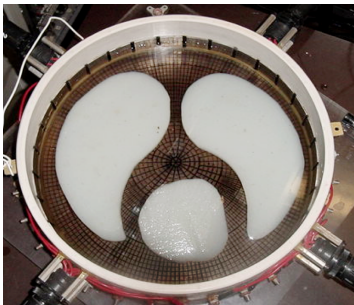


# Mathematics of Electrical Impedance Tomography

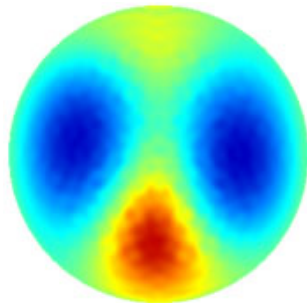
## Lecture 1: introduction



**Samuli Siltanen**

Department of Mathematics and Statistics  
University of Helsinki, Finland  
[samuli.siltanen@helsinki.fi](mailto:samuli.siltanen@helsinki.fi)  
[www.siltanen-research.net](http://www.siltanen-research.net)

35th Jyväskylä Summer School  
August 10–14, 2026



# Outline

## **Applications of Electrical Impedance Tomography (EIT)**

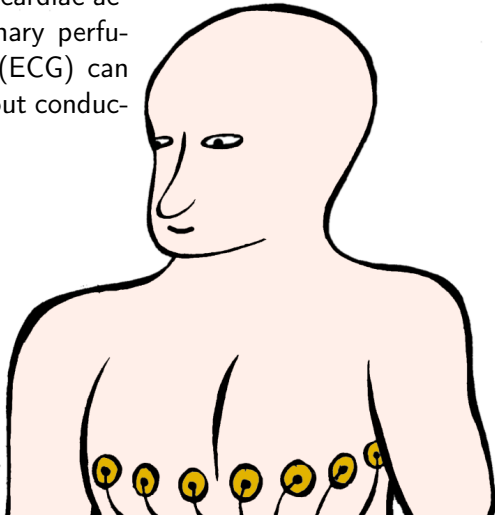
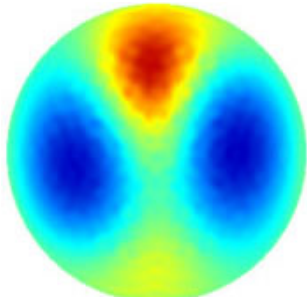
The principle of EIT

Non-uniqueness, ghosts, and ill-posedness

Regularized inversion for EIT

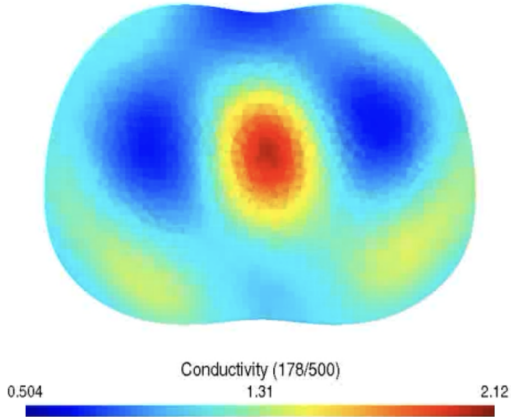
# Chest imaging is the standard application example of EIT in this talk

Medical applications: monitoring cardiac activity, lung function, and pulmonary perfusion. Also, electrocardiography (ECG) can be enhanced using knowledge about conductivity distribution.



# D-bar reconstruction of *in vivo* chest data

[Montoya & Mueller 2012]





# Medical application of EIT and the D-bar method: quantifying air-trapping in cystic fibrosis patients

[Muller, Mueller, Mellenthin, Murthy, Capps, Wagner, Alsaker, Deterding, Sagel & Hoppe 2018]

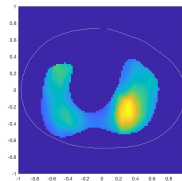
Images: **ventilation-perfusion index** maps, computed from three subjects at Children's Hospital Colorado using EIT and the D-bar method.

Dark blue regions are well-perfused but poorly ventilated.

Radiologist's report for Subject B: extensive regions of air trapping, regional to the lung areas affected by plugging, approx. 50% of both lungs.

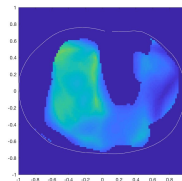
**Healthy control**

Average index 0.46



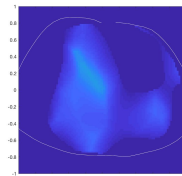
**CF Subject A**

Average index 0.34

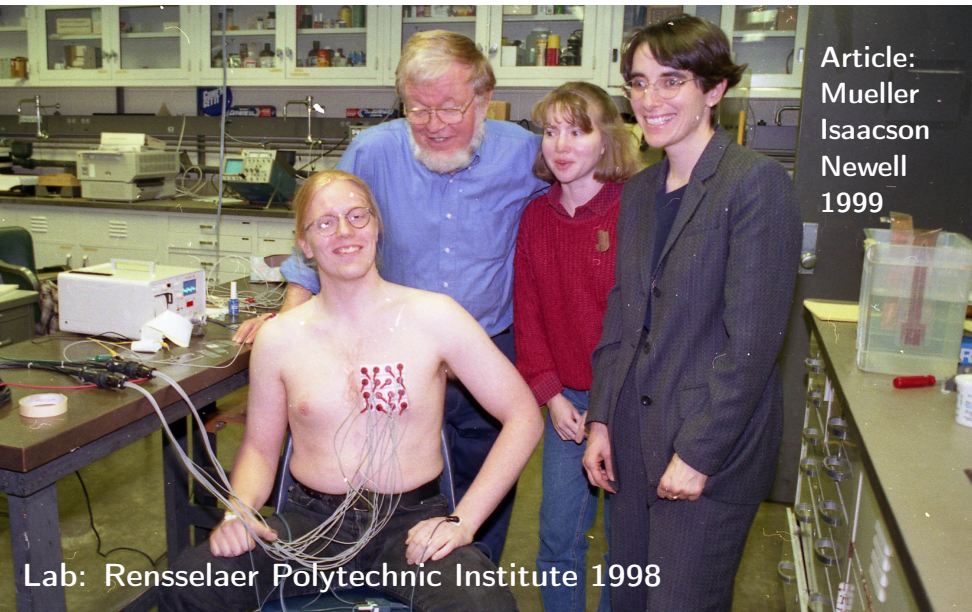


**CF Subject B**

Average index 0.10



# EIT can be used for heart imaging



Article:  
Mueller  
Isaacson  
Newell  
1999

Lab: Rensselaer Polytechnic Institute 1998

# EIT can potentially be used for imaging changes in vocal folds due to excessive voice use

Sao Paulo, February 27, 2013



Laukkanen  
León  
Lima  
Liu  
Moura  
Seppänen  
S

# EIT can perhaps be used for imaging changes in vocal folds due to dehydration

Fort Collins, March 10, 2015

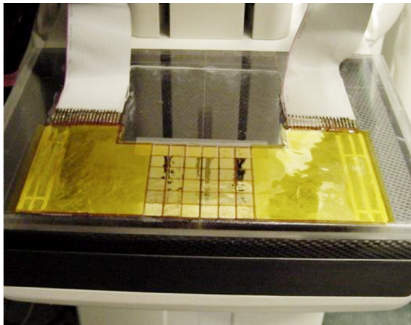


# The most promising use of EIT is detection of breast cancer in combination with mammography

ACT4 and mammography devices

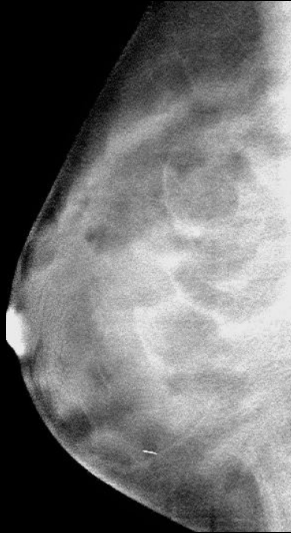
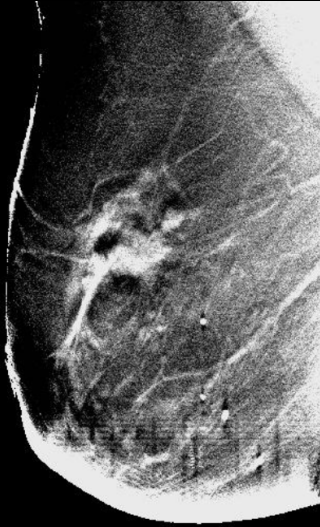


