Mathematics of Electrical Impedance Tomography
Lecture 2: Ohm's Law and the 1D diffusion model

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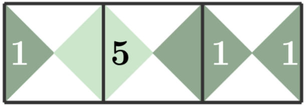

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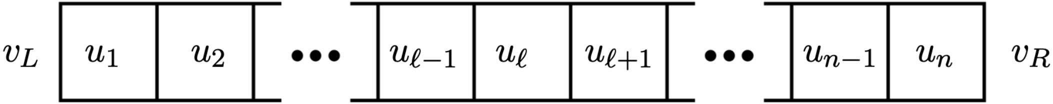
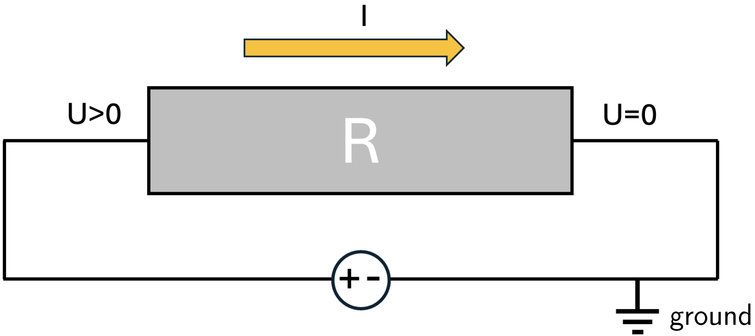


Ohm's Law: $U = RI$

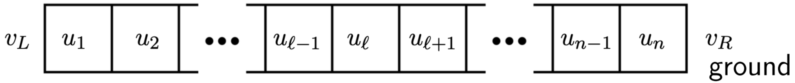


Voltage U is measured in volts (V), current in amperes (A) and resistance in ohms (Ω).

In EIT we consider conductance instead of resistance. Denote conductance by $G = 1/R$, measured in siemens $S = \Omega^{-1}$. Now Ohm's Law takes the form

$$I = GU.$$

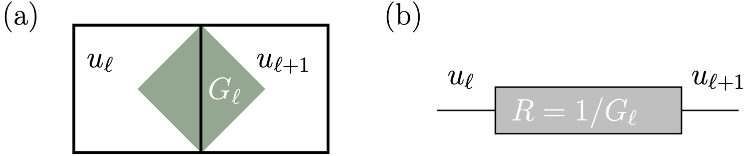
One-dimensional diffusion model



Our diffusion model contains a horizontal linear array of n pixels. The left edge of the leftmost pixel and the right edge of the rightmost edges are thought of as electrodes through which we inject currents.

Voltages at the electrodes are denoted by v_L (left end) and v_R (right end). Variables $u_1, u_2, u_3, \ldots, u_n$ are voltages inside the pixels.

One-dimensional diffusion model with conductivity



Our array of pixels is a coarse model of a rod of conductive material, allowed to be non-homogeneous. So we assign a conductance $G_\ell > 0$ to every vertical edge between neighboring pixels.

The current flowing from pixel u_ℓ to $u_{\ell+1}$ can be calculated using Ohm's Law as

$$I_\ell = G_\ell(u_\ell - u_{\ell+1}), \qquad \ell = 1, 2, 3, \ldots, n-1.$$

Kirchhoff's Law and the balance equation

Kirchhoff's current law states that the amount of electric current flowing into a pixel from left must equal the current exiting the pixel on the right. In mathematical terms, the balance equation for pixel ℓ is $I_{\ell-1} = -I_\ell$. Using Ohm's Law we get

$$G_{\ell-1}(u_{\ell-1} - u_\ell) = G_\ell(u_\ell - u_{\ell+1}), \qquad \ell = 2, 3, \ldots, n-1.$$

The above equation describes Kirchhoff's current law at all the internal pixels ($\ell = 2, 3, \ldots, n-1$) but not at pixels 1 and n. For those two we need to consider currents through the electrodes into the linear model. We will take the conductance at the outermost edges to be $G_0 = 1S = G_n$. So then the currents applied at the left and right ends are, ignoring units,

$$I_L = v_L - u_1, \qquad I_R = v_R - u_n.$$

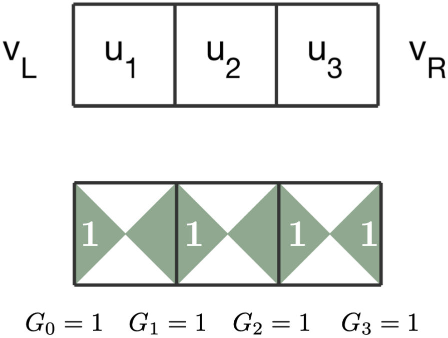
Note that the sign convention for I_R is different from all other pixels. We want to think that *current flowing from outside into the rod is positive*.

