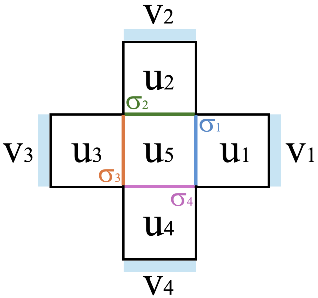
Mathematics of Electrical Impedance Tomography
Lecture 3: EIT with a 2D diffusion model



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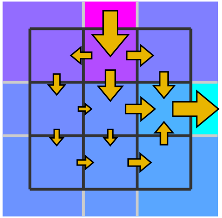
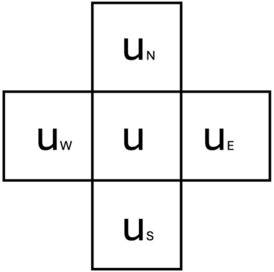
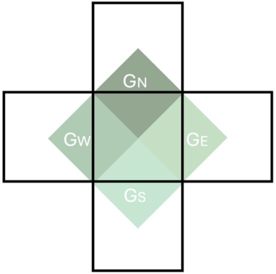


Outline

Basic building block of the two-dimensional diffusion model

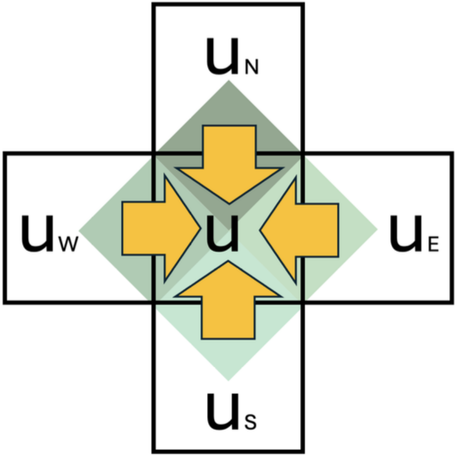


Consider the flow of electricity between one central pixel and its neighbors on the North, West, South and East sides.



Conductances between the central pixel and the neighbors are G_N, G_W, G_S, G_E.

Kirchhoff's Law gives a balance equation



The diffusion model is based on requiring conservation of charge: total sum of electric current into the central pixel should be zero.

For example, the current from the North towards the center is $I_N = G_N(u_N - u)$. The other neighbors have similar formulas.

We arrive at the balance equation

$$
\begin{aligned}
0 &= I_W + I_S + I_E + I_N \\
&= G_W(u_W - u) + G_S(u_S - u) + \\
&\quad + G_E(u_E - u) + G_N(u_N - u).
\end{aligned}
$$

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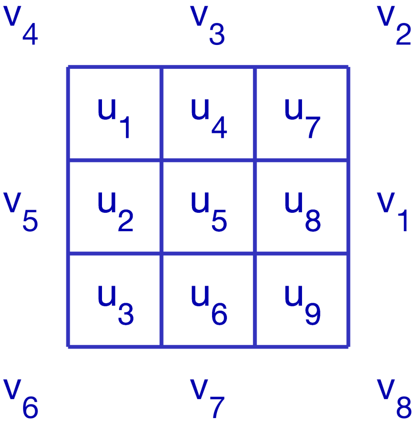
Grid system for diffusion-model EIT: case 3×3

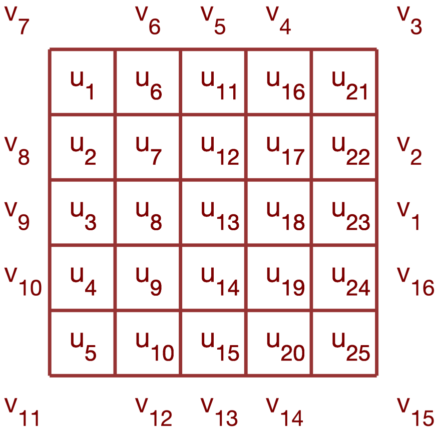


There are two kinds of variables in our model:

1. Voltages u_j at the pixels, $j = 1, 2, \ldots, n^2$.
2. Voltages v_i measured at the electrodes. Each outer edge of a boundary pixel is an electrode. The corner pixels have one electrode each. The indexing runs like this: $i = 1, 2, \ldots, L$ with $L = 4(n-1)$.

