Pair 5: D-bar Reconstruction

Choose exactly one: Problem A or Problem B.

Problem 5A (Hand Computations) [2 points]

In this problem we illustrate three stages of the D-bar pipeline: forming a difference DN map, regularizing the scattering data, and recovering the conductivity from $\mu(z,0)$. The three parts use separate illustrative datasets; their numerical outputs are not inputs to one another.

(a) Suppose data from a four-electrode EIT system have been expressed in a complete orthonormal basis of zero-mean current patterns and normalized to the unit disk with reference conductivity 1. In simplified scaled units, the discrete ND maps for the target and reference are

$$R_\sigma = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad R_{\mathrm{ref}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Compute the discrete DN maps Λ_σ and Λ_{ref}, and then form $\Delta\Lambda = \Lambda_\sigma - \Lambda_{\mathrm{ref}}$. Why are these matrices 3×3 even though the system has four electrodes?

(b) Suppose (for a dataset unrelated to (a)), the following illustrative sample of $t^{\exp}(k)$ is available at four spectral points, with

$$\mathbf{k} = \begin{bmatrix} 1 \\ 1+i \\ 2i \\ 2+2i \end{bmatrix}, \qquad \mathbf{t}^{\exp} = \begin{bmatrix} 2 \\ 1+i \\ -1 \\ 3-2i \end{bmatrix}.$$

Using the cutoff $\rho = 2$, form the regularized values by retaining $t^{\exp}(k)$ when $|k| < \rho$ and setting it to zero otherwise. In the real-world reconstruction setting, what happens to stability and spatial detail if ρ is made smaller?

(c) On an even k-grid, suppose that solving the D-bar equation (for a dataset unrelated to (a) and (b)) gives four central values of $\mu(z,k)$ at one fixed spatial point z: 1.08, 1.10, 1.12, and 1.10. Approximate $\mu(z,0)$ by their average and compute the normalized conductivity $\sigma(z) = \mu(z,0)^2$. If the measured reference conductivity was $\sigma_{\mathrm{ref}} = 0.25$ S/m, restore physical units. Is this point more or less conductive than the reference? Why can different spatial points be processed in parallel in a D-bar reconstruction?

Problem 5B (MATLAB) [2 points]

This problem follows the same three stages as Problem 5A using an eight-electrode system. Write the MATLAB commands and report the results needed to answer the following.

(a) The code below constructs a normalized trigonometric current basis along with target and reference voltage matrices. Run the code and verify that Phi has orthonormal columns. Form $R_\sigma = \Phi^\top V_\sigma$ and $R_{\text{ref}} = \Phi^\top V_{\text{ref}}$, then compute the corresponding DN maps and $\Delta\Lambda = \Lambda_\sigma - \Lambda_{\text{ref}}$. Report the size and Frobenius norm of $\Delta\Lambda$. Explain why the size of $\Delta\Lambda$ makes sense given the number of electrodes.

```
L = 8;                               % Number of electrodes
theta = 2*pi*(0:L-1)'/L;             % Electrode-center angles
raw = [cos(theta*(1:L/2)), ...       % Unnormalized current patterns
       sin(theta*(1:L/2-1))];
Phi = raw./sqrt(sum(raw.^2,1));      % Orthonormal current basis

% Reference ND map used to generate the example voltage data
Rref_data = diag([1.0 1.2 1.4 1.6 1.8 2.0 2.2]);

% Symmetric perturbation distinguishing the target from the reference
B = diag([-.10 -.08 -.06 -.05 -.04 -.03 -.02]) ...
    + .02*diag(ones(6,1),1) + .02*diag(ones(6,1),-1);

Rtemp = Rref_data + B;                % Target ND map
Vref = Phi*Rref_data;                 % Reference voltage responses
Vsigma = Phi*Rsigma_data;             % Target voltage responses

% Now clear things so we can pretend the voltage matrices come from measured data :-)
clear raw theta Rref_data Rsigma_data B
```

(b) Run the following, which provides an illustrative sample of a scattering transform (from data unrelated to part (a))

```
% Samples in the nonphysical spectral k-plane
k = [-3-2i; -2-1i; -1-1i; -.5i; .5i; 1+1i; 2+1i; 2+2i; 3i];

% Corresponding samples of the approximate scattering transform
texp = [2+1i; -1+2i; .5-.5i; 1; 1+1i; -2+1i; 3-1i; 2-2i; -1-1i];
```

Take $\rho = 2.5$. Construct `texp_rho` by retaining the entries of `texp` for which $|k| < \rho$ and replacing all other entries by zero. Report `texp_rho`. Explain the main tradeoff in choosing a smaller versus a larger value of ρ in the real-world reconstruction scenario.

(c)

(d) The code below gives four illustrative central k-grid values of $\mu(z,k)$ at each of five spatial pixels:

```
mu_center = [1.00 1.00 1.00 1.00;
             1.08 1.10 1.12 1.10;
              .94  .96  .95  .95;
             1.04 1.06 1.05 1.05;
              .88  .90  .92  .90];
```

Average each row of `mu_center` to approximate $\mu(z,0)$ at the five pixels. Suppose $\sigma_{\text{ref}} = 0.25\text{S/m}$. Compute the normalized conductivities and then restore physical units using $\sigma_{\text{physical}}(z) = \sigma_{\text{ref}}\mu(z,0)^2$. Identify which pixels are more conductive, less conductive, or equal in conductivity to the reference. Why would a `parfor` loop be appropriate if many spatial pixels were reconstructed?