Mathematics of Electrical Impedance Tomography
Lecture 4: Ikehata's enclosure method

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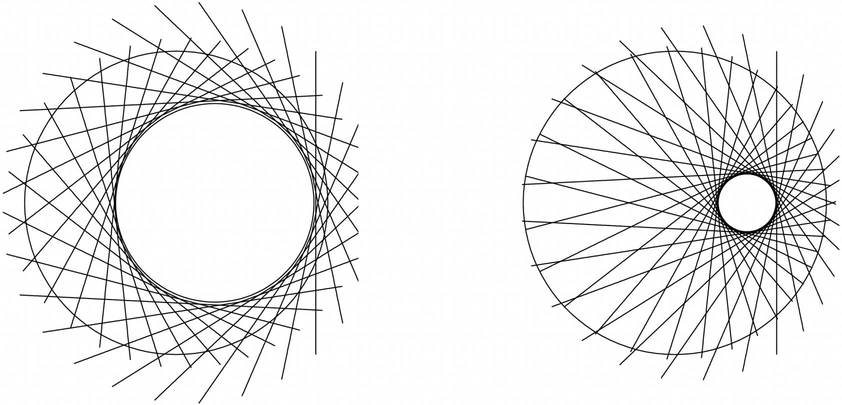


Figure 5. Left: the reconstruction of the jump of $\tilde{\gamma}_1$ of section 4.2 for 31 equispaced directions ω. Each line determines a half-space containing the inclusion D; this way the lines enclose the convex hull of D. Right: the same reconstruction for $\tilde{\gamma}_2$.

The paper trail of that visit to Japan in 1999

Ikehata and S 2000
Ikehata and Siltanen S 2004
Ikehata, Niemi and S 2012
S and Ide 2020
Sippola, Rautio, Hauptmann, Ide and S 2025
Rautio, S and Ide, submitted

The inclusion detection problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected C^∞ domain and

$$\sigma = 1 + \chi_D h.$$

Here $D \subset \Omega$ and $h \in L^\infty(D)$ is such that σ is bounded away from zero and has a jump along ∂D. We assume that the conductivity σ is strictly positive.

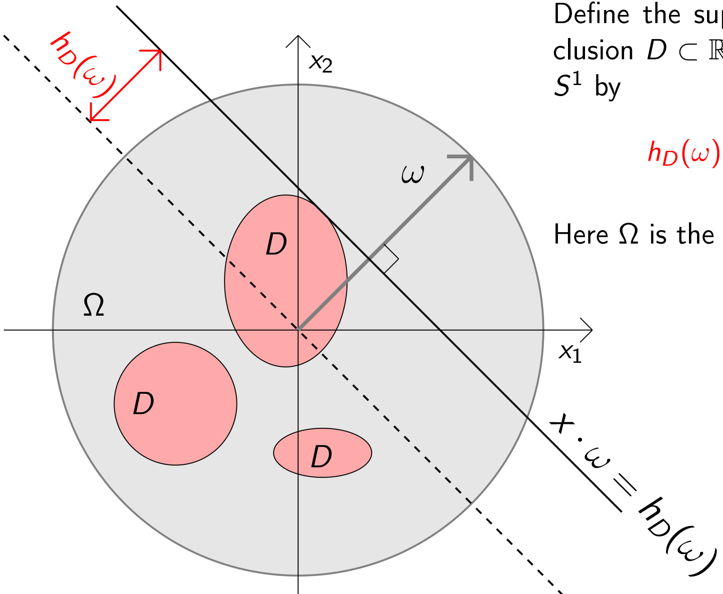
The goal of inclusion detection is to use EIT measurements, here Λ_σ, to extract information about D.

We discuss the enclosure method introduced in [Ikehata 2000]. Our main sources are [Ikehata-S 2000] and [S-Ide 2020]. However, see also [Brühl-Hanke 2000] and [Ikehata-Ohe 2002, 2007, 2008].

Some pointers to EIT inclusion detection literature

1997 Kang, Seo, Sheen
1998 Probe method **Ikehata**, Daido, Nakamura
1998 Factorization method **Kirsch**, Brühl, Hähner, Hakula, Hanke, Harrach, Hyvönen, Lechleiter, Schappel, Seo
1998 Point source method **Potthast**, Erhard
1998 Size estimation **Alessandrini**, Bilotta, Morassi, Rosset, Turco, Seo
1999 Linearization Isaacson, Mueller, Newell
2000 Enclosure method and its variants **Ikehata**, Brühl, Hanke, Ide, Isozaki, Nakata, Ohe, S, Uhlmann, many others
2004 Polarization tensors Ammari, Capdeboscq, Griesmaier, Hanke, Kang, Kwon, Seo, Vogelius, Woo, Yoon
2010 Monotonicity **Harrach**, Candiani, Dardé, Garde, Hyvönen, Mach, Seo, Staboulis, others

The support function $h_D(\omega)$



Define the support function of inclusion $D \subset \mathbb{R}^2$ with direction $\omega \in S^1$ by

$$h_D(\omega) := \sup_{x \in D} x \cdot \omega.$$

Here Ω is the unit disc.

The jump condition

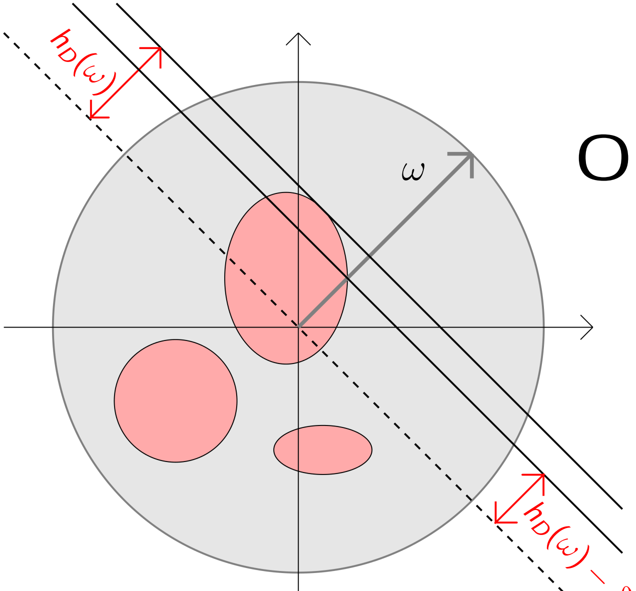
For $\omega \in S^1$ and $\delta > 0$, define

$$D_\omega(\delta) = \{x \in D \mid h_D(\omega) - \delta < x \cdot \omega \leq h_D(\omega)\}.$$

The conductivity $\sigma = 1 + \chi_D h$ satisfies the **jump condition** if for each $\omega \in S^1$, there exist positive constants C_ω and δ_ω such that

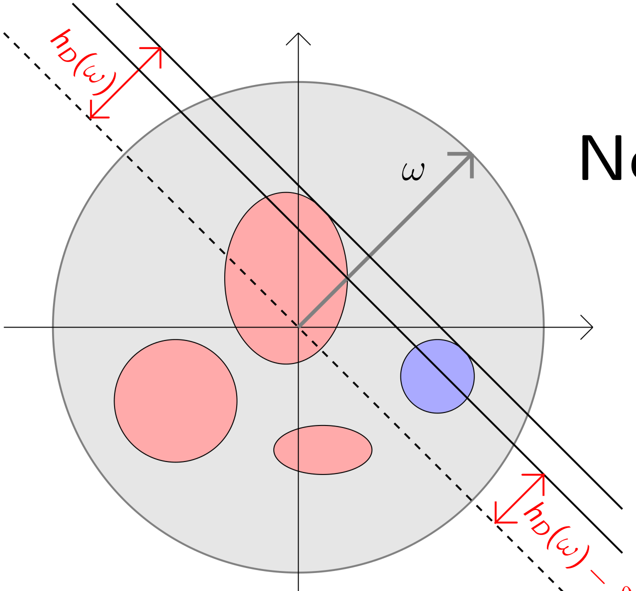
$$\begin{array}{rcll} h(x) & \geq & C_\omega & \text{for almost all } x \in D_\omega(\delta_\omega), \text{ or} \\ -h(x) & \geq & C_\omega & \text{for almost all } x \in D_\omega(\delta_\omega). \end{array}$$

Illustration of the jump condition



OK.

Illustration of the jump condition



Not OK!

Calderón's exponential

Let ω be a unit vector, and denote by $\omega^\perp$ the vector ω rotated counterclockwise by angle $\pi/2$. Then $\omega \cdot \omega^\perp = 0$.

For each $\tau > 0$ and $x \in \mathbb{R}^2$ set $f_\omega(x;\tau) := e^{\tau x\cdot\omega + i\tau x\cdot\omega^\perp}$.

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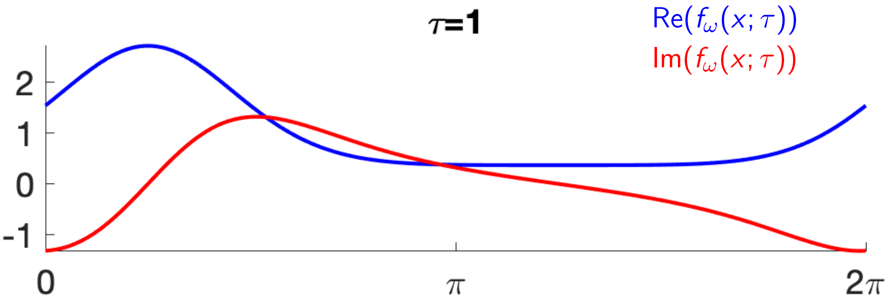
In the plot we have $\omega = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T$.