Radiolucent electrodes

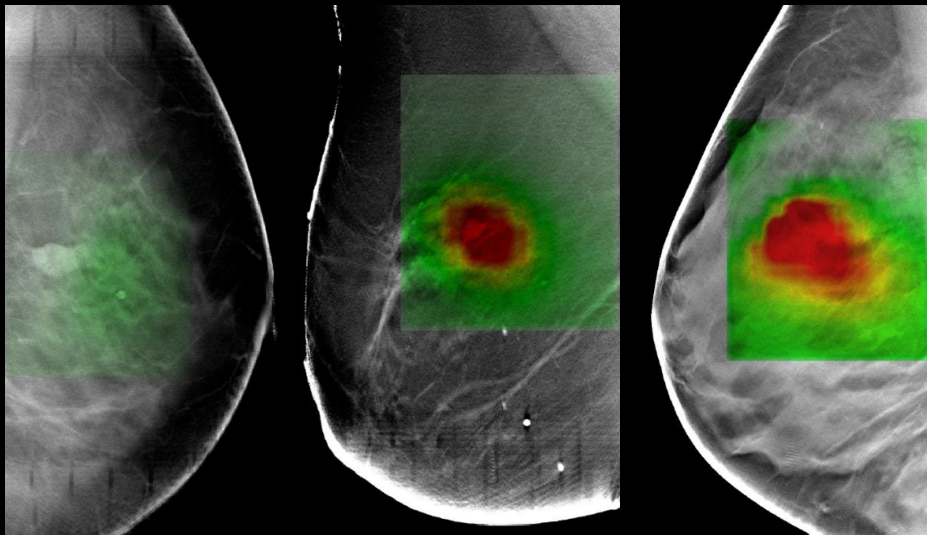


Cancerous tissue is up to four times more conductive than healthy breast tissue [Jossinet 1998]. The above experiment by **David Isaacson's** team measures 3D X-ray mammograms and EIT data at the same time.

Which of these three breasts have cancer?



# Spectral EIT can detect cancerous tissue

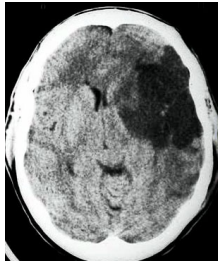


[Kim, Isaacson, Xia, Kao, Newell & Saulnier 2007]

# Imaging stroke with EIT. Same symptoms in both cases!

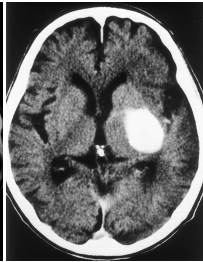
Ischemic stroke:  
low conductivity.

CT image from  
Jansen 2008



Hemorrhagic stroke:  
high conductivity.

CT image from  
Nakano *et al.* 2001



Difficulties: resistive skull layer and unknown background. However, see

- Romsauerova, McEwan, Horesh, Yerworth, Bayford & Holder 2006
- Shi, You, Xu, Wang, Fu, Liu & Dong 2009
- Bayford & Tizzard 2012
- Malone, Jehl, Arridge, Bettecke & Holder 2014
- Candiani, Hyvönen, Kaipio and Kolehmainen 2021
- Agnelli, Moura, Rautio, Alsaker, Murthy, Lassas and S 2026



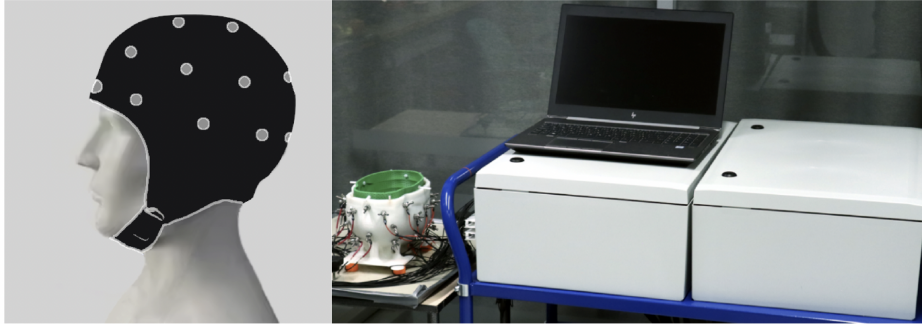
The idea would be to equip every ambulance with an EIT device for classifying strokes



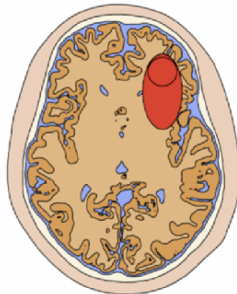
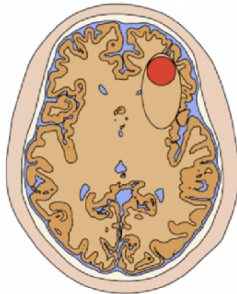
Monitoring a patient in an intensive care unit  
is an important application of stroke-EIT



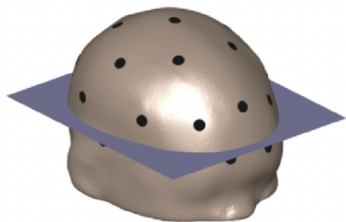
# Ville Kolehmainen's group has designed and built a prototype stroke-EIT device



**Figure 1:** Left: An illustration of EEG electrode headwear that can be used for EIT stroke monitoring measurements. Right: The prototype stroke EIT device with electrodes attached to surface of a 3D printed head model used for hardware development and validation in laboratory environment.



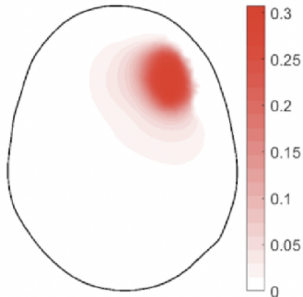
Transverse plane location



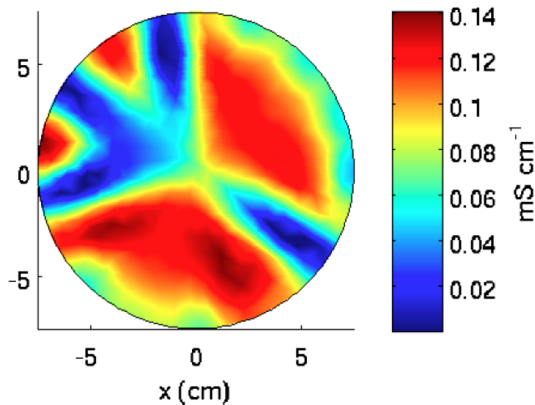
Change in tissues



Estimated change in conductivity



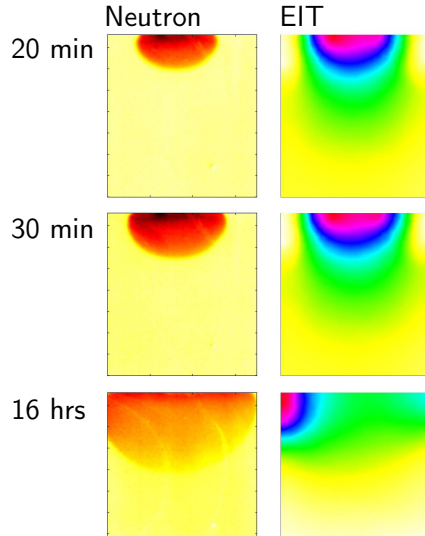
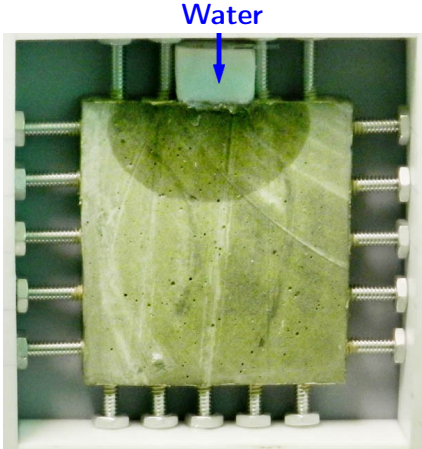
EIT can be used for nondestructive testing:  
here for crack detection in concrete structures



[Karhunen, Seppänen, Lehtikoinen, Monteiro & Kaipio 2010]

