Mathematics of Electrical Impedance Tomography Exercises

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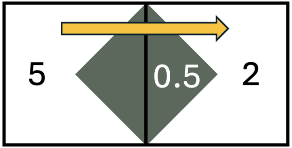
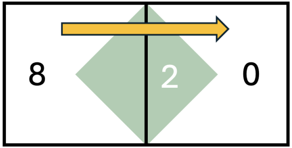
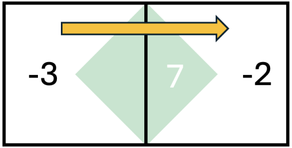
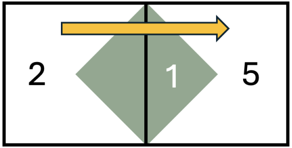
Exercise 1.1: Currents between pixels

Calculate the current flowing from the left pixel to the right pixel in the examples shown. Note that the result can be a negative number. In that case the direction of current flow is against the arrow direction.

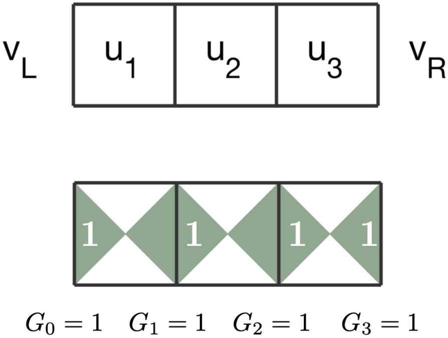








Exercise 1.2: Matrix model for a general 3-pixel example



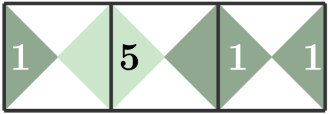
In the lecture we saw how to derive the matrix equation for the homogeneous diffusion model on the left:

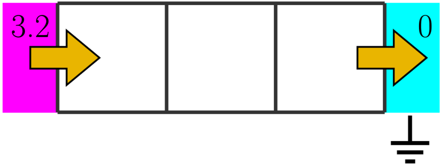
$$A\vec{v} = \left[\begin{array}{rrrrr} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ \hline 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right] \left[\begin{array}{c} v_L \\ u_1 \\ u_2 \\ u_3 \\ v_R \end{array}\right] = \left[\begin{array}{c} I_L \\ I_R \\ \hline 0 \\ 0 \\ 0 \end{array}\right].$$

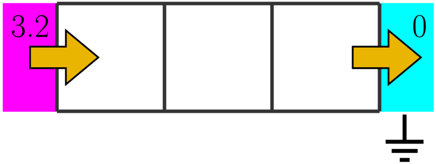
Form the matrix A in the general case when all the four conductances G_j with $j = 0, 1, 2, 3$ are variables.

Exercise 1.3 (Matlab): another example of nonuniqueness

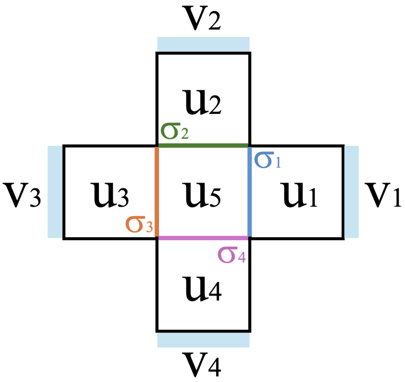
Modify the code in `FD_EIT_1D_demo_case3.m` to find a third conductivity distribution inside the diffusion model that results in the same voltage measurement than the two examples below.







Exercise 2.1: EIT forward problem in the "plus" model



Electrodes are marked as light blue rectangles.

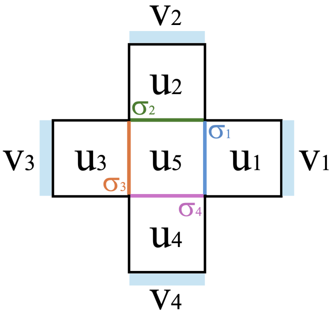
Assume the following values for variables:

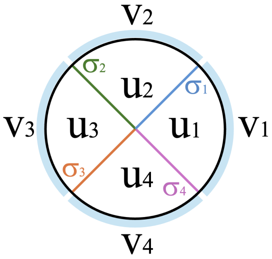
- $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 1$,
- $v_1 = 0$, $v_2 = 2$, $v_3 = 6$, $v_4 = 8$,
- $u_1 = 2$, $u_2 = 3$, $u_3 = 5$, $u_4 = 6$.

Use the balance equations to find the answers to these three questions:

1. What is the voltage u_5?
2. How much current flows in and out of pixel 5 through its four interfaces?
3. What are the currents through electrodes?

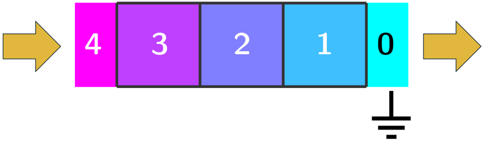
Exercise 2.2: EIT forward problem in small 2D models

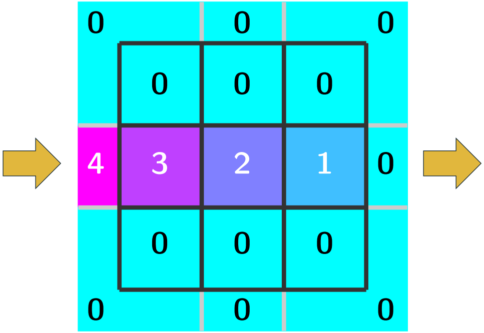




For both of the models above, take $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 1$ and use the grounding $v_1 = 0$. Consider the current pattern $[-2, 0, 2, 0]^T$. What is the measured voltage vector $[v_1, v_2, v_3, v_4]^T$ in each case? How much current flows through the interface with conductance σ_4 in each case?

