Electrical Impedance Tomography: Computation and Applications

Part 8: Linearized Difference Reconstruction

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A First EIT Solver

We will now look in-depth into a very basic EIT reconstruction method:

One-Step Linearized Difference Reconstruction

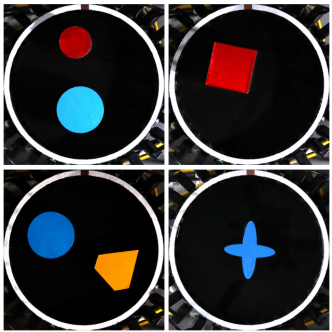
This method:

- Is a model-based method using the CEM for the forward problem
- Solves the forward problem using finite element methods (FEM)
- Uses a linear approximation of the EIT inverse problem
- Is a one-step (as opposed to iterative) method
- Is a form of *difference imaging*
  (we reconstruct a conductivity change from a baseline)

Our Data Source

Our data and base code are from the KTC2023 Open EIT Dataset, https://doi.org/10.5281/zenodo.10986692.

Data was collected using the KIT4 EIT machine at the University of Eastern Finland, Kuopio.



Red = metal,
Orange = conductive plastic,
Blue = resistive plastic

Image: Räsänen et al. 2024

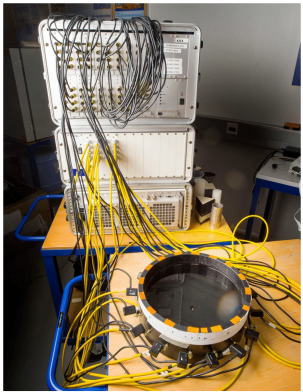


Image: Hauptmann et al. 2017

Problem Statement

Suppose we have voltage measurements from two physical states:

$$\sigma_{\text{ref}} \longrightarrow V_{\text{ref}}, \qquad \sigma \longrightarrow V$$

where here V and V_{ref} denote vectors of voltage measurements stacked over all current patterns.

We further have an approximation $\sigma_0 \approx \sigma_{\text{ref}}$.

Our Goal

Use the voltage difference $\Delta V = V - V_{\text{ref}}$
to reconstruct the conductivity difference $\Delta\sigma = \sigma - \sigma_0$.

This is known as *difference imaging*.

For notational simplicity, we will use σ for both the continuous conductivity field and its vector of nodal values after discretization; the intended meaning will be clear from context.

Difference Imaging Focuses on What Changed

Given

$$\boxed{\Delta V = V - V_{\mathrm{ref}}, \qquad \Delta\sigma = \sigma - \sigma_0,}$$

we consider the case where:

- V is measured on a real-world tank target.
- $\sigma = \sigma_0 + \Delta\sigma$ is the unknown conductivity distribution that gave rise to V.
- V_{ref} is measured on the real-world homogeneous saline tank.
- $\sigma_0 \equiv 1$ is our homogeneous model of that reference state, normalized to one.

Subtracting $V - V_{\mathrm{ref}}$ reduces systematic errors shared by the target and reference measurements.

Linearization Turns EIT into a Matrix Problem

We know the forward map $F : \sigma \mapsto V$ is nonlinear.

After discretization, the *Jacobian* $J(\sigma) = DF(\sigma)$ serves as the derivative of $F(\sigma)$.

So near the reference state σ_0 we have the first-order Taylor approximation

$$\underbrace{F(\sigma_0 + \Delta\sigma)}_{F(\sigma)} - F(\sigma_0) \approx J(\sigma_0)\Delta\sigma.$$

We have $V \approx F(\sigma)$ and for difference imaging we assume $V_{\text{ref}} \approx F(\sigma_0)$, so we use

$$\Delta V = V - V_{\text{ref}} \approx J(\sigma_0)\Delta\sigma.$$

This linearization is most accurate when σ is reasonably close to σ_0.

The Linearized Inverse Problem

Given

$$J(\sigma_0)\Delta\sigma \approx \Delta V,$$

- The vector ΔV is known.
- We compute the matrix $J(\sigma_0)$ by numerically solving the forward problem.
- Then we solve the above matrix equation for the vector $\Delta\sigma$.
- Because the measurements contain noise and the linearized model is only approximate, we seek the best least-squares fit:

$$\min_{\Delta\sigma} \|J(\sigma_0)\Delta\sigma - \Delta V\|_2^2.$$

Forming the Jacobian

Let $\sigma_0 = [\sigma_{0,1}, \ldots, \sigma_{0,K}]^\top$ denote the discretized reference conductivity, evaluated at K points in the spatial domain.

