Electrical Impedance Tomography: Computation and Applications

Part 9: D-bar Reconstruction

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D-bar Turns a Uniqueness Proof into an Algorithm

The D-bar reconstruction method comes from Adrian Nachman's 1996 global uniqueness proof for the two-dimensional inverse conductivity problem.

The proof was *constructive*: it described a sequence of mathematical steps that recover the conductivity from the DN map $\Lambda_\sigma : v \mapsto j$.

The resulting D-bar method is

- direct and non-iterative,
- nonlinear, and
- based on complex geometrical optics (CGO) solutions and a special nonlinear Fourier analysis.

Today we will focus on the computational path, not the deep mathematics.

Reference: A. I. Nachman, Annals of Mathematics 143 (1996), 71–96.

The Reconstruction Pipeline

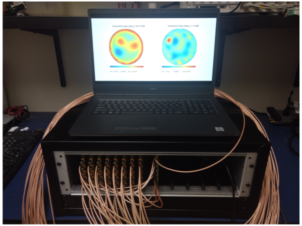
We will consider the case of D-bar difference imaging, beginning once again with current and voltage data from a target and a homogeneous reference.

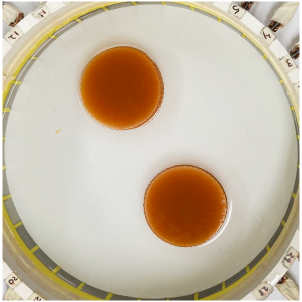
$$[\text{currents}, \text{voltages}] \underset{①}{\longrightarrow} \Delta\boldsymbol{\Lambda} \underset{②}{\longrightarrow} \mathbf{t}^{\text{exp}}(k) \underset{③}{\longrightarrow} \mu(z,k) \underset{④}{\longrightarrow} \sigma(z).$$

1. Process EIT data into a discrete *difference* DN map $\Delta\boldsymbol{\Lambda}$.
2. Use $\Delta\boldsymbol{\Lambda}$ to construct a Born-type approximation $\mathbf{t}^{\text{exp}}(k)$ to a nonlinear Fourier transform in a nonphysical frequency variable k.
3. $\mathbf{t}^{\text{exp}}(k)$ is used as a parameter in a D-bar equation. We solve this to obtain $\mu(z,k)$, which is directly related to the conductivity $\sigma(z)$.
4. Restrict to $k = 0$ to find the final reconstruction in the spatial z-domain.

Today's Data: A 32-Electrode ACT5 Tank Experiment

- We will be using data from the ACT5 machine at Colorado State University.
- The target: two conductive agar disks in a saline background.
- Trigonometric current patterns were applied on 32 injection electrodes.
- Resulting single-ended voltages were measured on all 32 electrodes.
- Reference measurements are from the same tank containing homogeneous saline.





Images: Shishvan et al. 2021 (top), Colorado State University EIT Laboratory (bottom)

Zero-Mean Current Pattern Space

- D-bar reconstruction requires approximating the Dirichlet-to-Neumann map $\Lambda_\sigma : v \mapsto j$, which necessitates a complete basis of current patterns and corresponding voltages.
- For L electrodes, current injections must satisfy $\sum_{\ell=1}^{L} I_\ell = 0$, so the space of possible current patterns has dimension $L - 1$.
- Additional patterns *can* be used, but their measurements must first be combined into an improved estimate of the same DN map (e.g. by averaging repeated patterns or using least squares).

The Trigonometric Current Basis

Assume L is even. For each electrode e_ℓ centered at angle θ_ℓ, $\ell = 1, \dots L$, we have $L - 1$ independent current injections of the form

$$I_\ell^{(n)} = \begin{cases} I_0 \cos(n\theta_\ell), & n = 1, \dots, L/2, \\ I_0 \sin\left((n - L/2)\theta_\ell\right), & n = L/2 + 1, \dots, L - 1. \end{cases}$$

Let $\mathbf{I}^{(n)} = [I_1^{(n)}, \dots, I_L^{(n)}]^\top$ be the n^{th} current pattern.
Normalize and collect these patterns into a matrix Φ with orthonormal columns:

$$\varphi^{(n)} = \frac{\mathbf{I}^{(n)}}{\|\mathbf{I}^{(n)}\|_2}, \qquad \Phi = [\varphi^{(1)}, \dots, \varphi^{(L-1)}] \in \mathbb{R}^{L \times (L-1)}.$$

Forming the Discrete ND Map

Let the matrix

$$\mathsf{V}_\sigma = [\mathbf{V}_\sigma^{(1)}, \ldots, \mathbf{V}_\sigma^{(L-1)}]$$

contain the zero-mean voltage response to each normalized current pattern.