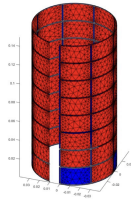
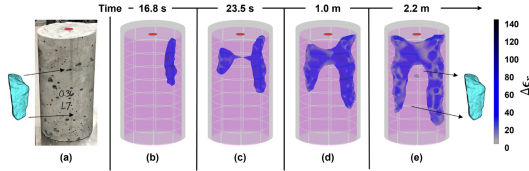
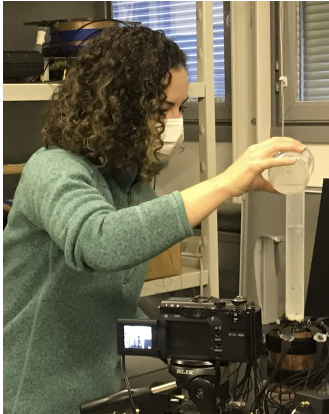
[Karhunen, Seppänen, Lehtikoinen, Monteiro, Kaipio, Blunt, Hyvönen]

EIT can be used for nondestructive testing:  
here for water in concrete structures



[Hallaji, Seppänen & Pour-Ghaz 2015]  
[Hallaji, Seppänen & Pour-Ghaz 2015]

# Material characterization: Imaging of water transport in concrete using ECT



$$\nabla \cdot \varepsilon(\mathbf{x}) \nabla u(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathbb{R}^3,$$

$$u(\mathbf{x}) + \zeta_\ell \varepsilon(\mathbf{x}) \frac{\partial u(\mathbf{x})}{\partial \hat{n}} = U_\ell, \quad \mathbf{x} \in e_\ell, \ell = 1, \dots, N_{el}$$

$$\int_{e_\ell} \varepsilon(\mathbf{x}) \frac{\partial u(\mathbf{x})}{\partial \hat{n}} dS = q_\ell, \quad \ell = 1, \dots, N_{el}$$

$$\frac{\partial u(\mathbf{x})}{\partial \hat{n}} = 0, \quad \mathbf{x} \in \partial\Omega \setminus \partial\Omega_{el},$$

# Electromagnetic flow tomography (EMFT)

- ❖ In EMFT, two pairs of coils and 16 electrodes are used.
- ❖ The aim is to reconstruct the velocity field based on the boundary voltage measurements created with applied  $\mathbf{B}$ -fields and the flowing conductive fluid.

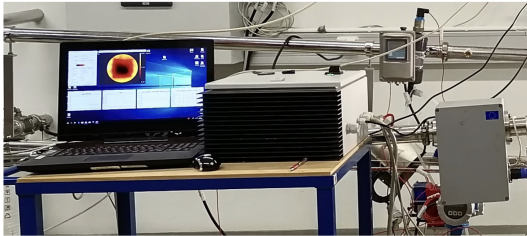


Fig 1. (a) The EMFT device.

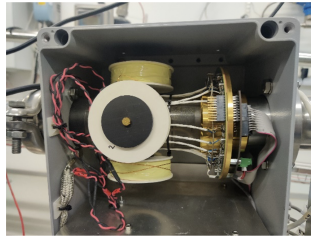
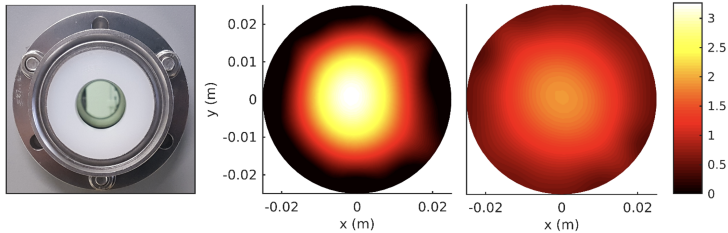


Fig 1. (b) EMFT sensors.

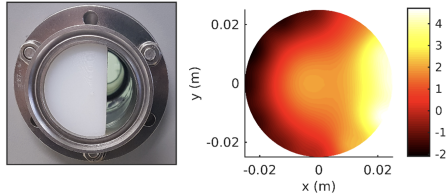


# Results, symmetric single-phase flows...



*Figure 2. The pipe with a hollow cylinder. Middle: The flow field reconstruction with the block inside the pipe. Right: The flow field reconstruction without the blockage.*

## ...and asymmetric single phase flow



*Figure 3. The pipe with a piece of plastic blocking 2/3 of the pipe diameter.  
Right: The flow field reconstruction with the blockage.*

# Outline

Applications of Electrical Impedance Tomography (EIT)

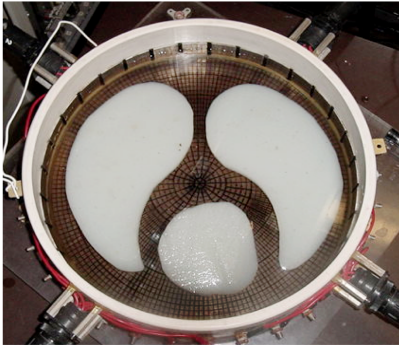
**The principle of EIT**

Non-uniqueness, ghosts, and ill-posedness

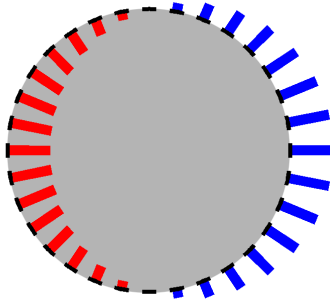
Regularized inversion for EIT

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



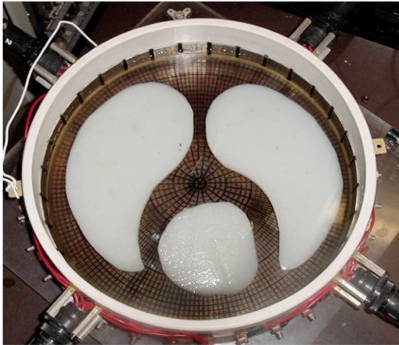
Apply current pattern  $\cos \theta$



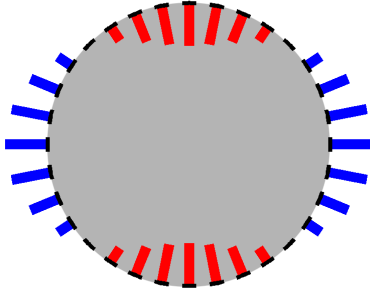
Measure the resulting voltages at the 32 electrodes

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



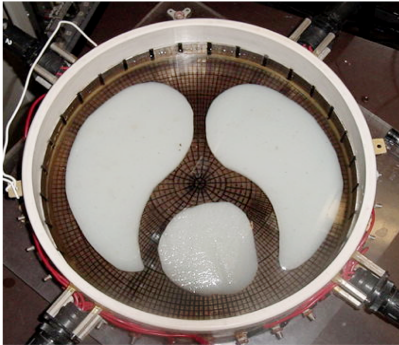
Apply current pattern  $\cos 2\theta$



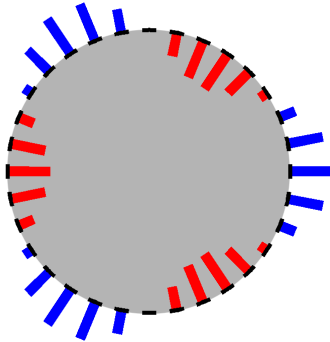
Measure the resulting voltages at the 32 electrodes

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



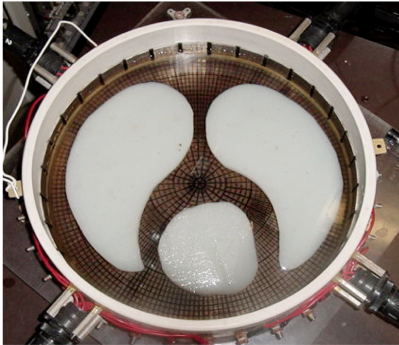
Apply current pattern  $\cos 3\theta$



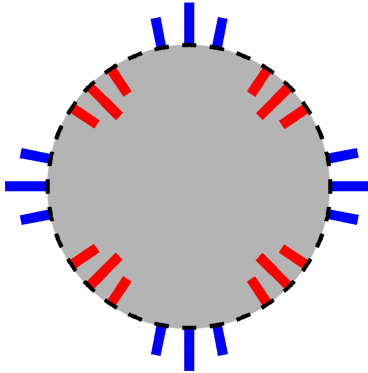
Measure the resulting voltages at the 32 electrodes

