Pair 4: Linearization and Regularization

Choose exactly one: Problem A or Problem B.

Problem 4A (Hand Computations) [2 points]

In Pair 3, we used the linearized EIT model $\Delta\mathbf{V} \approx J(\boldsymbol{\sigma}_0)\Delta\boldsymbol{\sigma}$ to predict voltage changes from a proposed conductivity change. Here we use the same model in the inverse direction to estimate the conductivity change from voltage data and examine the effect of regularization.

Near a reference conductivity $\boldsymbol{\sigma}_0$, suppose

$$\boldsymbol{\sigma}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \Delta\mathbf{V} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad J = J(\boldsymbol{\sigma}_0) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

(a) Treating the linearized model as an equality, compute the unregularized estimate $\Delta\boldsymbol{\sigma}$ and the reconstructed conductivity $\boldsymbol{\sigma}_0 + \Delta\boldsymbol{\sigma}$.

(b) Using the convention in this exercise, the zero-order Tikhonov estimate with regularization parameter $\alpha = \frac{1}{4}$ satisfies

$$\left(J^\top J + \alpha\mathbb{I}\right)\Delta\boldsymbol{\sigma}_\alpha = J^\top \Delta\mathbf{V},$$

where $\mathbb{I}$ is the identity matrix of the appropriate size. Compute $\Delta\boldsymbol{\sigma}_\alpha$ and the regularized reconstruction $\boldsymbol{\sigma}_0 + \Delta\boldsymbol{\sigma}_\alpha$.

(c) Suppose measurement error changes the voltage data to $\Delta\mathbf{V}_{\text{noisy}} = [1\ 1.1]^\top$. Using this perturbed data, recompute both the unregularized and Tikhonov estimates of $\Delta\boldsymbol{\sigma}$. Compare the resulting changes in the second component of each estimate with the measurement error of 0.1. Briefly explain how regularization reduces the effect of this error.

Problem 4B (MATLAB) [2 points]

Consider the linearized EIT model $\Delta\mathbf{V} \approx J\Delta\boldsymbol{\sigma}$ for a simplified three-pixel conductivity image. The reference conductivity, Jacobian, voltage data, and regularization parameter are defined below.

```
sigma0 = [1; 1; 1];
J = [1 0 0; 0 1/2 0; 0 0 1/10];
dV = [1; 1; 1];
alpha = 1/4;
```

Use MATLAB for each part. Include the commands you write and the requested outputs.

(a) Treating the linearized model as an equality, compute the unregularized estimate $\Delta\boldsymbol{\sigma}$ and the reconstructed conductivity $\boldsymbol{\sigma}_0 + \Delta\boldsymbol{\sigma}$.

(b) Compute the zero-order Tikhonov estimate with $\alpha = \frac{1}{4}$ by solving

$$\left(J^\top J + \alpha\mathbb{I}\right)\Delta\boldsymbol{\sigma}_\alpha = J^\top\Delta\mathbf{V},$$

where $\mathbb{I}$ is the identity matrix of the appropriate size. Report $\Delta\boldsymbol{\sigma}_\alpha$ and the regularized reconstruction $\boldsymbol{\sigma}_0 + \Delta\boldsymbol{\sigma}_\alpha$.

(c) Replace the voltage data by $\Delta\mathbf{V}_{\text{noisy}} = [1\ 1\ 1.1]^\top$ and recompute both estimates. For each method, report the change in the third component relative to the corresponding estimate obtained from the original data. Explain why the measurement error is strongly amplified by the unregularized reconstruction but has less effect on the Tikhonov estimate.