Writing $F(\sigma_0) = [F_1(\sigma_0), \ldots, F_M(\sigma_0)]^\top$, where M is the number of voltage measurements, the Jacobian is given by

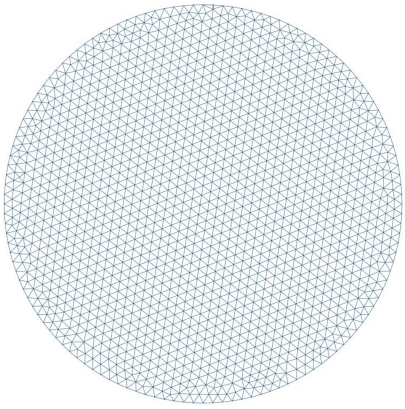
$$J(\sigma_0) = \left[\frac{\partial F}{\partial \sigma_1} \quad \frac{\partial F}{\partial \sigma_2} \quad \cdots \quad \frac{\partial F}{\partial \sigma_K} \right] \Bigg|_{\sigma=\sigma_0} \in \mathbb{R}^{M \times K}.$$

To form $J(\sigma_0)$ we must first approximate $F(\sigma_0)$ by numerically solving the forward problem using the complete electrode model (CEM).

This is done using finite element methods (FEM). We won't try to cover all of FEM here, but let's look at a few basic points.

FEM in One Slide

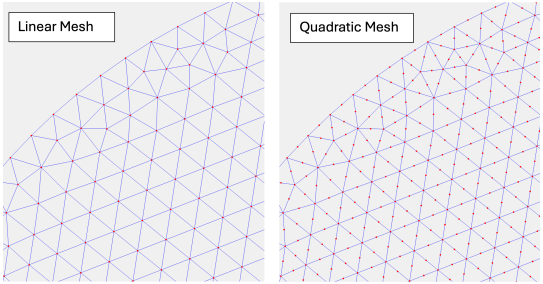
1. Discretize the spatial domain into many small triangles (the *mesh*).
2. Approximate σ by a linear function on each triangle.
3. Assemble the local CEM equations into a sparse matrix system.
4. Solve for the interior potential u, approximated by a quadratic function on each triangle, and extract the electrode voltages $F(\sigma)$.
5. Differentiate the discrete $F(\sigma)$ to obtain $J(\sigma)$.



For today, we will use a FEM forward solver as a supplied computational tool, without worrying about its inner workings.

Two Meshes Have Two Different Jobs

Since σ and u are approximated using linear and quadratic basis functions, respectively, we will use two different meshes:



Both meshes use the same triangular elements and tank geometry.

The linear mesh has nodes (computation points) at each triangle vertex. The quadratic mesh has additional nodes at the midpoint of each triangle edge, allowing us to interpolate a quadratic approximation of u over each triangle.

MATLAB Time

Let's go to the code!

We will start with the naïve unregularized inversion.

Why Naïve Inversion Fails

We saw that unregularized inversion doesn't work very well.

Why?

- Very different conductivity changes can produce almost the same boundary voltages.
- Small noise and modeling errors can therefore cause large changes in the reconstructed image.
- The unregularized solver can chase these tiny discrepancies using large, oscillatory conductivity patterns.

Bottom Line

An excellent voltage fit need not give a meaningful image!

What can we do about this?

Regularization Permits a Controlled Mismatch

The unregularized method cares only about matching the measured voltage change. This can require an enormous, highly oscillatory conductivity change.

Instead of fitting the voltage data "at any cost," solve

$$\min_{\Delta\sigma} \underbrace{\|J(\sigma_0)\Delta\sigma - \Delta V\|_2^2}_{\text{fit the data}} + \underbrace{\alpha^2 \|R\Delta\sigma\|_2^2}_{\text{control the solution}} .$$

- The matrix R determines which conductivity changes are penalized.
- The *regularization parameter* α controls how strongly we regularize:
  - Smaller α: fit the data more closely, but risk instability.
  - Larger α: give the penalty encoded by R greater influence.

This is known as *Tikhonov regularization.*

Tikhonov Regularization: A Controlled Compromise

As a first pass, we'll take R to be the identity matrix $\mathbb{I}$ and solve

$$\min_{\Delta\sigma} \|J(\sigma_0)\Delta\sigma - \Delta V\|_2^2 + \alpha^2 \|\Delta\sigma\|_2^2.$$

Let's go to the code again and try it out!

A Smoothness Prior Changes What We Penalize

Basic Tikhonov is like a blanket suppression of large $\Delta\sigma$ over the entire spatial domain.

A more sophisticated approach suppresses rapidly varying spatial changes in the solution $\Delta\sigma$.

A *Gaussian smoothness prior* is one way to do this. Consider

$$\min_{\Delta\sigma} \|J(\sigma_0)\Delta\sigma - \Delta V\|_2^2 + \alpha^2 \|R\Delta\sigma\|_2^2.$$

where R penalizes changes that vary sharply between nearby mesh nodes, favoring spatially correlated reconstructions.

This is still Tikhonov regularization; only the penalty has become spatially informed.

Let's go to the code one more time!