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



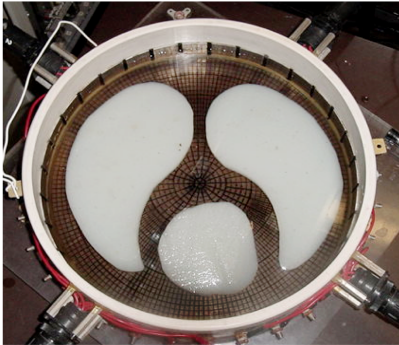
Apply current pattern  $\cos 4\theta$



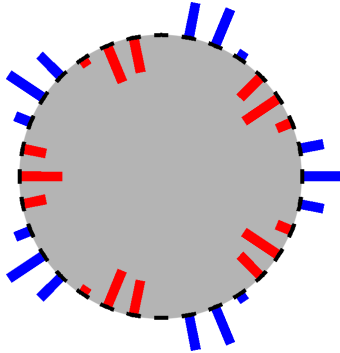
Measure the resulting voltages at the 32 electrodes

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



Apply current pattern  $\cos 5\theta$

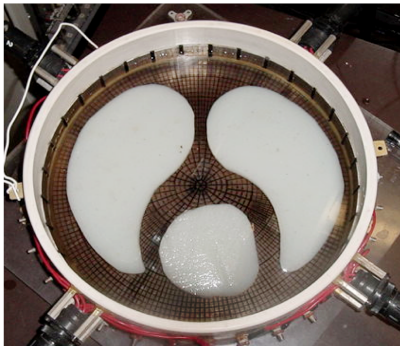


Measure the resulting voltages at the 32 electrodes

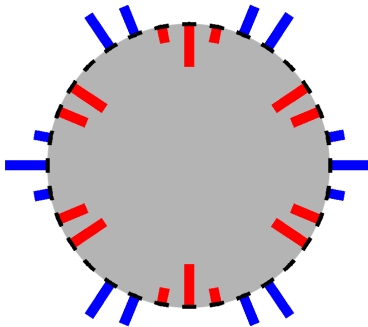


Note that EIT data collection involves applying several current patterns

Saline and agar phantom



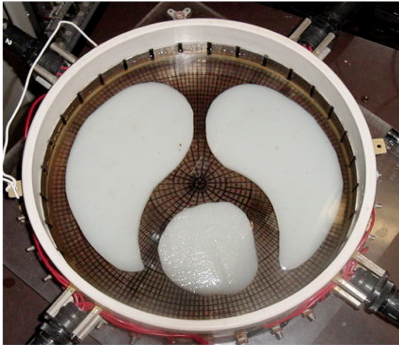
Apply current pattern  $\cos 6\theta$



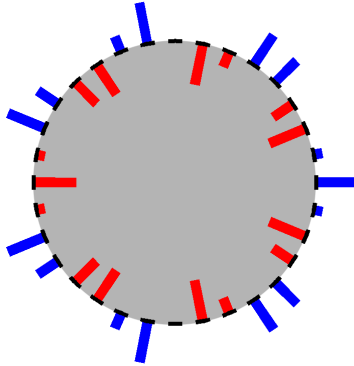
Measure the resulting voltages at the 32 electrodes

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



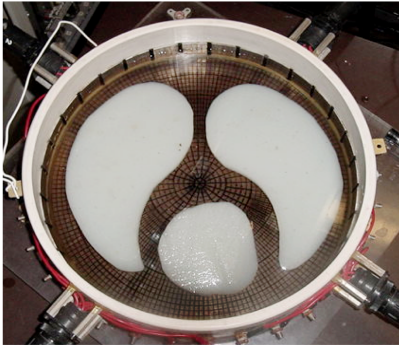
Apply current pattern  $\cos 7\theta$



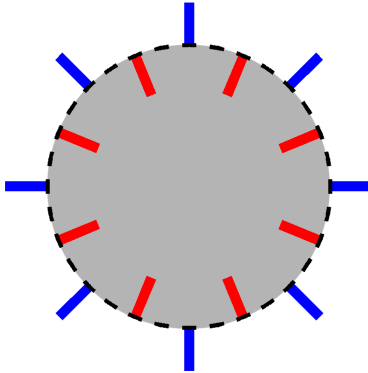
Measure the resulting voltages at the 32 electrodes

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



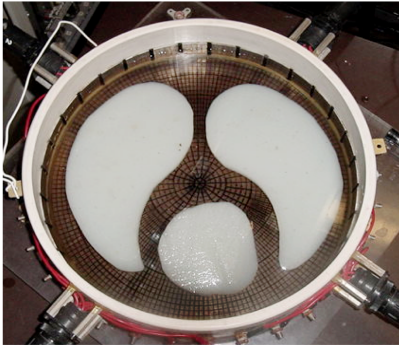
Apply current pattern  $\cos 8\theta$



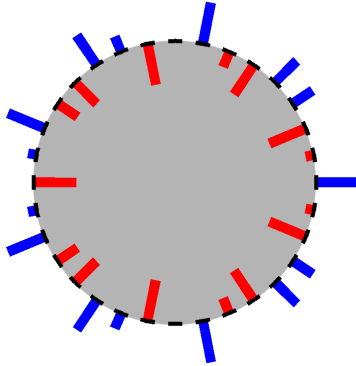
Measure the resulting voltages at the 32 electrodes

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



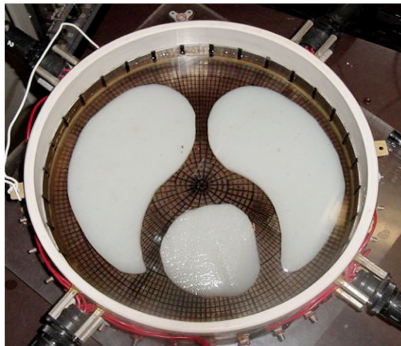
Apply current pattern  $\cos 9\theta$



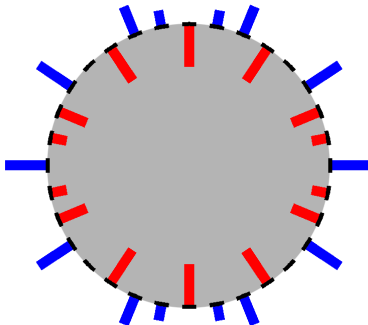
Measure the resulting voltages at the 32 electrodes

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



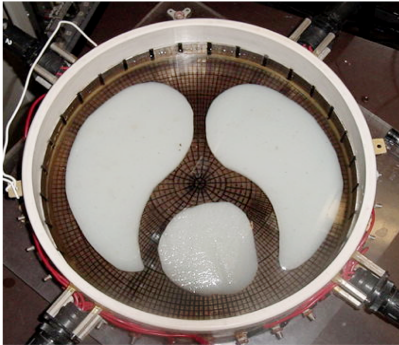
Apply current pattern  $\cos 10\theta$



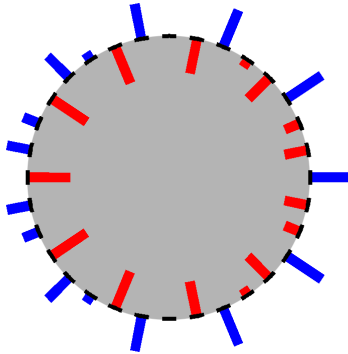
Measure the resulting voltages at the 32 electrodes

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



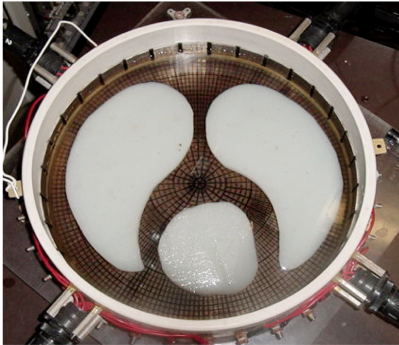
Apply current pattern  $\cos 11\theta$



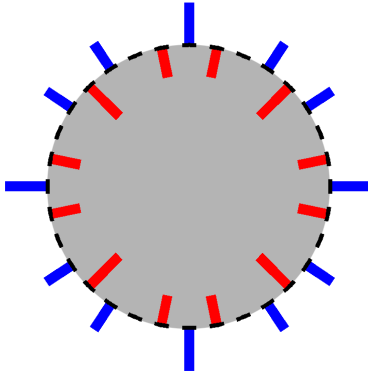
Measure the resulting voltages at the 32 electrodes

