}{\partial r}(R, \theta) = \frac{\partial u^{\text{ext}}}{\partial r}(R, \theta).$$

Recall the form of u^{ext}:

$$u^{\text{ext}}(r, \theta) = a_n r^{|n|} e^{in\theta} + b_n r^{-|n|} e^{in\theta}.$$

So, in terms of constants a_n, b_n and c_n, we arrive at

$$\begin{aligned}(1 + A)|n| c_n R^{|n|-1} = \\ |n| a_n R^{|n|-1} - |n| b_n R^{-|n|-1}.\end{aligned}$$

Solving for the constants a_n, b_n and c_n

We have derived three equations, which after a little manipulation look like this:

$$\begin{aligned} a_n + b_n &= (2\pi)^{-1/2}, \\ c_n &= a_n + b_n R^{-2|n|}, \\ (1+A)c_n &= a_n - b_n R^{-2|n|}. \end{aligned}$$

Let's eliminate c_n. Multiply the middle equation through by $1 + A$ and subtract it from the bottom equation to yield

$$0 = -Aa_n - (2+A)R^{-2|n|} b_n.$$

Inserting $a_n = (2\pi)^{-1/2} - b_n$ from the top equation gives us b_n:

$$\begin{aligned} b_n &= \frac{A(2\pi)^{-1/2}}{A - (2+A)R^{-2|n|}} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{1 - (1 + \frac{2}{A})R^{-2|n|}}. \end{aligned}$$

Now it is straightforward to solve for a_n and c_n by substitutions. But we do not need them here.

Determining the eigenvalues of Λ_σ

Calculating $\Lambda\varphi_n$ involves differentiating u^{ext} at the unit circle:

$$\begin{aligned}\Lambda\varphi_n &= \sigma\frac{\partial u}{\partial\vec{n}}|_{\partial\Omega} = \left.\frac{\partial u^{\text{ext}}}{\partial r}\right|_{r=1}\\ &= (a_n - b_n)|n|e^{in\theta}.\end{aligned}$$

By the equations we derived before we have $a_n = (2\pi)^{-1/2} - b_n$, so

$$a_n - b_n = (2\pi)^{-1/2} - 2b_n.$$

Also, recall that

$$b_n = \frac{1}{\sqrt{2\pi}}\frac{1}{1-(1+\frac{2}{A})R^{-2|n|}}.$$

We can conclude that $\Lambda_\sigma\varphi_n = \lambda_n\varphi_n$ with

$$\lambda_n = |n| - \frac{2|n|}{1-(1+\frac{2}{A})R^{-2|n|}}.$$

Since many theoretical EIT constructions use the difference map $\Lambda_\sigma - \Lambda_1$, it is worth noting that

$$\begin{aligned}(\Lambda_\sigma - \Lambda_1)\varphi_n &= (\lambda_n - |n|)\varphi_n\\ &= -\frac{2|n|}{1-(1+\frac{2}{A})R^{-2|n|}}\varphi_n.\end{aligned}$$

Outline

What is monotonicity in EIT?

Physically, the monotonicity result says that when the same voltage is imposed on the boundary, the more conductive medium permits a larger current to flow and therefore dissipates more power.

See [Harrach and Ullrich 2013] and its follow-up particles for great examples of using the monotonicity for EIT reconstructions.

Recall the quadratic form of the DN map

For a prescribed boundary voltage f, let u_σ be the unique solution of

$$\begin{cases} \nabla \cdot \sigma \nabla u_\sigma = 0 & \text{in } \Omega, \\ u_\sigma = f & \text{on } \partial\Omega. \end{cases}$$

The quadratic form of the Dirichlet-to-Neumann map is equal to the energy dissipated inside the disc:

$$\left\langle \Lambda_\sigma f, f \right\rangle = \int_\Omega \sigma |\nabla u_\sigma|^2 \, dx_1 dx_2.$$

Moreover, the solution u_σ minimizes this energy among all functions having the same boundary values:

$$\left\langle \Lambda_\sigma f, f \right\rangle = \min_{\substack{v \in H^1(\Omega) \\ v|_{\partial\Omega} = f}} \int_\Omega \sigma |\nabla v|^2 \, dx_1 dx_2.$$

Inequality for quadratic forms

Let $\sigma_1, \sigma_2 > 0$ be two rotationally symmetric conductivities. Assume

$$\sigma_1(z) \geq \sigma_2(z) \qquad \text{for all } z \in \Omega.$$

Since $\sigma_1 \geq \sigma_2$, every admissible function v satisfies

$$\int_\Omega \sigma_1 |\nabla v|^2 \, dx_1 dx_2 \geq \int_\Omega \sigma_2 |\nabla v|^2 \, dx_1 dx_2.$$

Taking the minimum over the same set of admissible functions on both sides gives

$$\langle \Lambda_{\sigma_1} f, f \rangle \geq \langle \Lambda_{\sigma_2} f, f \rangle.$$

Monotonicity of Dirichlet-to-Neumann maps

In the rotationally symmetric case $\varphi_n(\theta)$ are eigenfunctions of both Dirichlet-to-Neumann maps. Write

$$\Lambda_{\sigma_j}\varphi_n = \lambda_n^{(j)}\varphi_n, \qquad j = 1, 2.$$

Since $\|\varphi_n\|_{L^2(\partial\Omega)} = 1$, applying the quadratic-form inequality to $f = \varphi_n$ gives

$$\lambda_n^{(1)} = \langle \Lambda_{\sigma_1}\varphi_n, \varphi_n \rangle \geq \langle \Lambda_{\sigma_2}\varphi_n, \varphi_n \rangle = \lambda_n^{(2)}.$$

Consequently,

$$\sigma_1 \geq \sigma_2 \quad \Longrightarrow \quad \lambda_n^{(1)} \geq \lambda_n^{(2)} \quad \text{for every } n \in \mathbb{Z}.$$