The discrete Neumann-to-Dirichlet (current-to-voltage) matrix is then

$$\boxed{\mathbf{R}_\sigma = \Phi^\top \mathsf{V}_\sigma} \in \mathbb{R}^{(L-1)\times(L-1)}$$

The discrete Dirichlet-to-Neumann (voltage-to-current) matrix is the inverse map:

$$\boxed{\mathbf{\Lambda}_\sigma = \mathbf{R}_\sigma^{-1}} \in \mathbb{R}^{(L-1)\times(L-1)}$$

Compute DN maps for both the target and the homogeneous reference and then subtract to form a difference DN map:

$$\Delta\mathbf{\Lambda}^{\mathrm{meas}} = \mathbf{\Lambda}_\sigma^{\mathrm{meas}} - \mathbf{\Lambda}_{\mathrm{ref}}^{\mathrm{meas}}.$$

Normalize Before Using the D-bar Equations

Our D-bar implementation takes the domain to be the unit disk $\mathbb{D}$ with reference conductivity $\sigma \equiv 1$. Our code therefore:

1. rescales the circular tank to $\mathbb{D}$,
2. divides conductivity by the measured saline value σ_{ref}, and
3. applies the corresponding geometric and electrode scaling:

$$\boxed{\Delta\boldsymbol{\Lambda} = \frac{r_\Omega\,\Delta\theta}{A_e\,\sigma_{\mathrm{ref}}}\Delta\boldsymbol{\Lambda}^{\mathrm{meas}}}$$

where

- $r_\Omega =$ real-world tank radius,
- $\Delta\theta =$ angular distance between electrode centers,
- $A_e =$ area of each electrode.

So now spatial points are $z \in \mathbb{D}$ and the homogeneous reference conductivity is 1.

A Crucial Change of Variables

Begin with the conductivity equation

$$\nabla \cdot (\sigma \nabla u) = 0.$$

Define

$$q = \frac{\Delta \sqrt{\sigma}}{\sqrt{\sigma}}, \qquad \widetilde{u} = \sqrt{\sigma}\, u.$$

Here Δ is the spatial Laplacian; it does not denote a conductivity difference.

Then $\widetilde{u}$ satisfies a zero-energy Schrödinger equation:

$$\boxed{(-\Delta + q)\widetilde{u} = 0.}$$

The unknown conductivity is now encoded in the potential q.

Write spatial domain points as $z = x + iy$, and introduce $k = k_1 + ik_2$.
Assume $\sigma \equiv 1$ near $\partial\mathbb{D}$ and extend it as 1 outside $\mathbb{D}$.

For each nonzero $k \in \mathbb{C}$, we seek a special *complex geometrical optics* (CGO) solution satisfying

$$(-\Delta + q)\psi(z,k) = 0, \qquad \psi(z,k) \sim e^{ikz} \quad \text{as } |z| \to \infty.$$

We will also be interested in the related function

$$\boxed{\mu(z,k) = e^{-ikz}\psi(z,k).}$$

The parameter k is nonphysical but plays a role analogous to frequency.

A Special Nonlinear Fourier Transform

The *scattering transform* is defined by

$$\boxed{\mathbf{t}(k) = \int_{\mathbb{R}^2} e^{i\bar{k}\,\bar{z}}\, q(z)\psi(z,k)\, dz.}$$

If we replace $\psi(z,k)$ by its leading behavior e^{ikz}, then

$$\mathbf{t}(k) \approx \int_{\mathbb{R}^2} e^{i(kz+\bar{k}\,\bar{z})} q(z)\, dz,$$

which is an ordinary Fourier transform of q, up to a rotation and scaling of frequency coordinates.

Because ψ itself depends on q, the exact transform $q \mapsto \mathbf{t}$ is nonlinear.

Connecting $\mathbf{t}(k)$ to the Boundary Data

By [Alessandrini, 1988] the scattering transform may be written equivalently as a boundary integral

$$\mathbf{t}(k) = \int_{\partial\mathbb{D}} e^{i\bar{k}\,\bar{z}}\, \delta\Lambda\, \psi(\cdot, k)\, ds.$$

Here

$$\delta\Lambda = \Lambda_\sigma - \Lambda_0,$$

where Λ_0 is the DN map corresponding to $\sigma \equiv 1$ in $\mathbb{D}$.

After normalization, our $\mathbf{\Lambda}_{\text{ref}}$ becomes an approximation for Λ_0.

This provides the connection to the EIT data.

The $\mathbf{t}^{\exp}$ Approximation

The exact boundary formula for $\mathbf{t}(k)$ contains the unknown CGO trace $\psi(\cdot, k)|_{\partial\mathbb{D}}$.

The Born-type approximation replaces that trace by its leading exponential:

$$\psi(z, k) \approx e^{ikz}, \qquad z \in \partial\mathbb{D}.$$

This gives

$$\boxed{\mathbf{t}^{\exp}(k) = \int_{\partial\mathbb{D}} e^{i\bar{k}\,\bar{z}}\, \delta\Lambda\, e^{ikz}\, ds(z).}$$

In our discrete formulation, we further replace $\delta\Lambda$ by $\boldsymbol{\Delta\Lambda}$.

Our code expresses both boundary exponential functions in the same trigonometric basis used to represent $\boldsymbol{\Delta\Lambda}$. The boundary integral can then be approximated by matrix multiplication involving $\boldsymbol{\Delta\Lambda}$.