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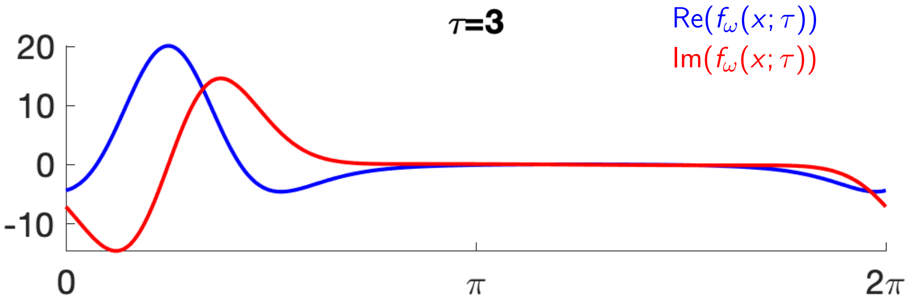

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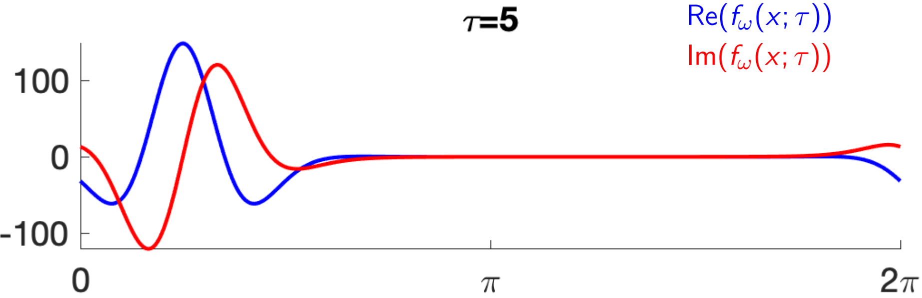

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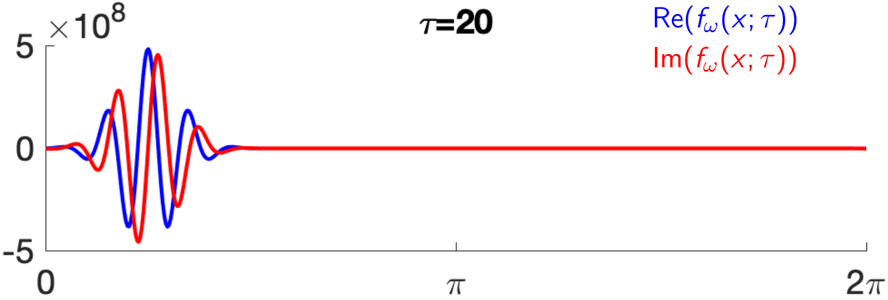

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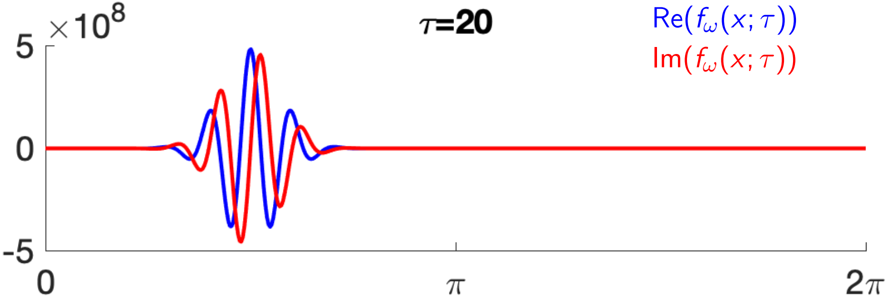
In the plot we have $\omega = [0,1]^T$.



Calderón's exponential and Ikehata's indicator function

Let ω be a unit vector, and denote by $\omega^\perp$ the vector ω rotated counterclockwise by angle $\pi/2$. Then $\omega \cdot \omega^\perp = 0$.

For each $\tau > 0$ and $x \in \mathbb{R}^2$ set

$$f_\omega(x;\tau) := e^{\tau x\cdot\omega + i\tau x\cdot\omega^\perp}.$$

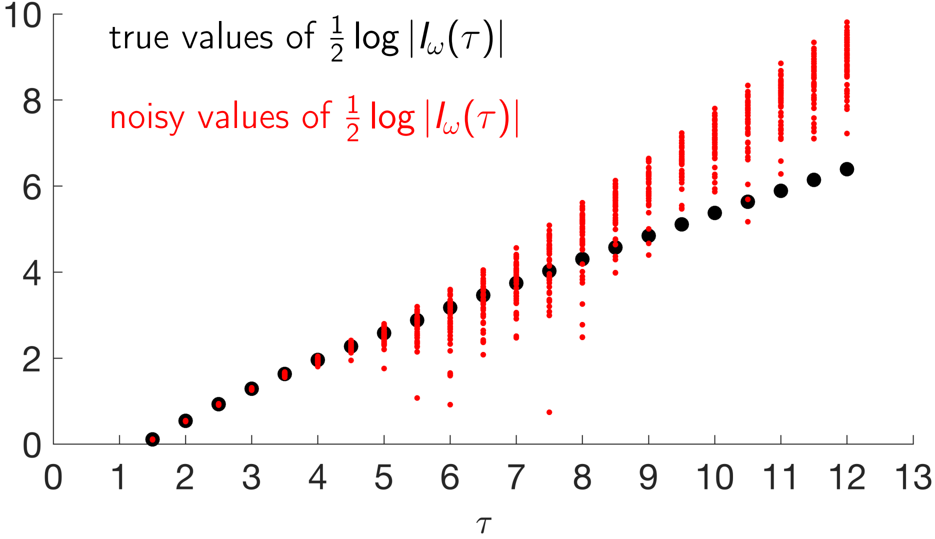
The indicator function $I_\omega(\tau)$ is defined by

$$I_\omega(\tau) = \langle(\Lambda_\sigma - \Lambda_1)\overline{f_\omega(\,\cdot\,;\tau)}, f_\omega(\,\cdot\,;\tau)\rangle.$$

The fundamental theorem of enclosure method [Ikehata 2000]:

$$h_D(\omega) = \lim_{\tau\longrightarrow\infty} \frac{\log|I_\omega(\tau)|}{2\tau}.$$

Measurement noise makes the evaluation of the indicator function unstable for large values of τ



Outline

Least-squares computation

Neural network improvement

[Ikehata and S 2000]

Inverse Problems **16** (2000) 1043–1052. Printed in the UK PII: S0266-5611(00)12515-8

Numerical method for finding the convex hull of an inclusion in conductivity from boundary measurements

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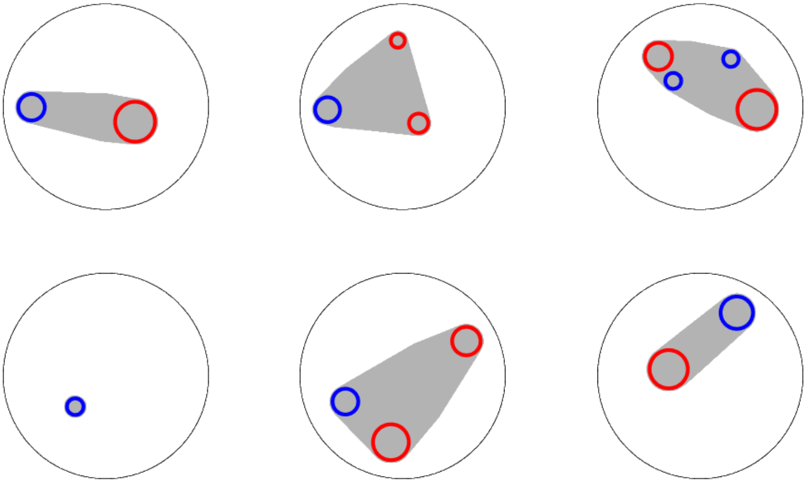
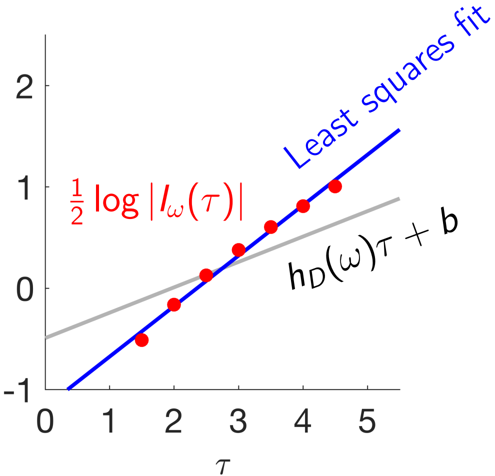
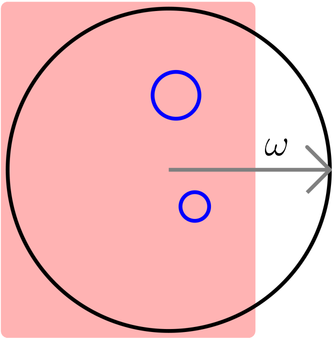
Abstract. We consider the 2D inverse conductivity problem for conductivities of the form $\gamma = 1 + \chi_D h$ defined in a bounded domain $\Omega \subset \mathbb{R}^2$ with C^∞ boundary $\partial\Omega$. Here $D \subset \Omega$ and $h \in L^\infty(D)$ are such that γ has a jump along ∂D. It was shown by Ikehata (Ikehata M *J. Inverse and Ill-Posed Problems* at press) that the Dirichlet–Neumann map determines the indicator function $I_\omega(\tau, t)$ that can be used to find the convex hull of D. In this paper we find numerically the indicator function for examples with constant h and recover the convex hull of D.

Here are a few examples of convex hulls calculated with 45 uniformly distributed vectors ω



We recover the support function approximately using least-squares fitting of a line to $\frac{1}{2}\log|I_\omega(\tau)|$





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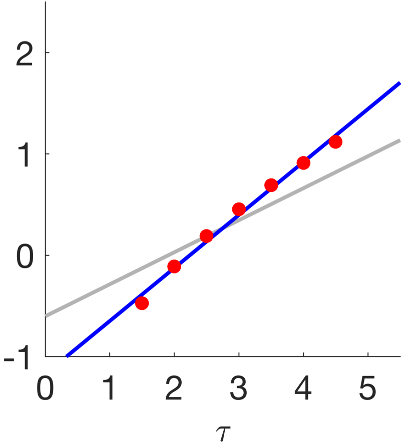

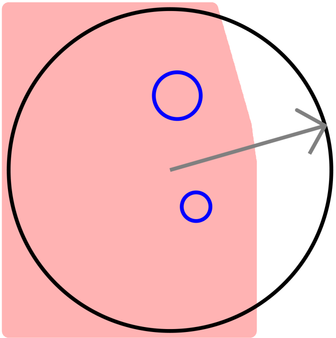

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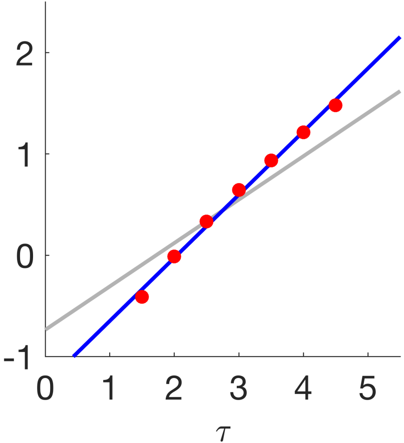