Outline

Simple three-pixel model

Non-uniqueness in the three-pixel model

Matrix model for a homogeneous 3-pixel example

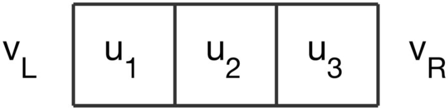


We have $n = 3$ pixels and all conductances equal to 1. We will build a matrix model

$$A\vec{v} = A \begin{bmatrix} v_L \\ u_1 \\ u_2 \\ u_3 \\ v_R \end{bmatrix} = \vec{b},$$

where A is a 5×5 matrix and the right hand side vector $\vec{b}$ depends on the current feeds.

Matrix model for a homogeneous 3-pixel example

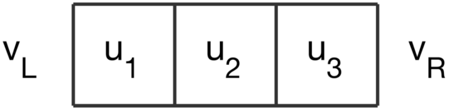


$G_0 = 1 \quad G_1 = 1 \quad G_2 = 1 \quad G_3 = 1$

Let's start by inserting the boundary conditions. They are given as current feeds, and we get the first two rows:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} v_L \\ u_1 \\ u_2 \\ u_3 \\ v_R \end{bmatrix} = \begin{bmatrix} I_L \\ I_R \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}.$$

Matrix model for a homogeneous 3-pixel example

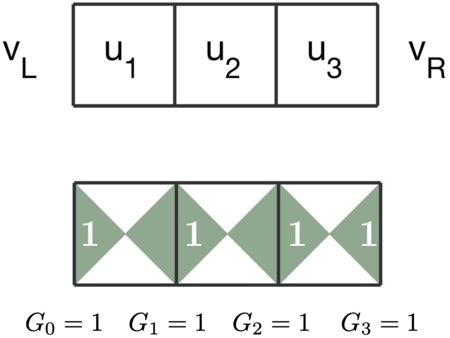


$G_0 = 1 \quad G_1 = 1 \quad G_2 = 1 \quad G_3 = 1$

Next we go to internal balance equations. From $G_{\ell-1}(u_{\ell-1} - u_\ell) = G_\ell(u_\ell - u_{\ell+1})$ and the identities $G_1 = 1 = G_2$ we get the equation for the middle pixel: $u_1 - u_2 = u_2 - u_3$. That gives us row 4 in the matrix model:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & -2 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} v_L \\ u_1 \\ u_2 \\ u_3 \\ v_R \end{bmatrix} = \begin{bmatrix} I_L \\ I_R \\ \cdot \\ 0 \\ \cdot \end{bmatrix}.$$

Matrix model for a homogeneous 3-pixel example

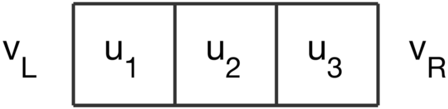


Similar balance equations hold for pixels 1 and 3, with voltage potentials v_L and v_R appropriately inserted. These three equations fill the final three rows and complete the construction of our matrix model:

$$A\vec{v} = \left[\begin{array}{rrrrr} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ \hline 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right] \begin{bmatrix} v_L \\ u_1 \\ u_2 \\ u_3 \\ v_R \end{bmatrix} = \left[\begin{array}{c} I_L \\ I_R \\ \hline 0 \\ 0 \\ 0 \end{array}\right].$$

Top block: boundary data. Bottom block: balance equations from Kirchhoff's current law.

Matrix model for a homogeneous 3-pixel example



$G_0 = 1 \quad G_1 = 1 \quad G_2 = 1 \quad G_3 = 1$

$$A\vec{v} = \left[\begin{array}{ccccc} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ \hline 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right] \begin{bmatrix} v_L \\ u_1 \\ u_2 \\ u_3 \\ v_R \end{bmatrix} = \left[\begin{array}{c} I_L \\ I_R \\ \hline 0 \\ 0 \\ 0 \end{array}\right]$$

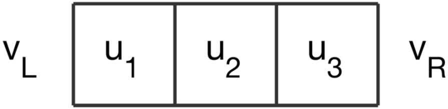
A straightforward idea would be to invert the square matrix A to solve our linear system. But it is not that simple as A is a singular matrix: $\det(A) = 0$, so the inverse does not exist.

Further analysis reveals that A has a one-dimensional kernel:

$$\ker(A) = \{[\lambda, \lambda, \lambda, \lambda, \lambda]^T \in \mathbb{R}^5 \,|\, \lambda \in \mathbb{R}\}.$$

This means non-uniqueness in the following sense. If a vector $\vec{v}$ is a solution of, then any vector $\vec{v}_0 \in \ker(A)$ produces another solution $\vec{v} + \vec{v}_0$. So the system has infinitely many solutions, and we need a way to pick out the one we want.