Electrodes are numbered in a counter-clockwise manner, starting from the "three o'clock" electrode.. The internal voltages are numbered column-wise.

Grid system for diffusion-model EIT: case 5×5



There are two kinds of variables in our model:

1. Voltages u_j at the pixels, $j = 1, 2, \ldots, n^2$.
2. Voltages v_i measured at the electrodes. Each outer edge of a boundary pixel is an electrode. The corner pixels have one electrode each. The indexing runs like this: $i = 1, 2, \ldots, L$ with $L = 4(n-1)$.

Electrodes are numbered in a counter-clockwise manner, starting from the "three o'clock" electrode.. The internal voltages are numbered column-wise.

Grid system for diffusion-model EIT: case 9×9

v_{13}		v_{12}	v_{11}	v_{10}	v_{9}	v_{8}	v_{7}	v_{6}		v_{5}
	u_{1}	u_{10}	u_{19}	u_{28}	u_{37}	u_{46}	u_{55}	u_{64}	u_{73}	
v_{14}	u_{2}	u_{11}	u_{20}	u_{29}	u_{38}	u_{47}	u_{56}	u_{65}	u_{74}	v_{4}
v_{15}	u_{3}	u_{12}	u_{21}	u_{30}	u_{39}	u_{48}	u_{57}	u_{66}	u_{75}	v_{3}
v_{16}	u_{4}	u_{13}	u_{22}	u_{31}	u_{40}	u_{49}	u_{58}	u_{67}	u_{76}	v_{2}
v_{17}	u_{5}	u_{14}	u_{23}	u_{32}	u_{41}	u_{50}	u_{59}	u_{68}	u_{77}	v_{1}
v_{18}	u_{6}	u_{15}	u_{24}	u_{33}	u_{42}	u_{51}	u_{60}	u_{69}	u_{78}	v_{32}
v_{19}	u_{7}	u_{16}	u_{25}	u_{34}	u_{43}	u_{52}	u_{61}	u_{70}	u_{79}	v_{31}
v_{20}	u_{8}	u_{17}	u_{26}	u_{35}	u_{44}	u_{53}	u_{62}	u_{71}	u_{80}	v_{30}
	u_{9}	u_{18}	u_{27}	u_{36}	u_{45}	u_{54}	u_{63}	u_{72}	u_{81}	
v_{21}		v_{22}	v_{23}	v_{24}	v_{25}	v_{26}	v_{27}	v_{28}		v_{29}

There are two kinds of variables in our model:

1. Voltages u_j at the pixels, $j = 1, 2, \ldots, n^2$.
2. Voltages v_i measured at the electrodes. Each outer edge of a boundary pixel is an electrode. The corner pixels have one electrode each. The indexing runs like this: $i = 1, 2, \ldots, L$ with $L = 4(n-1)$.

Electrodes are numbered in a counter-clockwise manner, starting from the "three o'clock" electrode.. The internal voltages are numbered column-wise.

Current patterns

In 2D we have more freedom for current feeds than in 1D. We can choose the magnitude and direction of current independently at every electrode.

If our EIT setup has L electrodes, then a current pattern is a nonzero vector $\vec{I} \in R^L$.

A positive element $\vec{I_\ell} > 0$ means that current is going into the domain through electrode ℓ, and a negative $\vec{I_\ell} < 0$ indicates that current of amplitude $|\vec{I_\ell}|$ amperes is exiting the domain through electrode ℓ. Due to conservation of charge it holds that

$$\sum_{\ell=1}^{L} \vec{I_\ell} = 0.$$

Therefore, we have at most $L - 1$ linearly independent current patterns.

The general linear system

Boundary conditions and balance equations are collected into a linear system

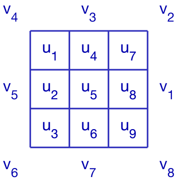
$$A\vec{v} = \left[\begin{array}{c|c} I & A' \\ \hline A'' & A''' \end{array}\right] \left[\begin{array}{c} v_1 \\ \vdots \\ v_L \\ \hline u_1 \\ \vdots \\ u_{n^2} \end{array}\right] = \left[\begin{array}{c} b_1 \\ \vdots \\ b_L \\ \hline 0 \\ \vdots \\ 0 \end{array}\right] = \vec{b},$$

where I is the $L \times L$ identity matrix. Boundary conditions are in block A', while blocks A'' and A''' contain balance equations.