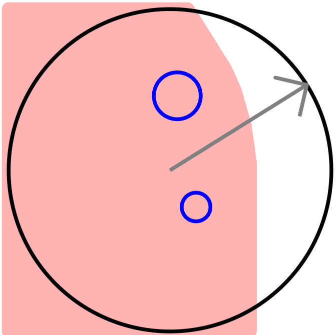

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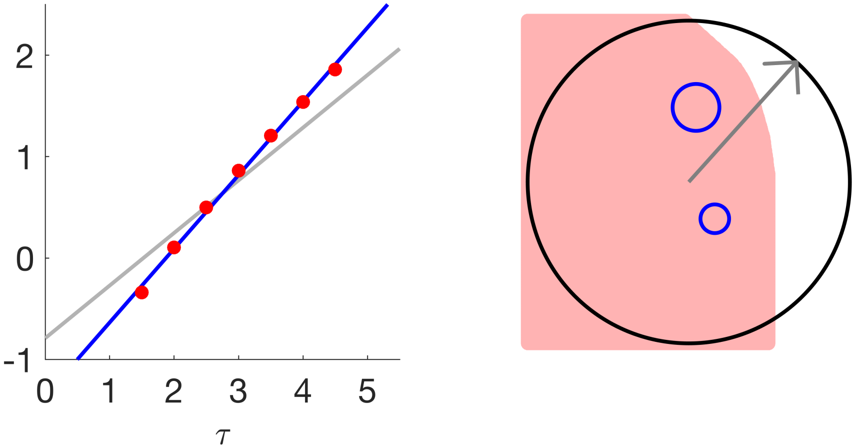

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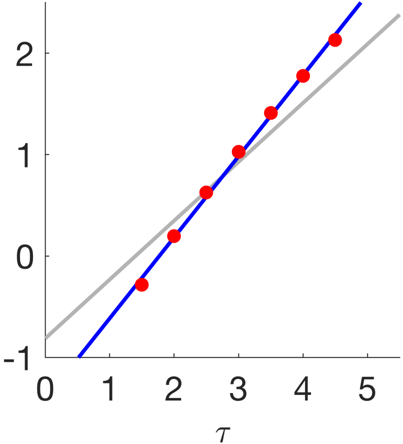

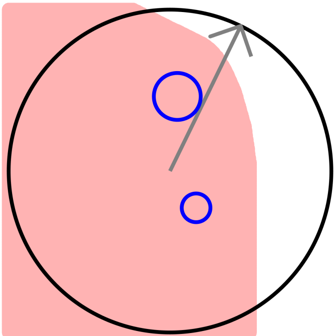

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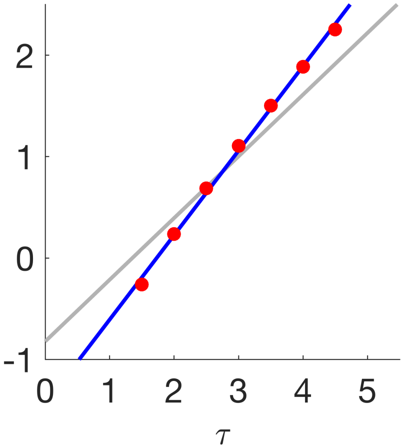

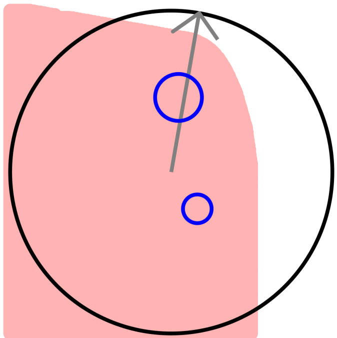

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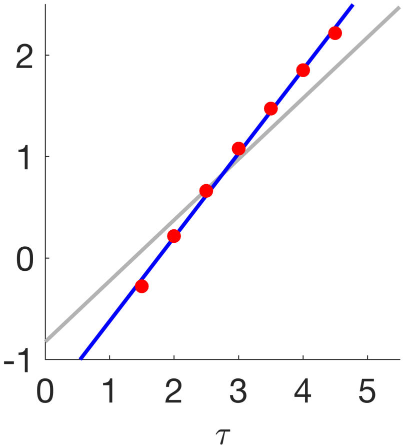

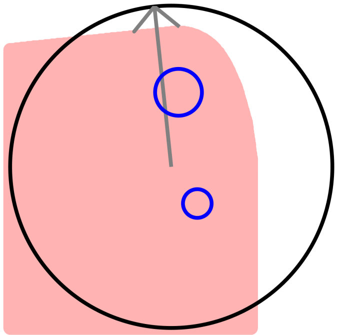

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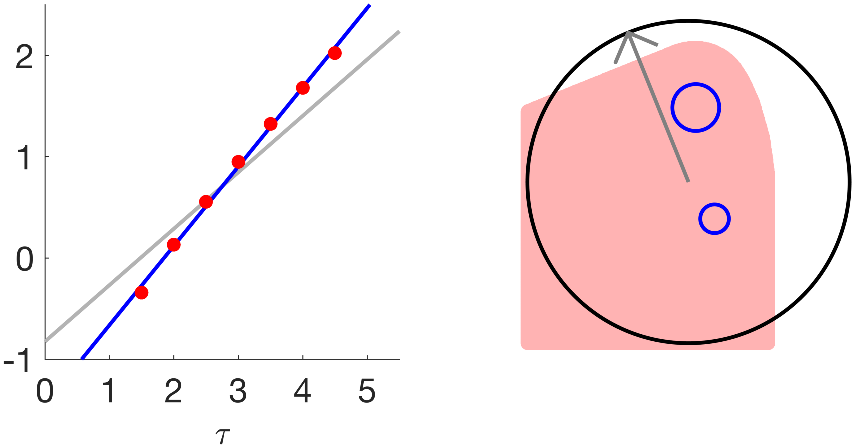

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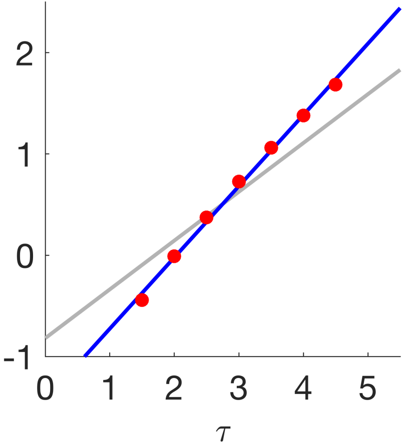

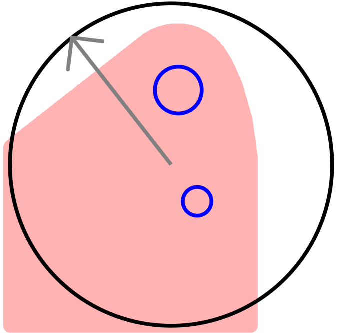

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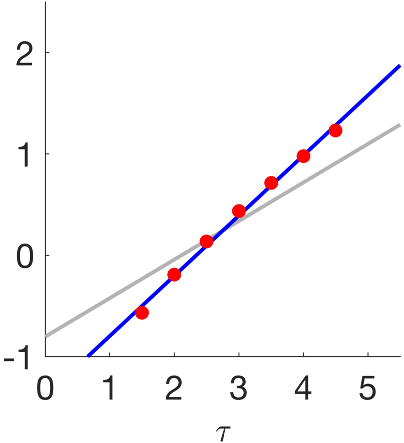

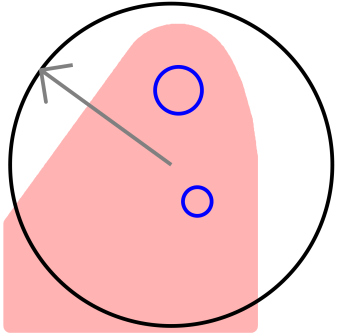

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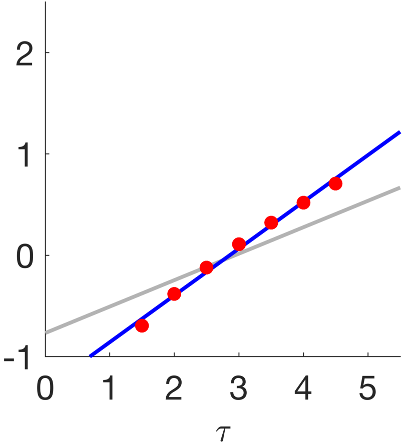

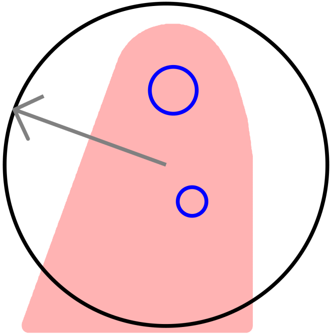

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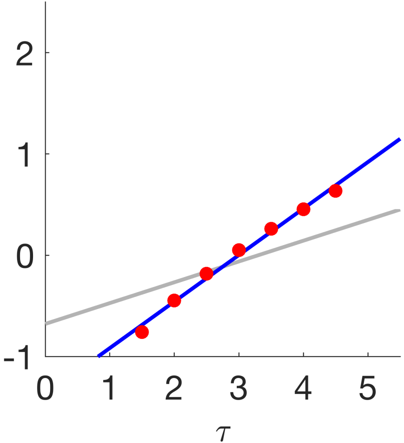

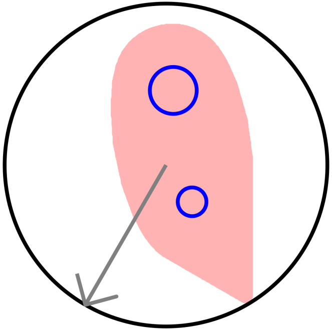

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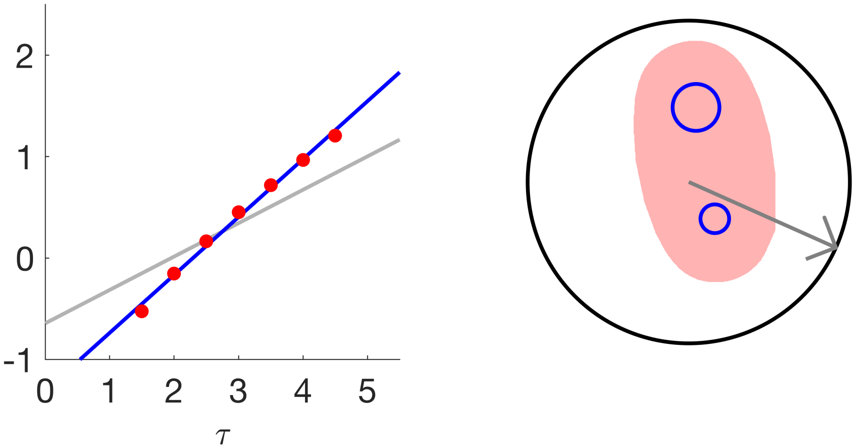