Matrix model for a homogeneous 3-pixel example



$G_0 = 1 \quad G_1 = 1 \quad G_2 = 1 \quad G_3 = 1$

$$A\vec{v} = \left[\begin{array}{ccccc} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ \hline 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right] \begin{bmatrix} v_L \\ u_1 \\ u_2 \\ u_3 \\ v_R \end{bmatrix} = \left[\begin{array}{c} I_L \\ I_R \\ \hline 0 \\ 0 \\ 0 \end{array}\right]$$

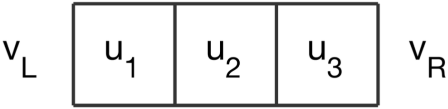
One approach is to ground the right electrode, setting the voltage potential there to zero:

$$v_R = 0.$$

Let's see how this works in a concrete example. Consider feeding a current $I_L = 1$ (mA) into the rod through the left electrode and taking the current out form the right electrode: $I_R = -1$. First calculate the least-squares solution as

$$\vec{v}^{\dagger} = (A^T A)^{\dagger} A^T \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Matrix model for a homogeneous 3-pixel example



$G_0 = 1 \quad G_1 = 1 \quad G_2 = 1 \quad G_3 = 1$

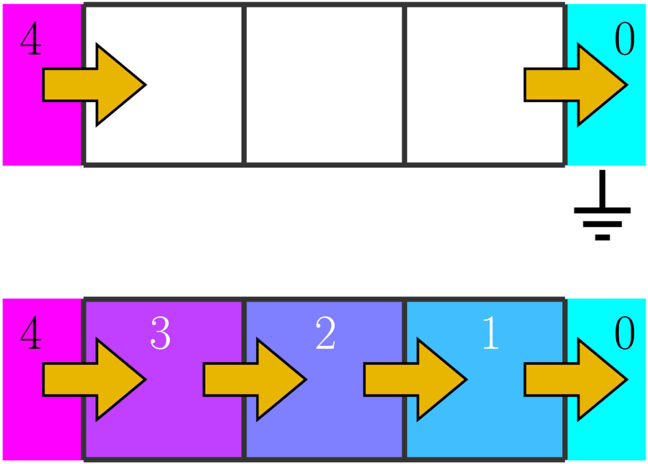
$$A\vec{v} = \left[\begin{array}{ccccc} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ \hline 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right] \begin{bmatrix} v_L \\ u_1 \\ u_2 \\ u_3 \\ v_R \end{bmatrix} = \left[\begin{array}{c} I_L \\ I_R \\ \hline 0 \\ 0 \\ 0 \end{array}\right]$$

Then the "grounding of the right electrode" can be modelled like this:

$$\vec{v}_\perp = \vec{v}^\dagger - \begin{bmatrix} v_R^\dagger \\ v_R^\dagger \\ v_R^\dagger \\ v_R^\dagger \\ v_R^\dagger \end{bmatrix} = \begin{bmatrix} v_L^\dagger - v_R^\dagger \\ u_1^\dagger - v_R^\dagger \\ u_2^\dagger - v_R^\dagger \\ u_3^\dagger - v_R^\dagger \\ 0 \end{bmatrix}.$$

We get the unique solution $\vec{v}_\perp = [4, 3, 2, 1, 0]^T$. Our measurement data will be just one number: voltage v_L at the left electrode. We have $v_L = 4$ millivolts.

Visualization of the solution



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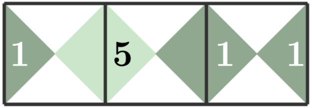
Matrix model for a non-homogeneous 3-pixel example



$G_0 = 1 \quad G_1 = 5 \quad G_2 = 1 \quad G_3 = 1$

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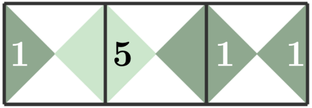
We now introduce changes in the conductive interior of the rod. We use two digital phantoms, where we insert a higher conductance $G = 5$ Siemens to one of the two internal edges in the model. Can we recover the conductance values from boundary measurements?

Matrix model for a non-homogeneous 3-pixel example



$G_0 = 1 \quad G_1 = 5 \quad G_2 = 1 \quad G_3 = 1$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & -6 & 5 & 0 & 0 \\ 0 & 5 & -6 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} v_L \\ u_1 \\ u_2 \\ u_3 \\ v_R \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let us again derive a matrix model. The two first rows in the system stay the same. From $G_1 = 5$ we get the balance equation for the left pixel:

$$(v_L - u_1) = 5(u_1 - u_2) \quad \Rightarrow \quad v_L - 6u_1 + 5u_2 = 0.$$

A similar calculation for the middle pixel gives

$$5(u_1 - u_2) = u_2 - u_3 \quad \Rightarrow \quad 5u_1 - 6u_2 + u_3 = 0.$$

The last row is the same as above. So we get the system on the left.

The solution including the grounding trick is $\vec{v}_\perp = [3.2 \;\; 2.2 \;\; 2 \;\; 1 \;\; 0]^T$. Our measurement data is then 3.2 millivolts.

Matrix model for a non-homogeneous 3-pixel example

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -6 & 5 & 0 \\ 0 & 0 & 5 & -6 & 1 \end{bmatrix} \begin{bmatrix} v_L \\ u_1 \\ u_2 \\ u_3 \\ v_R \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$G_0 = 1 \quad G_1 = 1 \quad G_2 = 5 \quad G_3 = 1$

Next step is to build the matrix model for the other case. It's on the left.

The solution is $\vec{v}_\perp = [3.2 \;\; 2.2 \;\; 1.2 \;\; 1 \;\; 0]^T$, and our measurement data is again $v_L = \vec{v}_1 = 3.2$ millivolts.

One-dimensional EIT is impossible with this linear model: two different rods produce the exact same measurement