Exercise 2.3: difference between 1D and 2D





Recall the one-dimensional homogeneous diffusion model shown in top left.

Inspired by the straight-shooting X-rays, one could imagine the two-dimensional homogeneous diffusion model to have the solution shown at bottom left, where the electric current is simply traveling through the middle squares.

Your task is to find out why the numbers in the 2D model are non-physical. Find at least one pixel where the balance equation fails.

Exercise 2.4: linearized reconstruction

Run the Matlab routine `FD_EIT_2D_demoE_recon9x9largecontrast_comp.m`. The final graph that shows up, entitled *True conductances (black) and reconstruction (red)*, shows that the reconstruction has some negative values and cannot quite reach the high conductance values of 9 in the conductive inclusion.

Adjust the conductance values in rows

```
condmatH(1,:)  = .4;
condmatV(1,:)  = .4;
condmatH(end,:)  = 9;
condmatV(end,:)  = 9;
```

closer towards 1 so that the reconstructed values are within 10% of the true conductances. How close to one do you need to take them?

You can also modify the threshold value 0.1 in the command

```
recon2 = pinv(Jcbn,.1)*(Uadj-Uadj0);
```

Exercise 3.1: harmonic function z^n

Recall the Laplace operator for functions $f(x, y)$ in dimension two:

$$\nabla \cdot \nabla f = \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

Functions that satisfy $\Delta f = 0$ are called *harmonic*. Consider the polar coordinates $x = r\cos\theta, y = r\sin\theta$, leading to

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan(\frac{y}{x}).$$

Show that the Laplacian in polar coordinates is given by

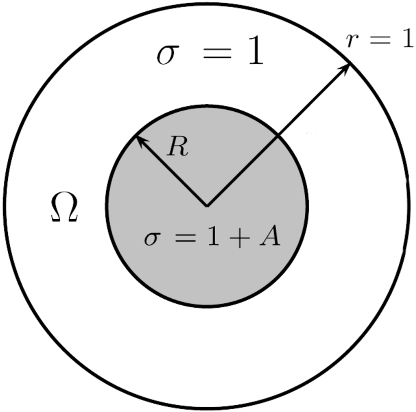
$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial f^2}{\partial \theta^2}.$$

Prove that z^n is harmonic for all $z \in \mathbb{C}$, where $z = (x + iy) = re^{i\theta}$ and $n > 0$. For which $z \in \mathbb{C}$ is z^n harmonic when $n < 0$?

Exercise 3.2: fill in the left half of L_1

$$L_1 = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}$$

Exercise 3.3: checking the jump-conductivity calculation



Write the solution of

$$\begin{cases} \nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= \varphi_n, \end{cases}$$

in the form $u = u^{\text{int}}(r, \theta)$ for $r < R$ and $u = u^{\text{ext}}(r, \theta)$ for $R < r < 1$. Then

$$\begin{aligned} u^{\text{ext}}(r, \theta) &= a_n r^{|n|} e^{in\theta} + b_n r^{-|n|} e^{in\theta}, \\ u^{\text{int}}(r, \theta) &= c_n r^{|n|} e^{in\theta}. \end{aligned}$$

Compute a_n and c_n.

Choose some $A > 0$ and $0 < R < 1$. Take $n = 3$. Plot the real part of u.

Exercise 3.4: small difference in eigenvalues

Let σ be as in Exercise 3.3. Then we have

$$(\Lambda_\sigma - \Lambda_1)\varphi_n = (\lambda_n - |n|)\varphi_n = -\frac{2|n|}{1-(1+\frac{2}{A})R^{-2|n|}}\varphi_n.$$

- What happens in the right hand side of the above equation when $R \to 0$?
- Choose some $A > 0$ and $0 < R < 1$. Calculate the multiplier of φ_n in the right hand side of the above equation for $n = 1, \ldots, 25$.
- Choose a smaller $R > 0$ and repeat the computation. Compare the multipliers.

Exercise 3.5: enclosure method

Recall the definition of the indicator function:

$$I_\omega(\tau) = \langle(\Lambda_\sigma - \Lambda_1)\overline{f_\omega(\,\cdot\,;\tau)}, f_\omega(\,\cdot\,;\tau)\rangle, \qquad f_\omega(x;\tau) := e^{\tau x\cdot\omega + i\tau x\cdot\omega^\perp}.$$

Use Fourier series to derive

$$I_\omega(\tau) = \int_0^{2\pi} \sum_{m=0}^{\infty} \frac{\tau^m\omega^m}{m!} e^{-im\theta} \sum_{n=0}^{\infty} \frac{\tau^n\overline{\omega}^n}{n!}(\lambda_n - |n|)e^{in\theta}\,d\theta.$$

Let σ be as in Exercise 3.3. Choose some $A > 0$ and $0 < R < 1$; then $h_D(\omega) = R$ for every direction ω. Evaluate the indicator function numerically for $\tau = 10, 11, 12, \ldots, 20$. What do you observe about the theoretical convergence

$$h_D(\omega) = \lim_{\tau\longrightarrow\infty} \frac{\log|I_\omega(\tau)|}{2\tau}?$$

Exercise 4.1: accuracy of Finite Element Method

Take the 25 eigenvalues you calculated in Exercise 3.4. Put the same A value and radius R in the file `FEMconductivity.m` and calculate the same eigenvalues using FEM. How many eigenvalues can you distinguish from those of conductivity one?

Try increasing the `Nrefine` from 5 to 6 or more in routine `FEM_DN02_mesh_comp.m`. Can you distinguish more eigenvalues with the improved accuracy?