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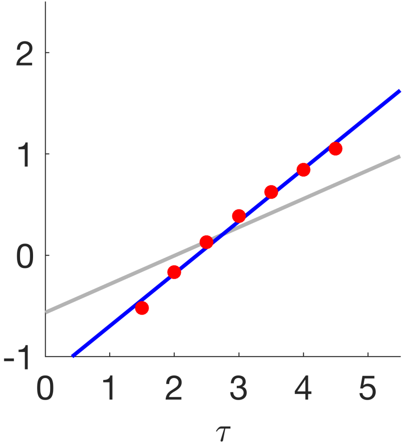
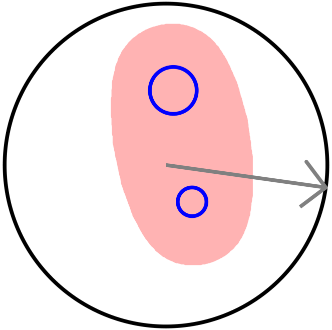
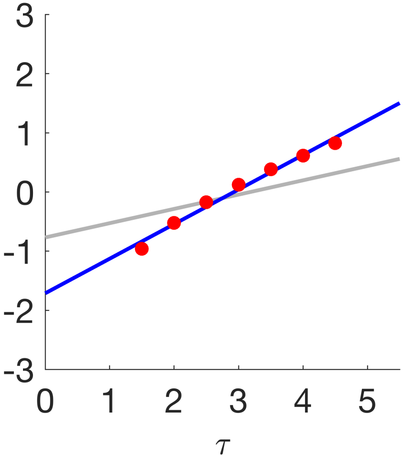
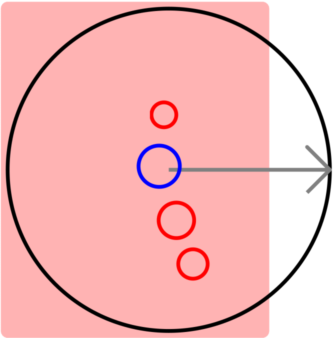




How about a more difficult case?

We recover the support function approximately using least-squares fitting of a line to $\frac{1}{2}\log|I_\omega(\tau)|$





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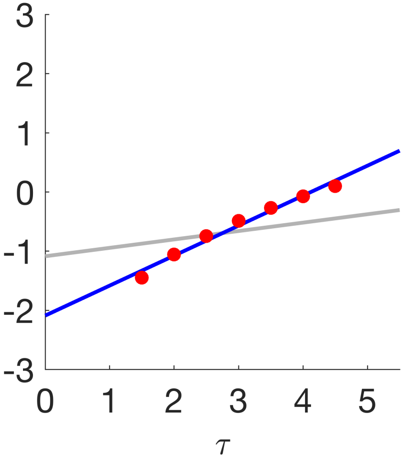

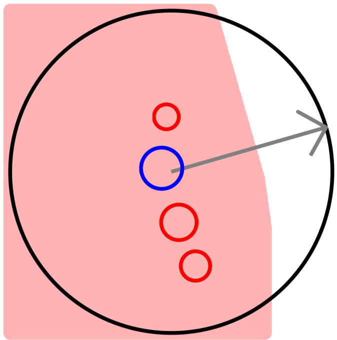

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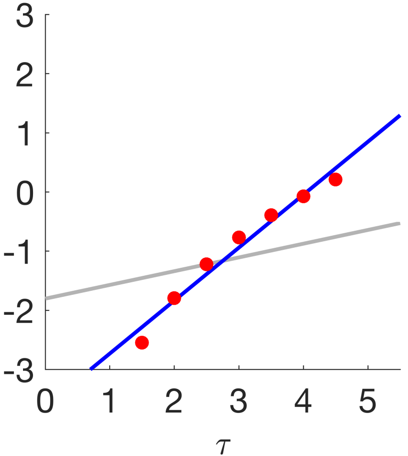

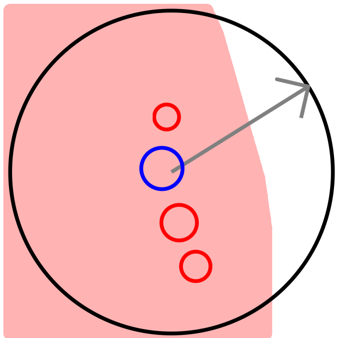

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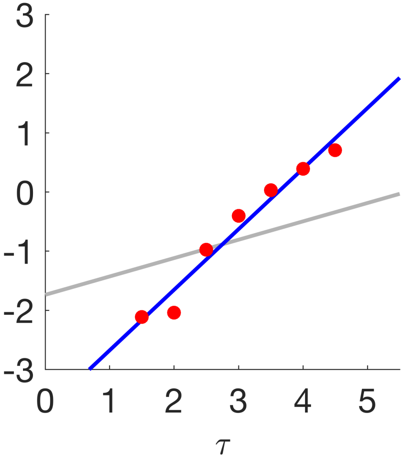

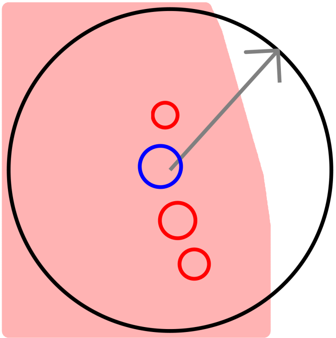

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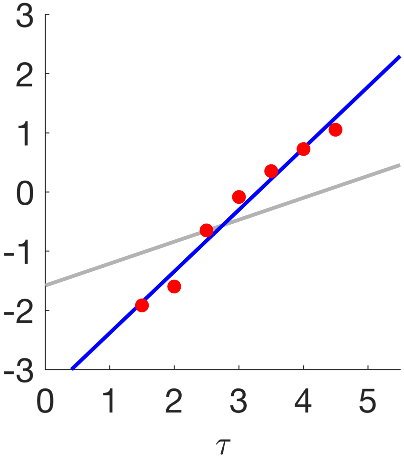

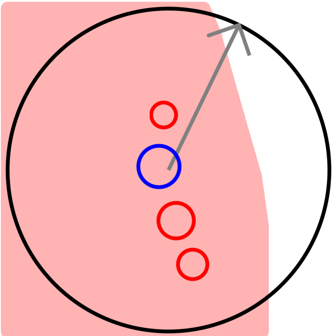

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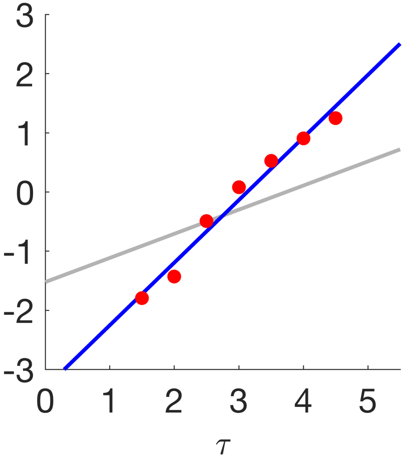

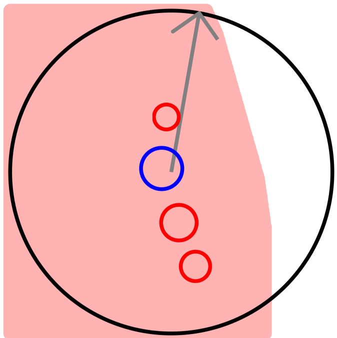

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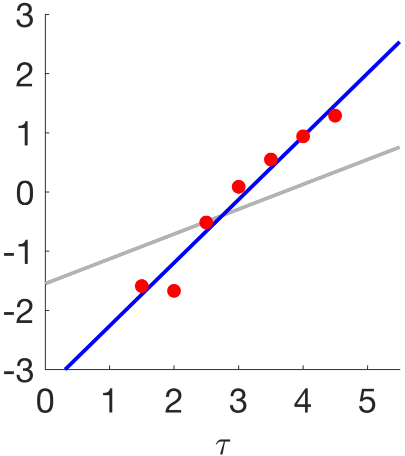

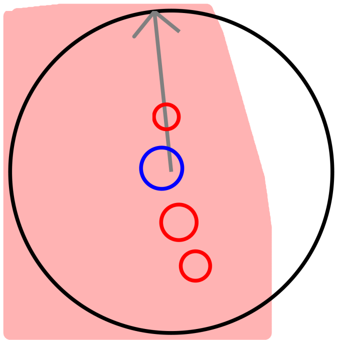

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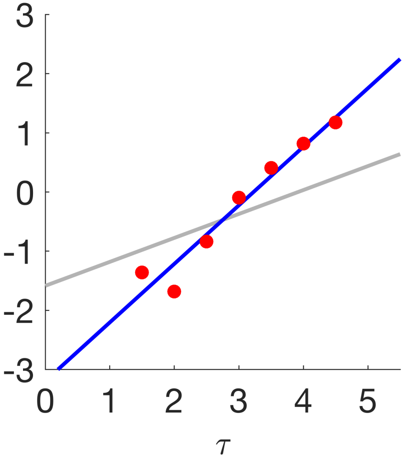

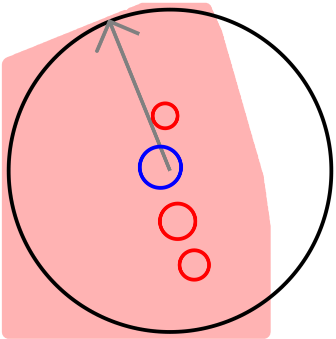

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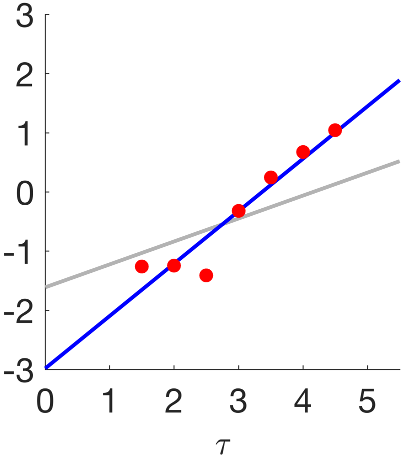

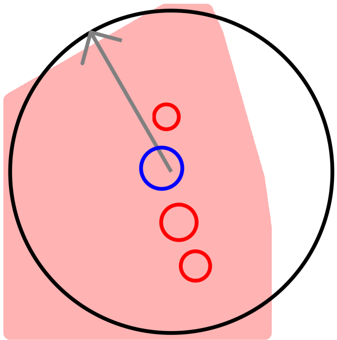

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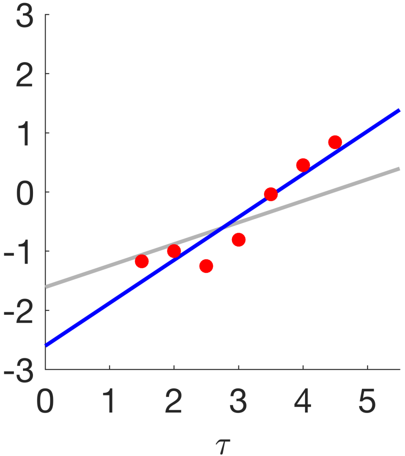