Here A is a square matrix of size $N \times N$ with with $N = n^2 + L = n^2 + 4(n-1)$.

The matrix equation for the homogeneous 3×3 model

$$
\left[\begin{array}{cccccccc|ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 3 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 3 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 3 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 3
\end{array}\right]
\left[\begin{array}{c}
v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ \hline u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9
\end{array}\right]
=
\left[\begin{array}{c}
-1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{array}\right]
$$



Minimum norm solution

Recall that a vector $\vec{w}$ is called a *least-squares solution* of the equation $A\vec{v} = \vec{b}$ if

$$\|A\vec{w} - \vec{b}\| = \min_{\vec{v}} \|A\vec{v} - \vec{b}\|.$$

Furthermore, a least-squares solution $\vec{v}^+$ is called the *minimum norm solution* if

$$\|\vec{v}^+\| = \inf\{\|\vec{w}\| \,:\, \vec{w} \text{ is a least-squares solution of } A\vec{v} = \vec{b}\}.$$

Unique solution for the matrix equation

The matrix A has a kernel of dimension one. So the solution is unique up to a constant, and we can find some representative of the solution space by solving the system in the least squares sense for the minimum norm solution. Denote the voltage potentials given by the minimum norm solution at all the pixels and electrodes by

$$\vec{v}^{+} = [v_1^{+}, \ldots, v_L^{+}, u_1^{+}, \ldots, u_{n^2}^{+}]^T.$$

Now calculate the mean value of the electrode voltages:

$$\overline{v} = \frac{1}{L} \sum_{j=1}^{L} v_j^{+}.$$

Then we simply subtract $\overline{v}$ from all the voltages in the solution: $\vec{v} = \vec{v}^{+} - \overline{v}$. This way we find the unique solution with voltages at the electrodes summing up to zero. You can try this out with the Matlab routine `FD_EIT_2D_demoA_3x3_case2_matrix.m`.

Outline

Example of nonlinearity of the forward map

Run the Matlab routine `FD_EIT_2D_demoA_3x3_case3_nonlinearity.m`.

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The voltage and current matrices

Consider applying the current pattern $\vec{I^j}$ and calculating the corresponding unique solution $\vec{v}^{(j)} = [v_1^{(j)}, \dots, v_L^{(j)}, u_1^{(j)}, \dots, u_{n^2}^{(j)}]^T$. Truncating the solution so that only the electrode voltages remain, we define

$$\vec{V^j} := \begin{bmatrix} v_1^{(j)} \\ \vdots \\ v_L^{(j)} \end{bmatrix}.$$

Furthermore, assume that we have chosen a set of $L-1$ linearly independent current patterns. Define the *current matrix* $\mathsf{I} \in \mathbb{R}^{L \times (L-1)}$ and voltage matrix $\mathsf{U} \in \mathbb{R}^{L \times (L-1)}$ by

$$\mathsf{I} := [\vec{I^1}, \vec{I^2}, \dots, \vec{I}^{(L-1)}], \qquad \mathsf{U} := [\vec{V^1}, \vec{V^2}, \dots, \vec{V}^{(L-1)}].$$

The current-to-voltage matrix

It is a linear process to go from the applied current vector I^j to the measured voltage vector V^j. Therefore, there must be a current-to-voltage matrix R_γ of size $L \times L$ that satisfies $V^j = R_\gamma I^j$.

We arrive at the formula

$$\mathrm{U} = R_\gamma \mathrm{I}.$$

It is often useful to think of R_γ as the EIT data.

The voltage matrix collects all measurements together

Run the Matlab routines

```
FD_EIT_2D_demoB1_voltmat1.m
FD_EIT_2D_demoB2_voltmat2.m
FD_EIT_2D_demoB3_voltmat_plot.m
FD_EIT_2D_demoB4_voltmat_trig1.m
FD_EIT_2D_demoB5_voltmat_trig2.m
FD_EIT_2D_demoB6_voltmat_trig_plot.m
```

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The voltage matrix collects all measurements together

Run the Matlab routine

`FD_EIT_2D_demoC_deeper_look_comp.m`

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