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



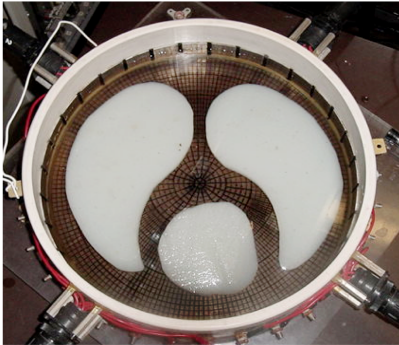
Apply current pattern  $\cos 12\theta$



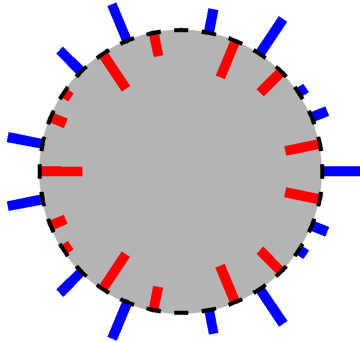
Measure the resulting voltages at the 32 electrodes

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



Apply current pattern  $\cos 13\theta$

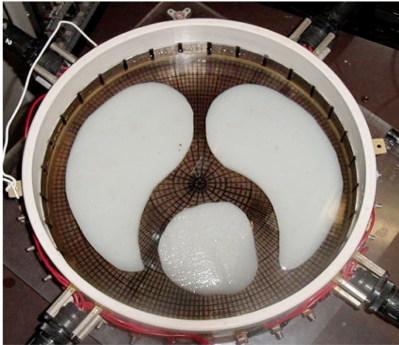


Measure the resulting voltages at the 32 electrodes

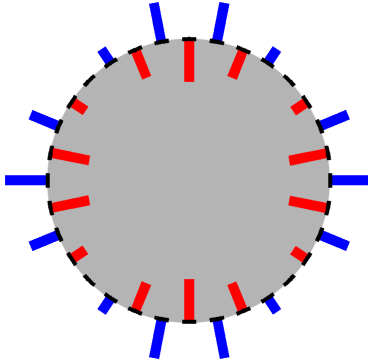


Note that EIT data collection involves applying several current patterns

Saline and agar phantom



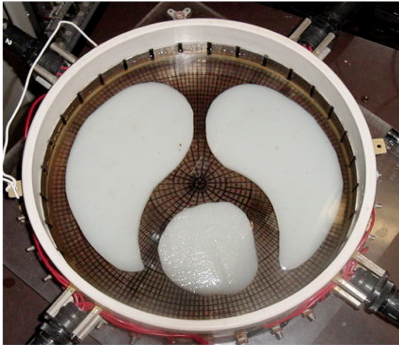
Apply current pattern  $\cos 14\theta$



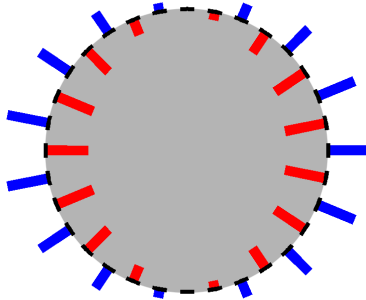
Measure the resulting voltages at the 32 electrodes

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



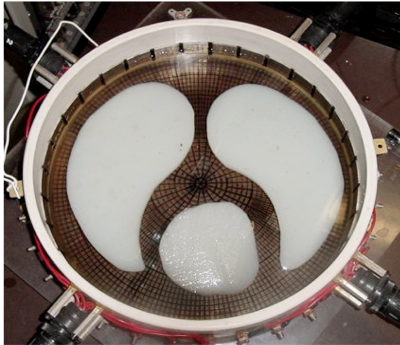
Apply current pattern  $\cos 15\theta$



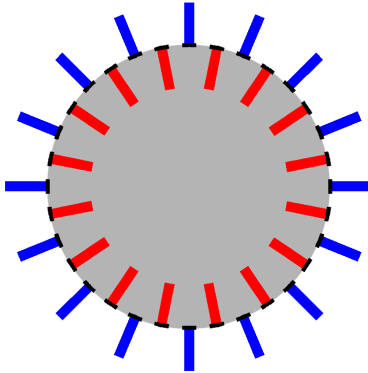
Measure the resulting voltages at the 32 electrodes

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



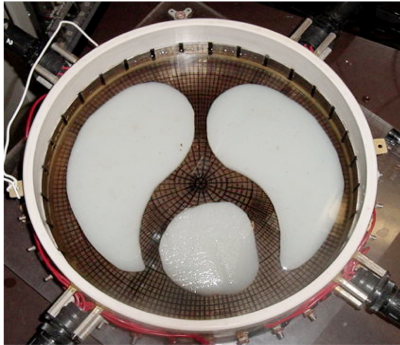
Apply current pattern  $\cos 16\theta$



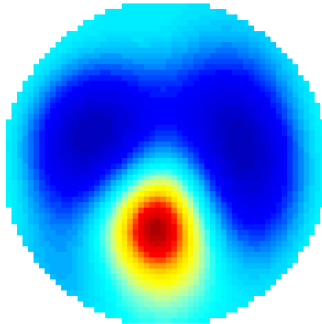
Measure the resulting voltages at the 32 electrodes

# The D-bar method works for real EIT data, such as laboratory phantoms and *in vivo* human data

Saline and agar phantom



Reconstruction ( $R = 4$ )



[Isaacson, Mueller, Newell & S 2004]

[Montoya 2012]

# The mathematical model of EIT is the inverse conductivity problem introduced by Calderón

Let  $\Omega \subset \mathbb{R}^2$  be the unit disc and let conductivity  $\sigma : \Omega \rightarrow \mathbb{R}$  satisfy

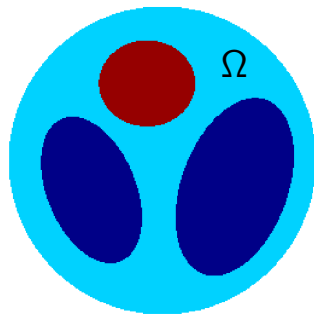
$$0 < M^{-1} \leq \sigma(z) \leq M.$$

Applying voltage  $f$  at the boundary  $\partial\Omega$  leads to the elliptic PDE

$$\begin{cases} \nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= f. \end{cases}$$

Boundary measurements are modelled by the Dirichlet-to-Neumann map

$$\Lambda_\sigma : f \mapsto \sigma \frac{\partial u}{\partial \vec{n}}|_{\partial\Omega}.$$



Calderón's problem is to recover  $\sigma$  from the knowledge of  $\Lambda_\sigma$ . It is a nonlinear and ill-posed inverse problem.

## Why is the forward map $F : \sigma \mapsto \Lambda_\sigma$ nonlinear?

Define a quadratic form  $\mathcal{P}_\sigma$  for functions  $f : \partial\Omega \rightarrow \mathbb{R}$  by

$$\mathcal{P}_\sigma(f) = \int_{\Omega} \sigma |\nabla u|^2 dz, \quad (1)$$

where  $u$  is the solution of the Dirichlet problem

$$\begin{cases} \nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= f. \end{cases}$$

Now the map  $\sigma \mapsto \mathcal{P}_\sigma$  is nonlinear because  $u$  depends on  $\sigma$  in (1). Physically,  $\mathcal{P}_\sigma(f)$  is the power needed for maintaining the voltage potential  $f$  on the boundary  $\partial\Omega$ .

Integrate by parts in (1):

$$\mathcal{P}_\sigma(f) = \int_{\partial\Omega} f \left( \sigma \frac{\partial u}{\partial \vec{n}} \right) ds = \int_{\partial\Omega} f (\Lambda_\sigma f) ds.$$

**Thus the map  $\sigma \mapsto \Lambda_\sigma$  cannot be linear in  $\sigma$ .**

# Outline

Applications of Electrical Impedance Tomography (EIT)

The principle of EIT

**Non-uniqueness, ghosts, and ill-posedness**

Regularized inversion for EIT

# Ill-posed inverse problems are defined as opposites of well-posed direct problems



Hadamard (1903): a problem is well-posed if the following conditions hold.

1. A solution exists,
2. The solution is unique,
3. The solution depends continuously on the input.

**Well-posed direct problem:**

Input  $\sigma$ , find infinite-precision data  $\Lambda_\sigma$ .

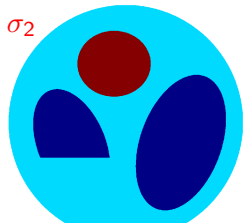
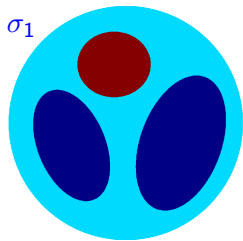
**Ill-posed inverse problem:**

Input noisy data  $\Lambda_\sigma^\delta$ , reconstruct  $\sigma$ .