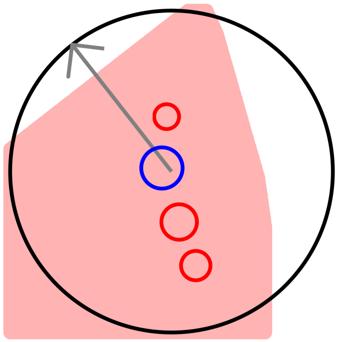

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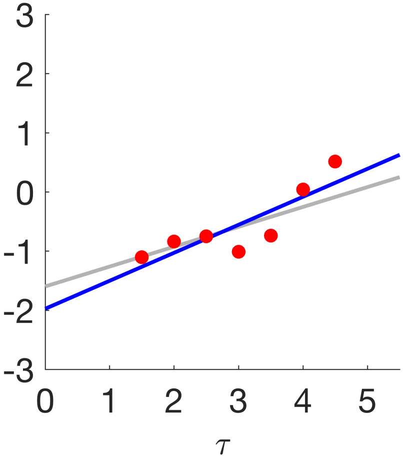

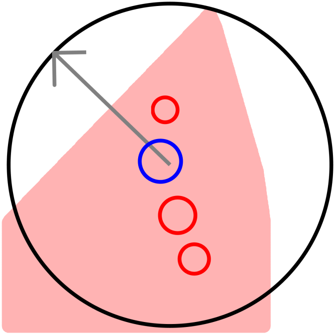

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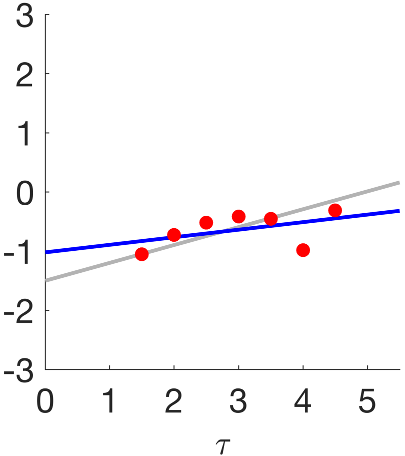

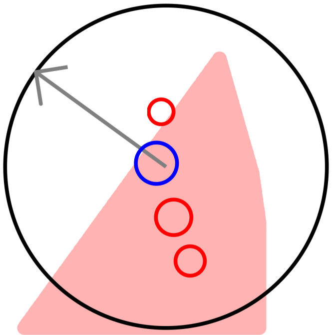

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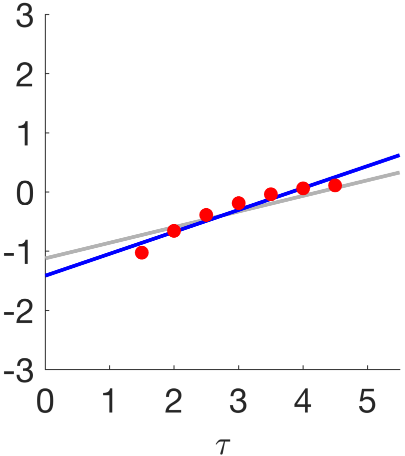

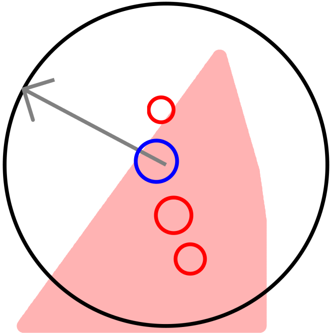

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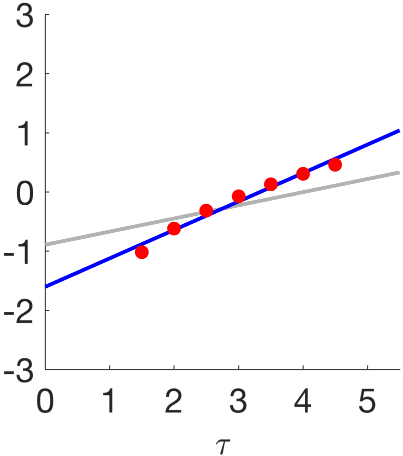

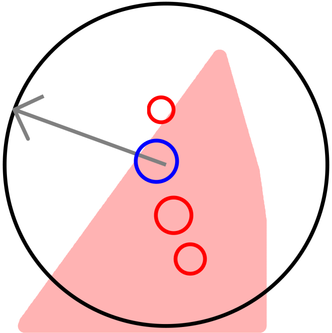

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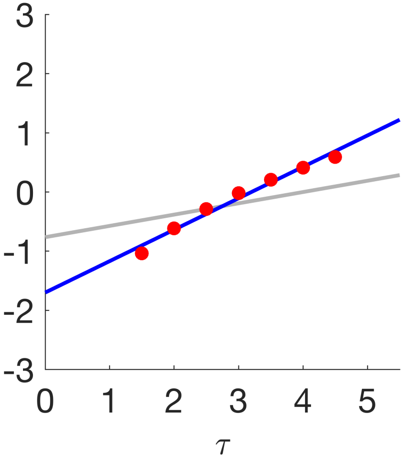

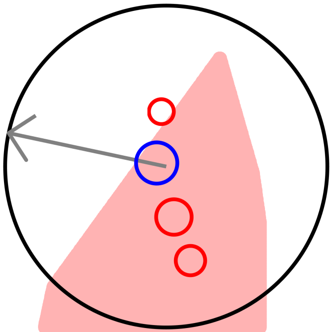

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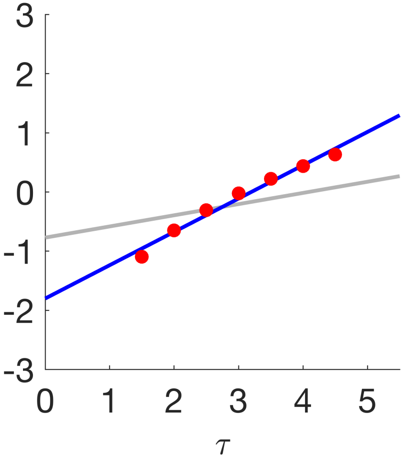

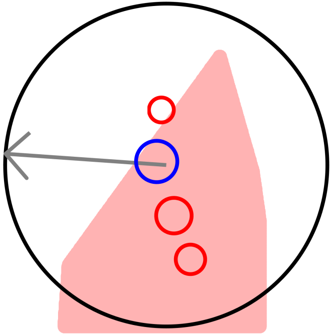

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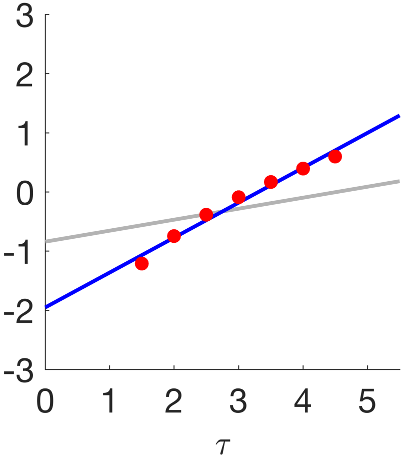

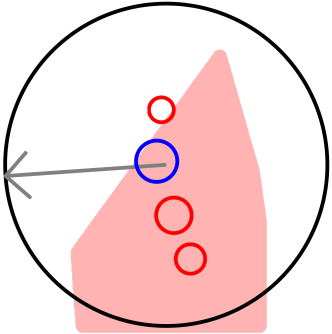

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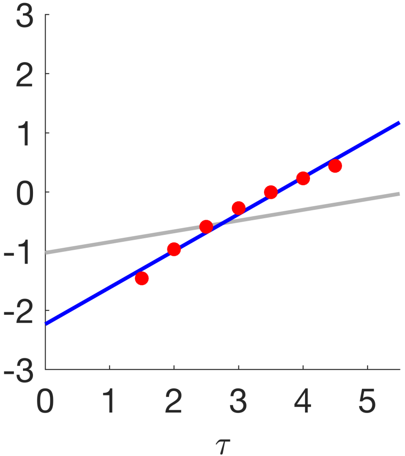

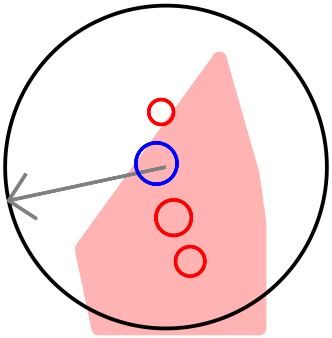

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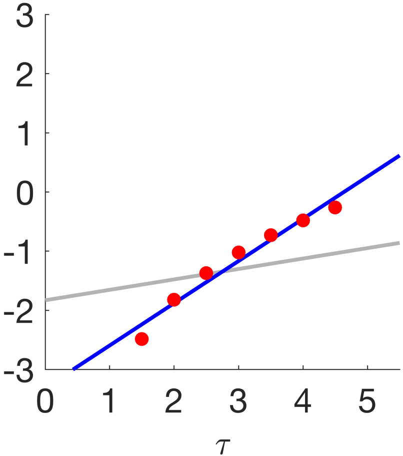

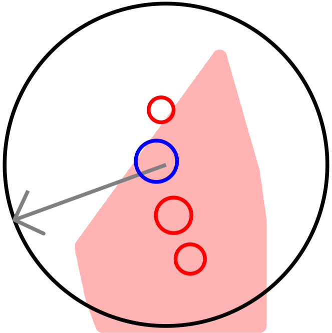

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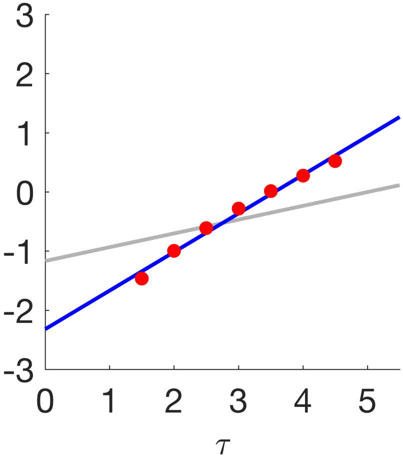