The D-bar Equation

Our approximation $\mathbf{t}^{\exp}(k)$ now encodes the EIT data. What next?

For each fixed spatial point z, the function $\mu(z,k) = e^{-ikz}\psi(z,k)$ satisfies the *D-bar equation*

$$\boxed{\overline{\partial}_k \mu(z,k) = \frac{\mathbf{t}(k)}{4\pi \overline{k}}\, e^{-i(kz+\overline{k}\,\overline{z})}\overline{\mu(z,k)}} \qquad \text{where} \qquad \overline{\partial}_k = \frac{1}{2}\left(\frac{\partial}{\partial k_1} + i\frac{\partial}{\partial k_2}\right).$$

We see $\mathbf{t}(k)$ acts as known data in this equation.

Written in an equivalent integral form:

$$\boxed{\mu(z,k) = 1 + \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{\mathbf{t}(\kappa)}{(k-\kappa)\overline{\kappa}}\, e^{-i(\kappa z+\overline{\kappa}\,\overline{z})}\overline{\mu(z,\kappa)}\, dA(\kappa)}$$

We substitute $\mathbf{t}^{\exp}$ for $\mathbf{t}$ into the above equation. This integral, once discretized, may be written as a real-linear matrix system.

One Value of μ Gives the Conductivity

Nachman's reconstruction formula is remarkably simple:

$$\sqrt{\sigma(z)} = \lim_{k \to 0} \mu(z, k).$$

Thus, in normalized variables,

$$\boxed{\sigma(z) = \mu(z, 0)^2.}$$

Practical Steps

The basic computational steps involved in D-bar are:

1. Discretize the spatial domain into a set of points $z_j \in \mathbb{D}$. Since no FEM is required, we can use a rectangular pixel grid.
2. Discretize the spectral k-plane on a separate rectangular grid.
3. Compute $\mathbf{t}^{\exp}(k)$ on the k-grid using the EIT data stored in $\Delta\boldsymbol{\Lambda}$.
4. Solve the D-bar integral equation at each z_j, yielding an approximation for $\mu(z,k)$.
5. Recover the conductivity: $\sigma(z) = \mu^2(z,0)$.

   (Our even k-grid omits $k=0$, so we approximate $\mu(z,0)$ by averaging the four central grid values.)

Our implementation adds two optional final steps:

- Restore physical units using $\sigma_{\mathrm{physical}}(z) = \sigma_{\mathrm{ref}}\, \mu(z,0)^2$ $[\mathrm{S/m}]$, where σ_{ref} was the measured conductivity of the saline in the tank.
- Interpolate the reconstruction onto a finer grid for display.

Our FFT-based Implementation

There are various approaches for solving the D-bar equation

$$\boxed{\mu(z,k) = 1 + \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{\mathbf{t}^{\exp}(\kappa)}{(k-\kappa)\overline{\kappa}}\, e^{-i(\kappa z + \overline{\kappa}\,\overline{z})}\overline{\mu(z,\kappa)}\, dA(\kappa).}$$

Our implementation uses the FFT-based method introduced in [Knudsen et al. 2004]:

First form the z-independent parts of the integrand.

Then, for each z_j:

1. Form the z-dependent phase $e^{-i(\kappa z + \overline{\kappa}\,\overline{z})}$.
2. Evaluate the convolution using `fft2` and `ifft2`.
3. Stack real and imaginary parts of $\mu(z,k)$ into one vector and solve the discrete real-linear system using a matrix-free linear solver (e.g. GMRES).

The solution gives us an approximation for $\mu(z_j, k)$

D-bar Reconstructs Pixel-by-Pixel

For every spatial reconstruction point z_j, we solve a separate equation for $\mu(z_j, \cdot)$.

$$z_j \longmapsto \text{one D-bar solve} \longmapsto \mu(z_j, 0)^2 = \sigma(z_j).$$

There is no coupling between different z_j, so the reconstruction is formed independently for each grid point / pixel.

Crucially, this means the spatial loop over the z_j is *trivially parallelizable*, permitting greatly enhanced computational speed with minimal effort.

In MATLAB, parallelization can be easily implemented using a `parfor` loop in place of a `for` loop.

Regularization by Low-Pass Filter

Noise and modeling error cause the computed scattering transform to become unstable as $|k|$ grows.

Following [Knudsen et al. 2009], we can regularize by applying a truncation in the spectral k domain, e.g.

$$\mathbf{t}^{\exp}_{\rho}(k) = \begin{cases} \mathbf{t}^{\exp}(k), & |k| < \rho, \\ 0, & |k| \geq \rho. \end{cases}$$

- Smaller ρ: stronger smoothing and greater stability, but loss of edges and fine detail.
- Larger ρ: potentially better preservation of edges and fine detail, but greater sensitivity to noise and modeling error.

This acts as a nonlinear low-pass Fourier filter.

MATLAB Time

Let's check out the code!