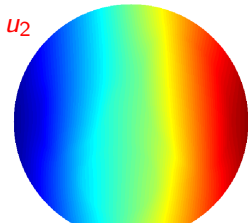
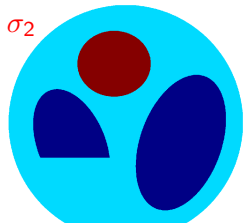
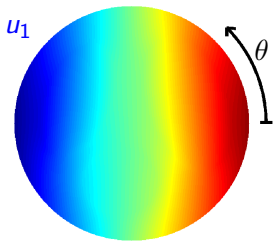
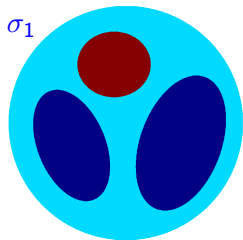
*J. Hadamard*



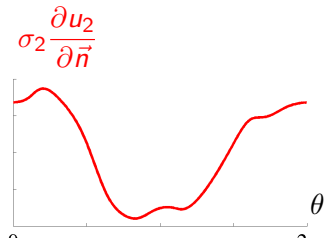
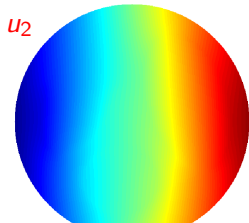
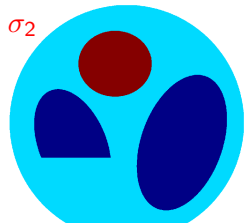
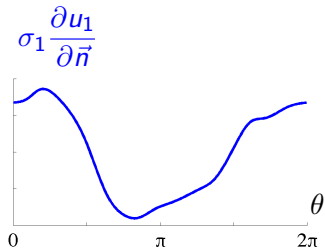
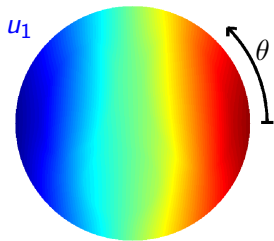
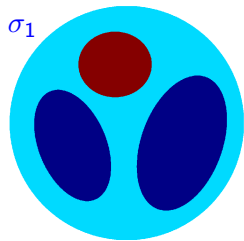
We illustrate the ill-posedness of EIT  
using a simulated example



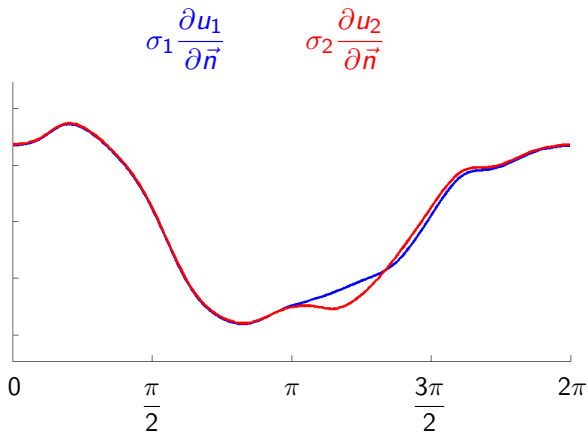
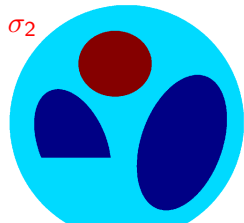
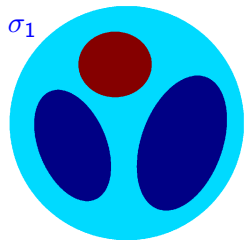
We apply the voltage distribution  $f(\theta) = \cos \theta$  at the boundary of the two different phantoms



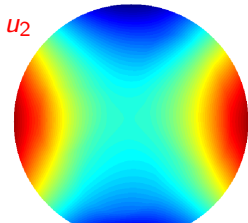
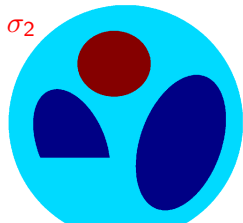
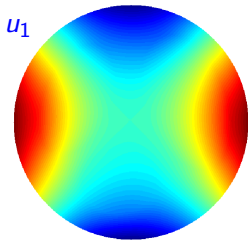
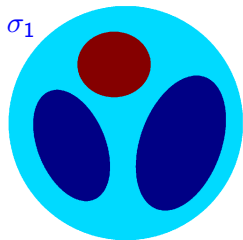
The measurement is the distribution of current through the boundary



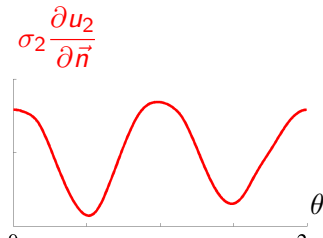
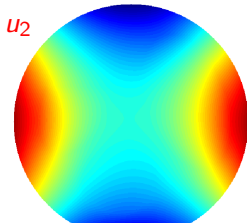
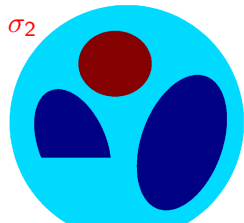
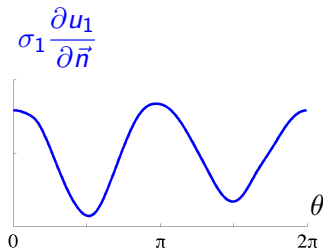
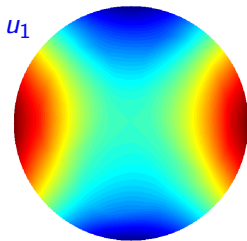
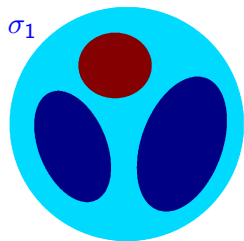
The current data are very similar,  
although the conductivities are quite different



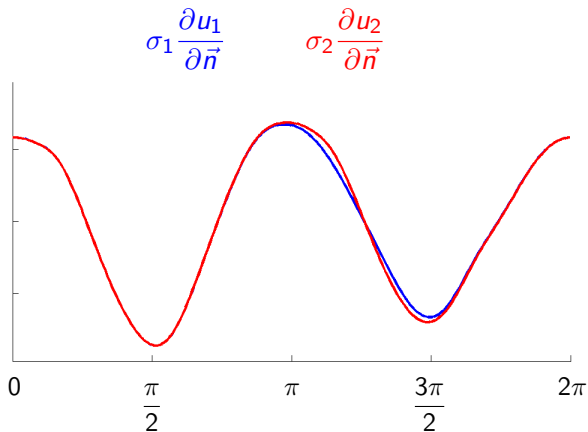
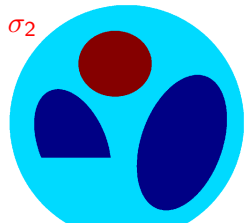
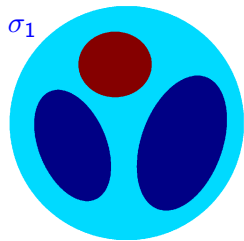
Let us apply the more oscillatory distribution  $f(\theta) = \cos 2\theta$  of voltage at the boundary



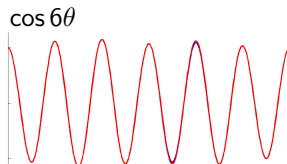
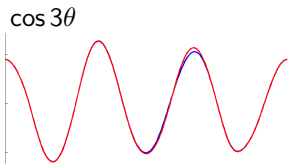
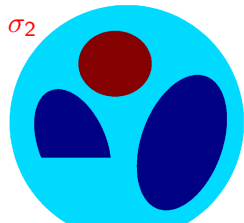
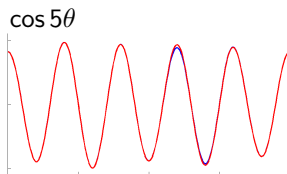
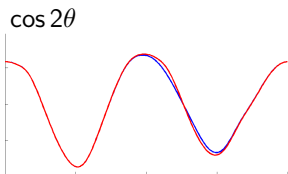
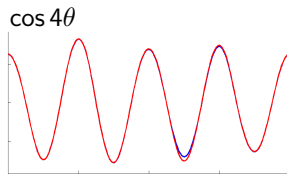
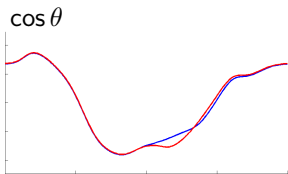
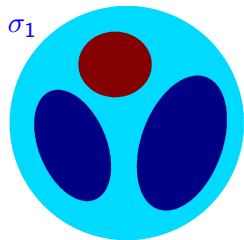
The measurement is again the distribution of current through the boundary