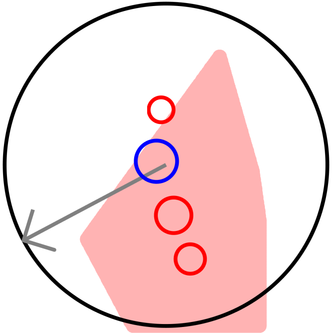

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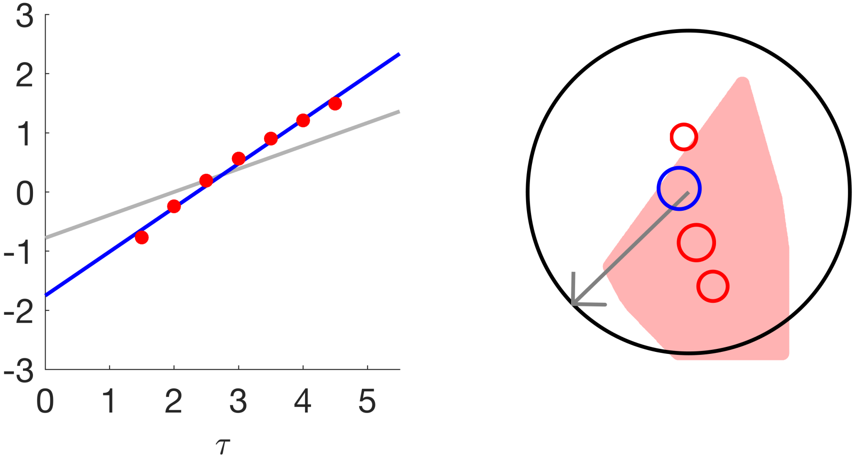

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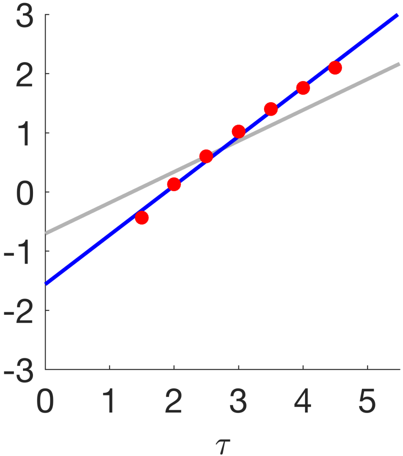

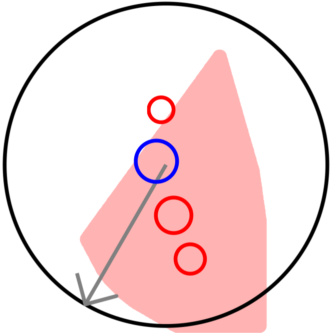

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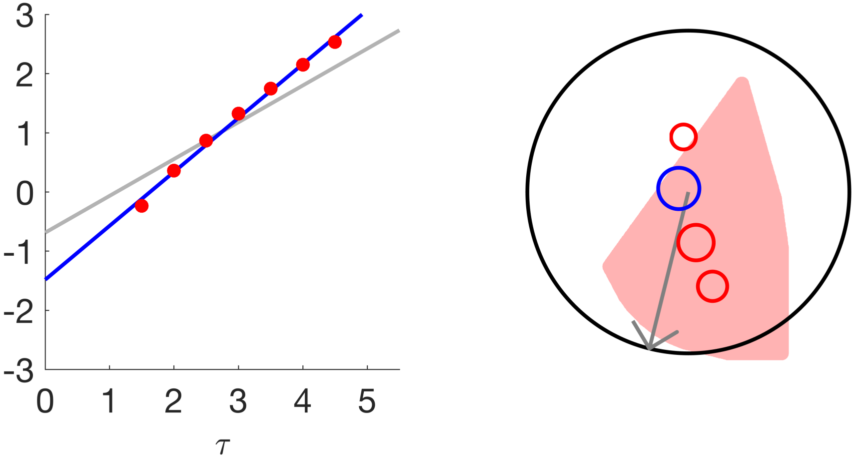

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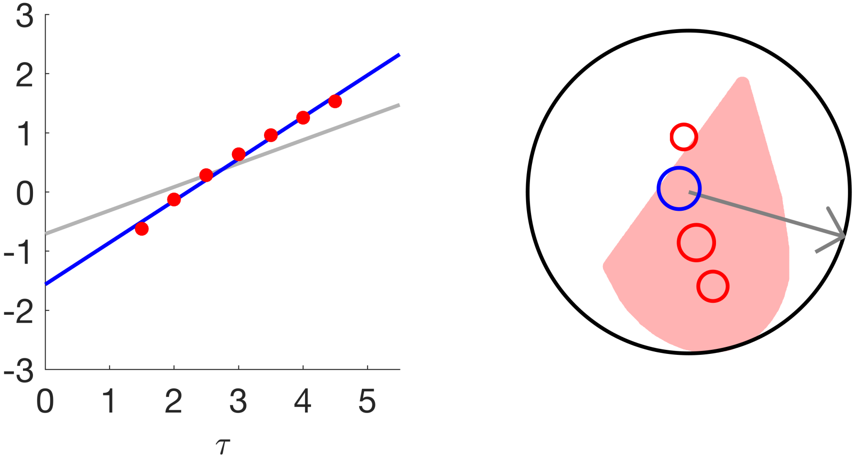

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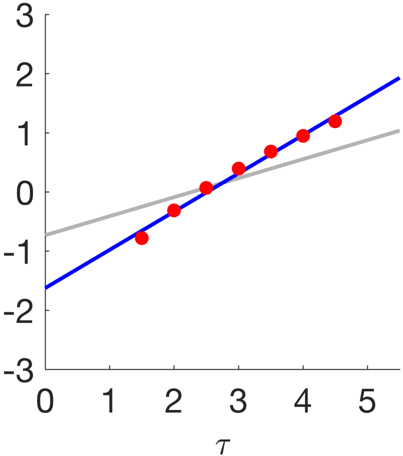

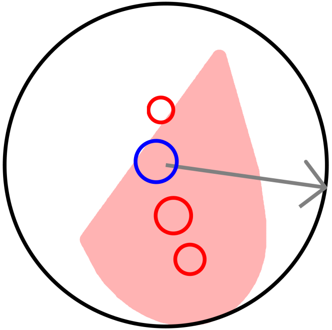

Outline

Least-squares computation

Neural network improvement

[S and Ide 2020]

2020 IEEE INTERNATIONAL WORKSHOP ON MACHINE LEARNING FOR SIGNAL PROCESSING, SEPT. 21–24, 2020, ESPOO, FINLAND

ELECTRICAL IMPEDANCE TOMOGRAPHY, ENCLOSURE METHOD AND MACHINE LEARNING

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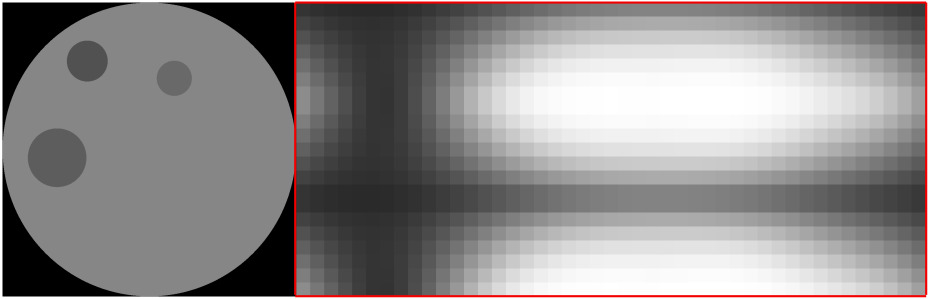
Tokyo, Japan
i24824_ide@aisin-aw.co.jp

We give the indicator functions to the neural net in the form of a 21×50 image



Horizontal axis: angles $1, 2, \ldots, 44, 45, 1, 2, 3, 4, 5.$

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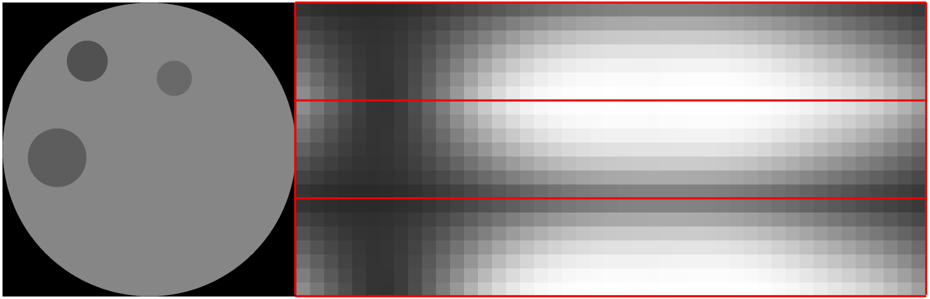

Horizontal axis: angles $1, 2, \ldots, 44, 45, 1, 2, 3, 4, 5$.

Vertical axis: 7 values of indicator function thrice

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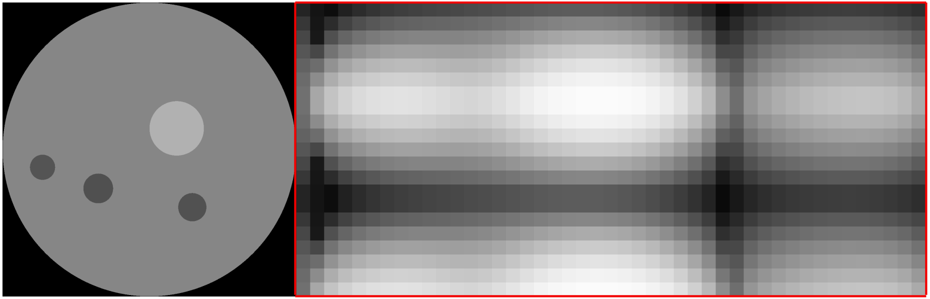


Horizontal axis: angles $1, 2, \ldots, 44, 45, 1, 2, 3, 4, 5$.
Vertical axis: 7 values of indicator function thrice

After many experiments we ended up choosing this network architecture

There are six layers in our convolutional neural network.
Layers that involve learnable parameters are marked with this color.

Image Input	21x50 images with zero mean normalization
Convolution	120 filters of size 6x4, three-pixel zero padding
Leaky ReLU	Leaky ReLU with scale 0.01
Fully Connected	Fully connected layer with 45 outputs
Tanh	Hyperbolic tangent layer
Regression Output	

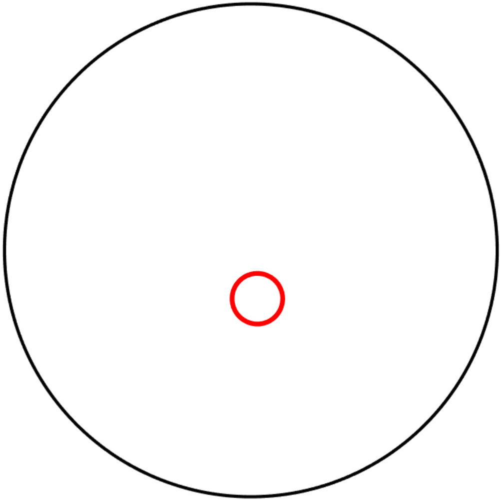
Training data: 19 000 cases with round inclusions

Background: homogeneous conductivity 1 defined in the unit disc.

There are 1–4 disc-shaped inclusions, random centers.

Disc radii are random, uniformly distributed in the interval [0.05, 0.2].

The conductivity inside each inclusion is random with uniform distribution in either [1.5, 5] or in [0.2, 0.667].



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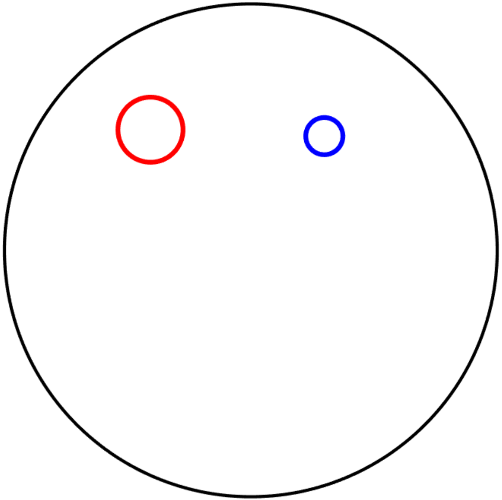

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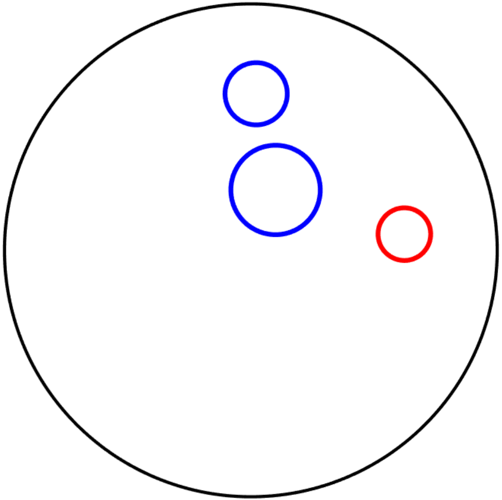

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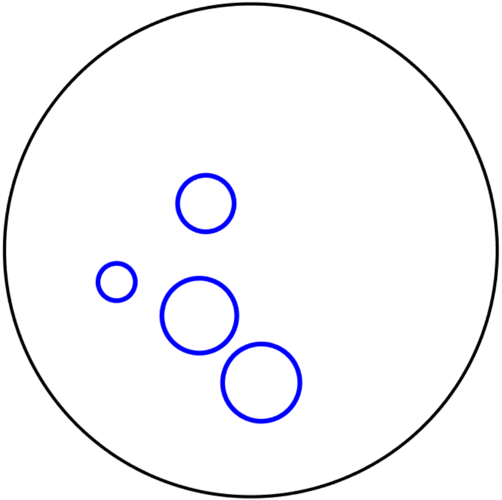


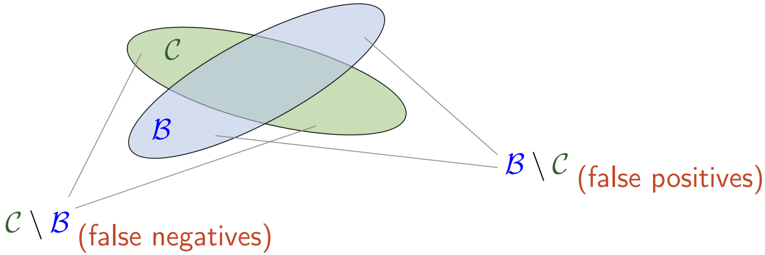
Relative error measurement for enclosure method

Denote the true convex hull by $\mathcal{C} \subset \mathbb{R}^2$ and the reconstructed convex hull by $\mathcal{B} \subset \mathbb{R}^2$. How to measure the quality of $\mathcal{B}$ as an approximation to $\mathcal{C}$ quantitatively?

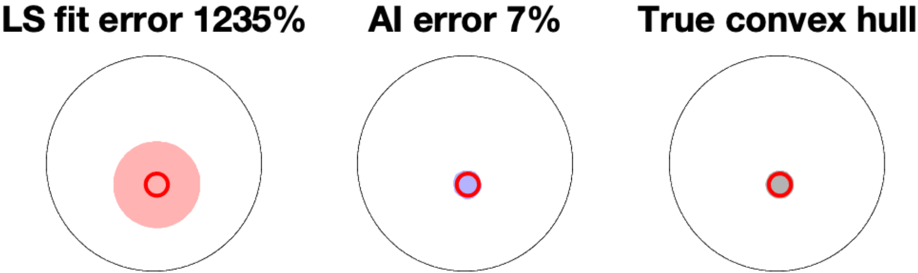
We use the relative error

$$\frac{|\mathcal{C} \setminus \mathcal{B}| + |\mathcal{B} \setminus \mathcal{C}|}{|\Omega|} \cdot 100\%,$$

where $| \cdot |$ is area. Here $|\Omega| = \pi$ since we work in the unit disc.



Reconstruction examples



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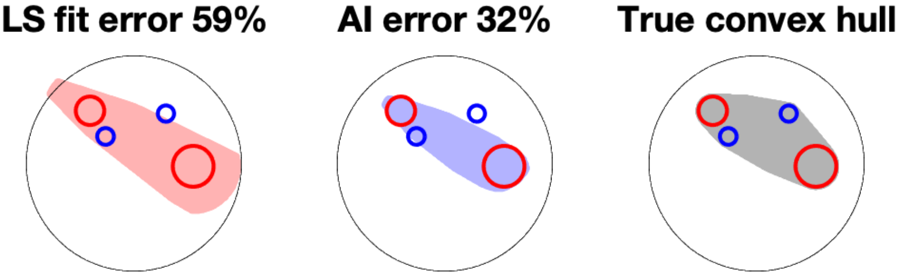

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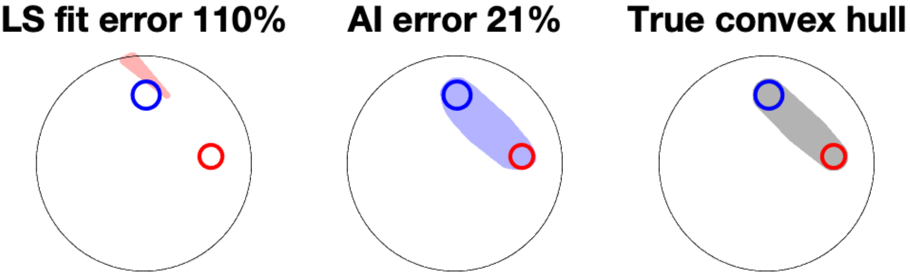

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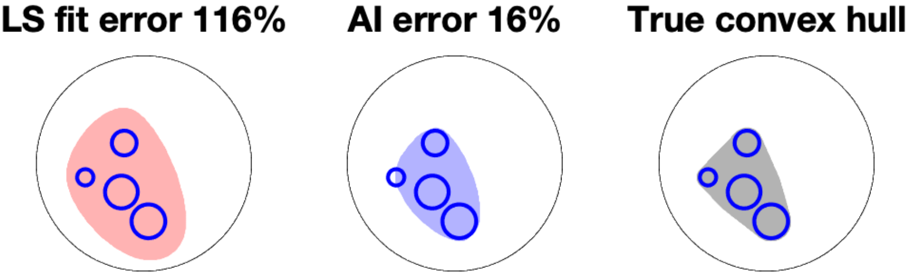

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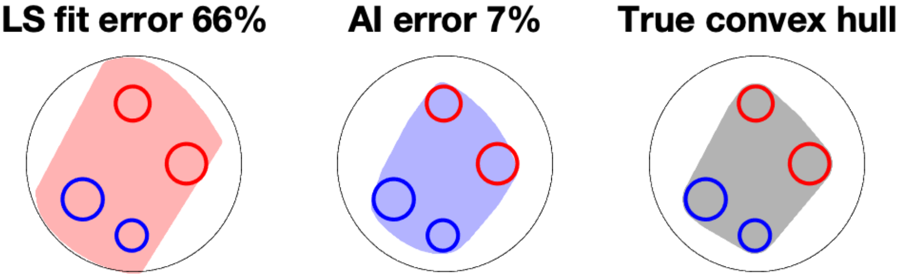


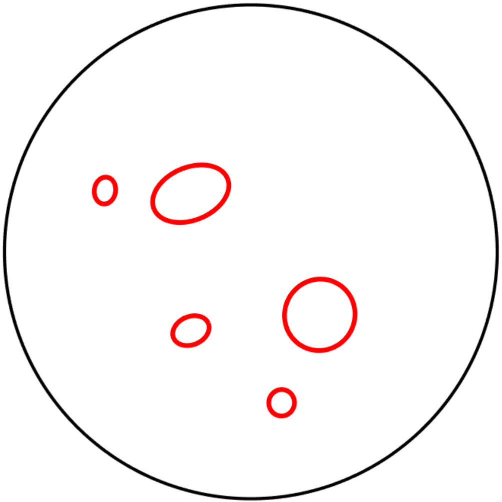
Test data: 1 000 cases with elliptic inclusions

Background: homogeneous conductivity 1 defined in the unit disc.

Five randomly placed and oriented elliptic inclusions.

Each semi-major axis random with half-length R uniformly distributed in the interval $[0.04, 0.22]$; each semi-minor axis r in the interval $[0.04, R]$

The conductivity inside each inclusion is random with uniform distribution in either $[1.3, 7]$ or in $[0.14, 0.77]$.