The current distribution measurements  
are almost the same

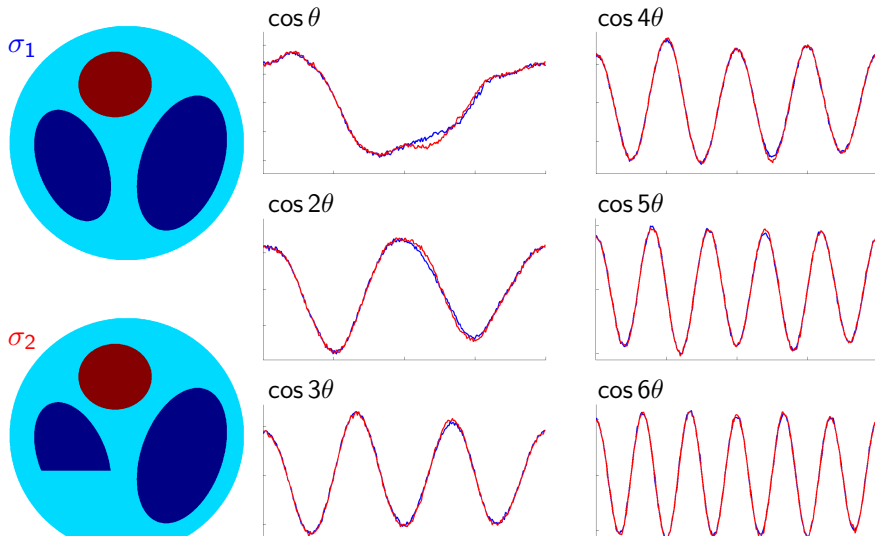


EIT is an ill-posed problem: big differences in conductivity cause only small effect in data



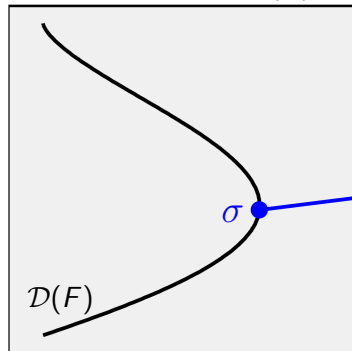


EIT is an ill-posed problem: noise in data causes serious difficulties in interpreting the data



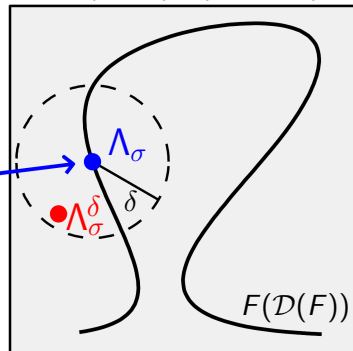
The forward map  $F : X \supset \mathcal{D}(F) \rightarrow Y$   
does not have a continuous inverse!

Model space  $X = L^\infty(\Omega)$



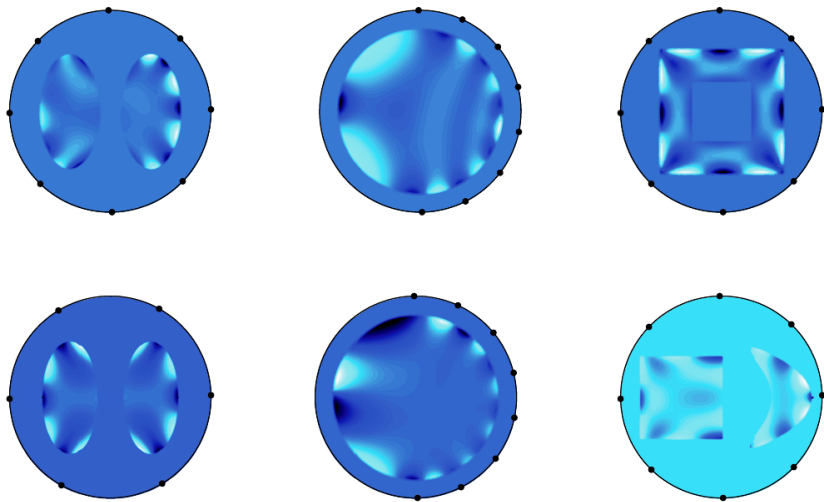
Data space

$Y = L(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$



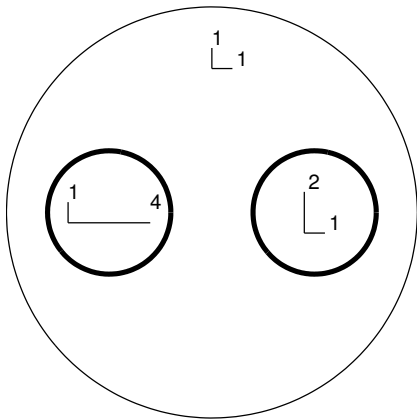
Furthermore, the noisy data  $\Lambda_\sigma^\delta$  does not belong to the range  $F(\mathcal{D}(F))$ . So Hadamard's conditions 1 and 3 fail for EIT. How about uniqueness?

# Ghosts, or invisible structures, when using point electrodes in electrical impedance tomography



[Chesnel, Hyvönen & Staboulis 2014]

# Anisotropic, or matrix-valued, conductivities $\sigma$ lead to non-uniqueness in EIT



Let  $\sigma(x) = [\sigma^{ij}(x)]$  be a symmetric and positive-definite  $2 \times 2$  matrix.

Define anisotropic DN map by

$$\Lambda_{\sigma}(f) = \nu \cdot \sigma \nabla u|_{\partial\Omega}.$$

Let  $F : \Omega \rightarrow \Omega$  be a diffeomorphism with  $F|_{\partial\Omega} = \text{Identity}$ . Then

$\Lambda_{F_*\sigma} = \Lambda_{\sigma}$ ,  
where  $F_*$  is the push-forward:

$$(F_*\sigma)^{ij}(y) = \frac{1}{\det \left[ \frac{\partial F^i}{\partial x^j}(x) \right]} \sum_{p,q=1}^2 \frac{\partial F^i}{\partial x^p}(x) \frac{\partial F^j}{\partial x^q}(x) \sigma^{pq}(x) \Big|_{x=F^{-1}(y)}$$

# Outline

Applications of Electrical Impedance Tomography (EIT)

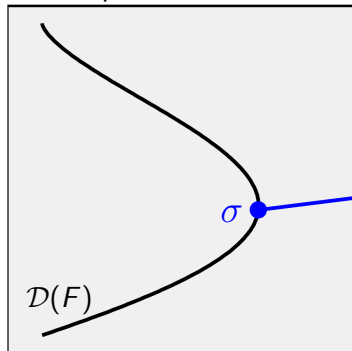
The principle of EIT

Non-uniqueness, ghosts, and ill-posedness

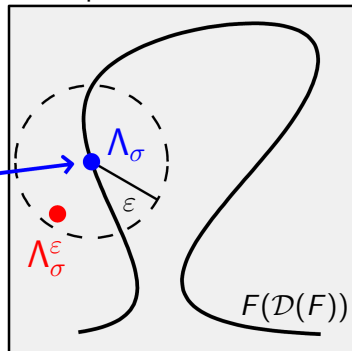
**Regularized inversion for EIT**

The forward map  $F : X \supset \mathcal{D}(F) \rightarrow Y$  of an ill-posed problem does not have a continuous inverse

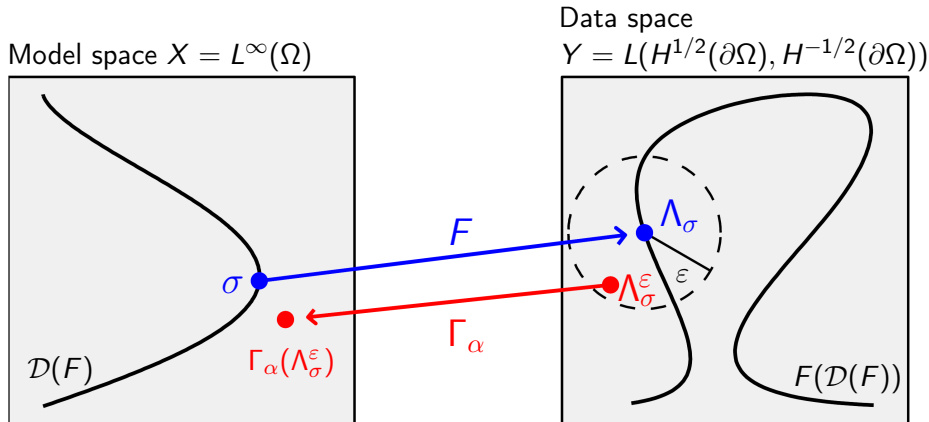
Model space  $X$



Data space  $Y$



Regularization means constructing a continuous map  $\Gamma_\alpha : Y \rightarrow X$  that inverts  $F$  approximately



Regularization must be based on combining the incomplete measurement data with *a priori* information about the conductivity.