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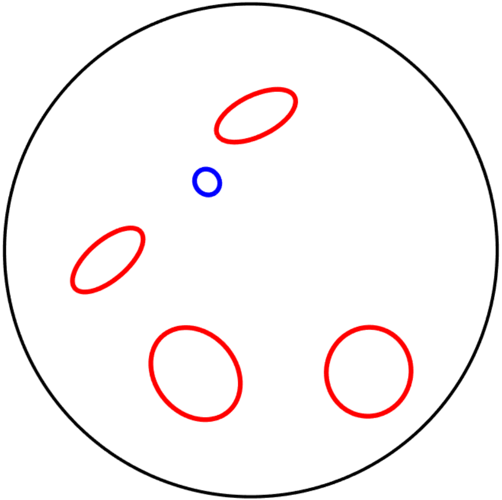

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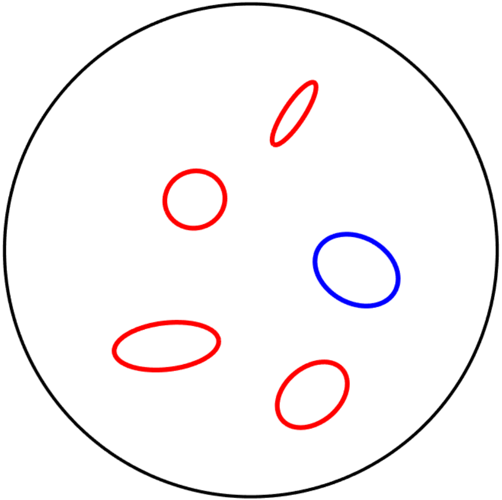

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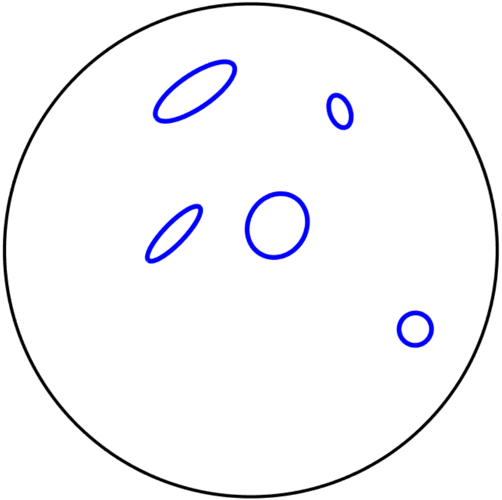
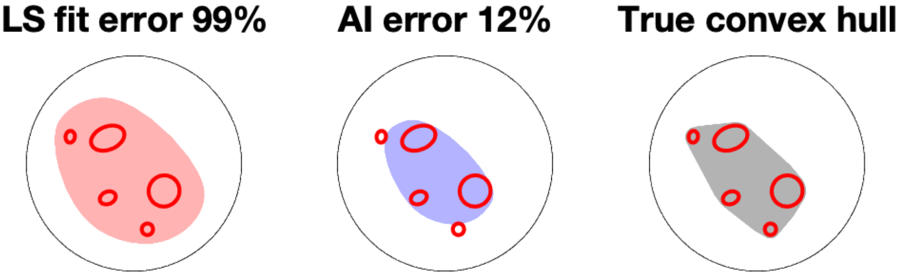


Reconstruction examples: test data



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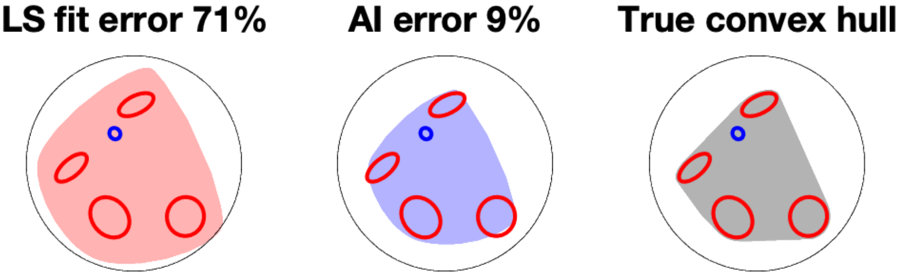

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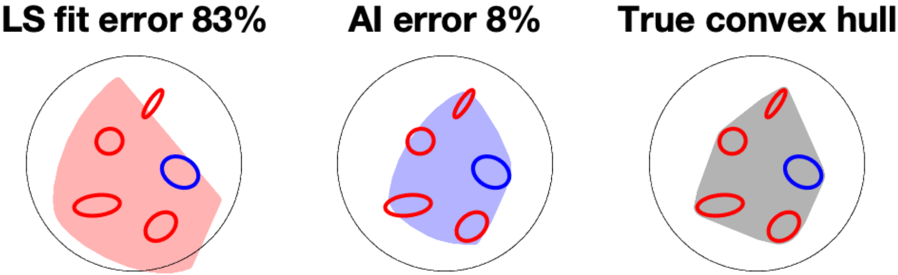

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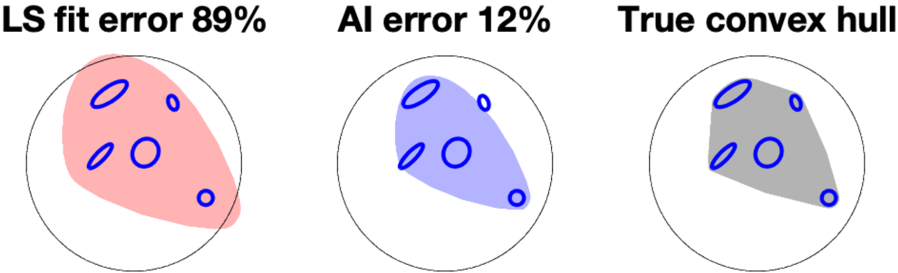
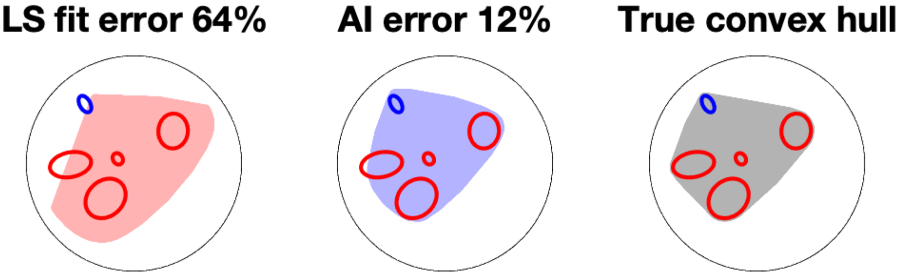
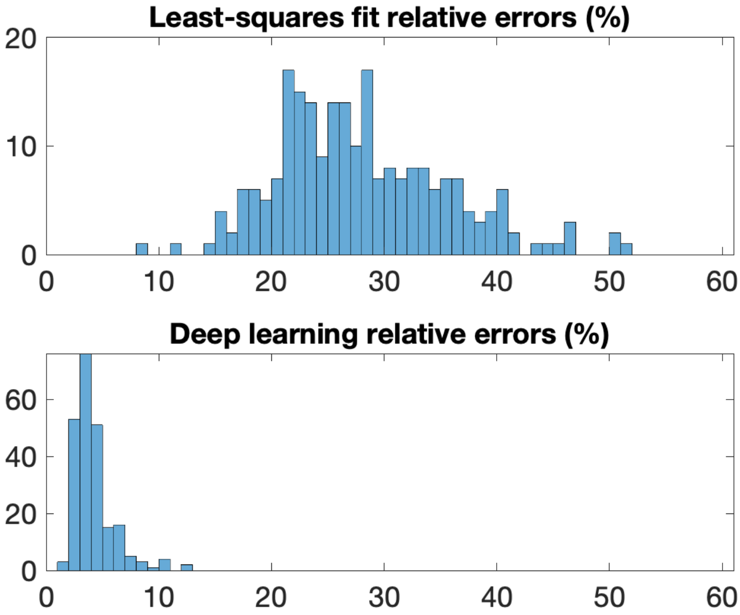
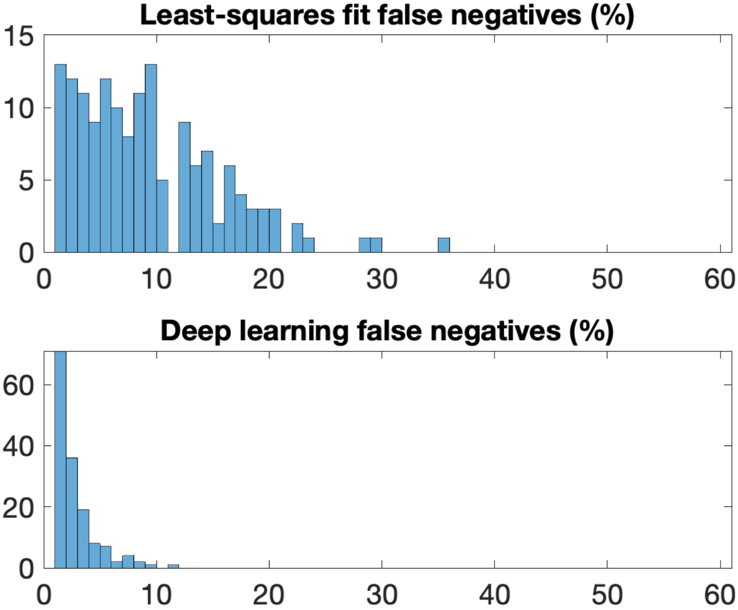


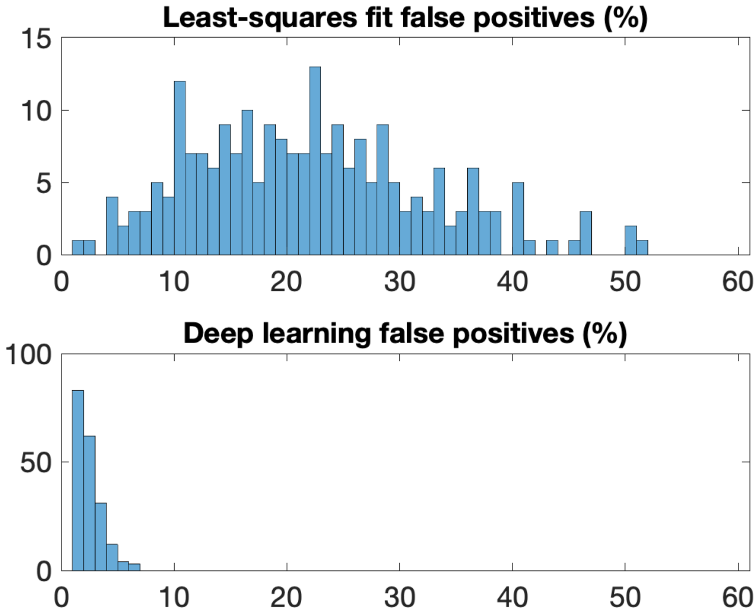
Reconstruction examples: test data





Least-squares fit relative errors (%)
20
10
0
0
10
20
30
40
50
60
Deep learning relative errors (%)
60
40
20
0
0
10
20
30
40
50
60


Least-squares fit false negatives (%)
0
5
10
15
0
10
20
30
40
50
60
Deep learning false negatives (%)
0
20
40
60
0
10
20
30
40
50
60


Least-squares fit false positives (%)
15
10
5
0
0
10
20
30
40
50
60
Deep learning false positives (%)
100
50
0
0
10
20
30
40
50
60

The paper trail of that visit to Japan in 1999

Ikehata and S 2000
Ikehata and Siltanen S 2004
Ikehata, Niemi and S 2012
S and Ide 2020
Sippola, Rautio, Hauptmann, Ide and S 2025
Rautio, S and Ide, submitted