# There are three major methodologies for solving nonlinear inverse problems

## 1. Variational regularization.

- + The same code applies to many problems.
- Repeated solution of direct problem needed.
- Can get stuck in local minima.

## 2. Direct nonlinear inversion.

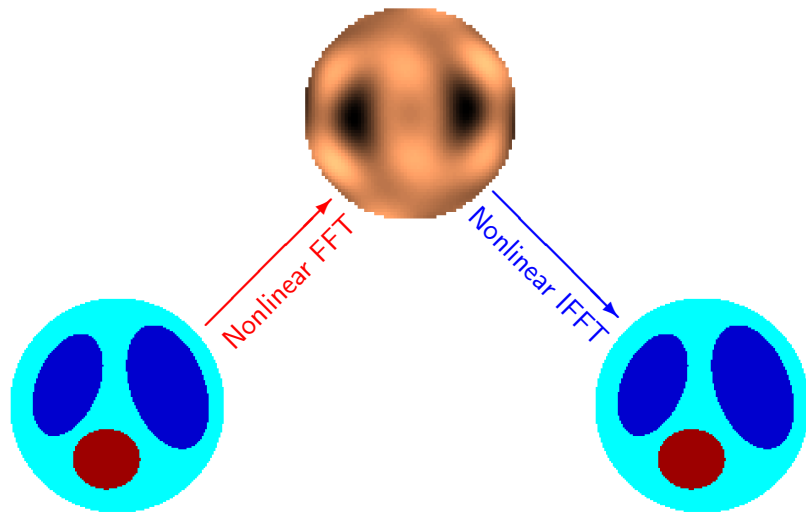
- + Can deal efficiently with a specific nonlinearity.
- Each formula applies to only one inverse problem.

## 3. Bayesian inversion (not covered in this course).

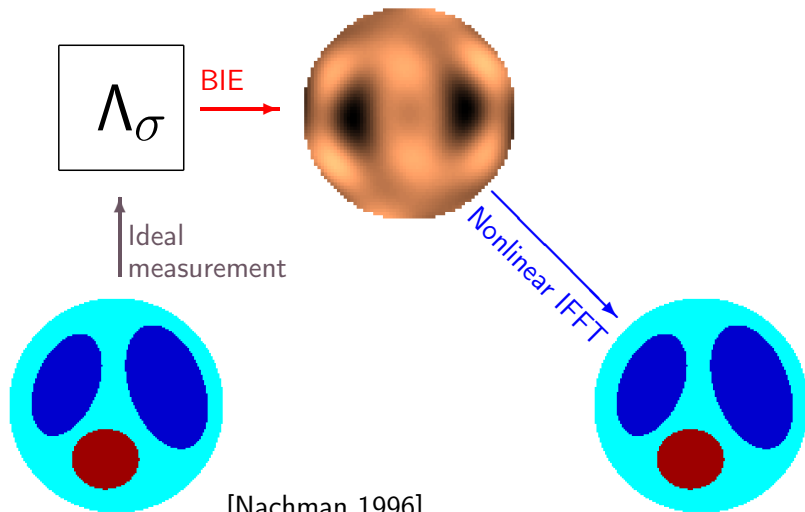
- + Very flexible framework.
- + Includes uncertainty quantification.
- Computationally heavy.



There exists a nonlinear Fourier transform adapted to electrical impedance tomography

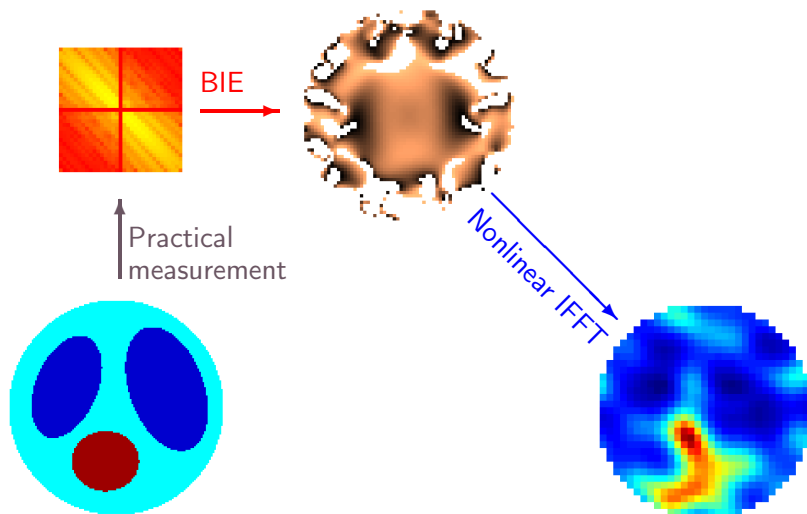


The nonlinear Fourier transform can be recovered from infinite-precision EIT measurements

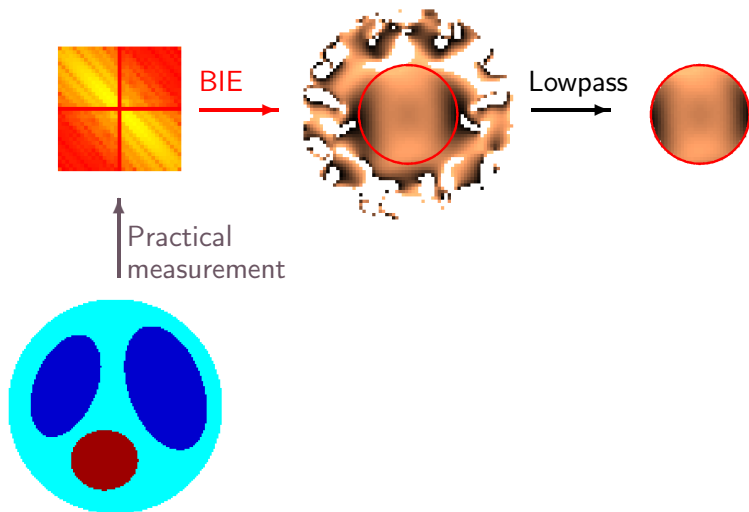


[Nachman 1996]

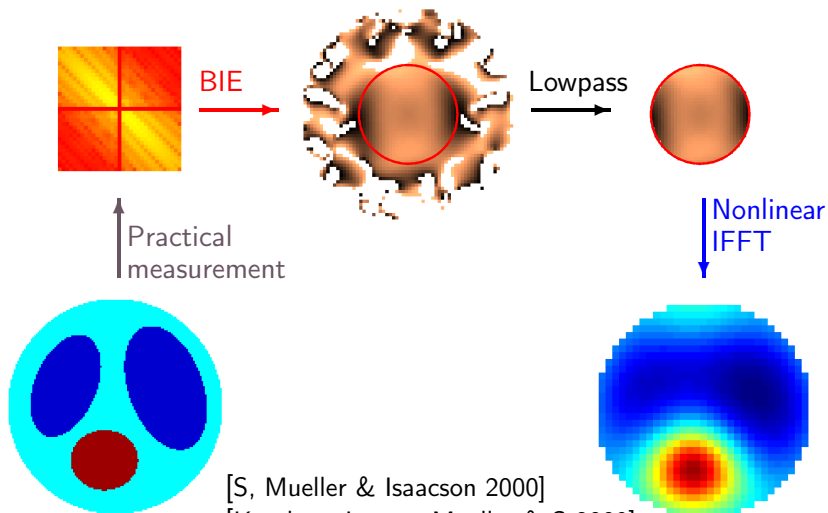
# Measurement noise prevents the recovery of the nonlinear Fourier transform at high frequencies



We truncate away the bad part in the transform; this is a nonlinear low-pass filter



There is currently only one regularized method for reconstructing the full conductivity distribution



[S, Mueller & Isaacson 2000]

[Knudson, Lassas, Mueller & S 2000]