Pair 3: The CEM and Linearized Reconstruction

Choose exactly one: Problem A or Problem B.

Problem 3A (Hand Computations) [2 points]

Consider a three-dimensional four-electrode EIT system modeled by the complete electrode model (CEM). The applied electrode currents are $\mathbf{I} = [5\ -3\ -2\ 0]^\top$ mA. Electrode e_1 has area $A_1 = 1\ \mathrm{cm}^2$ and contact impedance $z_1 = 10^{-3}\ \Omega\mathrm{m}^2$. Divide e_1 into four patches of equal area, and let $j_1(x) = \sigma \frac{\partial u}{\partial \mathbf{n}}(x)$ denote the outward normal current density on e_1. Suppose that solving the CEM forward problem gives the patchwise normal current densities $\mathbf{j}_1 = [20\ 40\ 60\ 80]^\top$ A/m^2 on these patches and the metal-electrode voltage $V_1 = 0.120$ V.

(a) Verify the applied currents $\mathbf{I}$ satisfy conservation of charge. Approximate $I_1 = \int_{e_1} j_1\, dS$ by treating j_1 as constant on each patch. Report the result in mA and verify that it agrees with the specified value of I_1.

(b) Use $u = V_1 - z_1 j_1$ to compute the tissue-side potential on the four patches. Compute the contact-resistance scale $R_{\text{contact}} = z_1/A_1$ and verify that $V_1 - \operatorname{mean}(u) = I_1 R_{\text{contact}}$, using I_1 in amperes.

(c) Suppose three voltage-change measurements are collected into $\Delta\mathbf{V}$, while $\Delta\boldsymbol{\sigma}$ describes changes in the conductivity values of two pixels in a difference image. In scaled units, suppose

$$\Delta\mathbf{V} = \begin{bmatrix} 0.40 \\ -0.10 \\ -0.40 \end{bmatrix}, \qquad J(\boldsymbol{\sigma}_0) = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}, \qquad \Delta\boldsymbol{\sigma} = \begin{bmatrix} 0.10 \\ -0.20 \end{bmatrix}.$$

Verify that $\Delta\mathbf{V} \approx J(\boldsymbol{\sigma}_0)\Delta\boldsymbol{\sigma}$. In words, what does the second column of $J(\boldsymbol{\sigma}_0)$ describe?

Problem 3B (MATLAB) [2 points]

Consider a three-dimensional four-electrode EIT system modeled by the complete electrode model (CEM). The applied electrode currents are $\mathbf{I} = [5 \;\; -3 \;\; -2 \;\; 0]^\top$ mA. Electrode e_1 has area $A_1 = 1\ \mathrm{cm}^2$ and contact impedance $z_1 = 10^{-3}\ \Omega\mathrm{m}^2$. Divide e_1 into eight patches of equal area, and let $j_1(x) = \sigma \frac{\partial u}{\partial \mathbf{n}}(x)$ denote the outward normal current density on e_1. Suppose that solving the CEM forward problem gives the patchwise normal current densities $\mathbf{j}_1 = [20\ 30\ 40\ 50\ 60\ 70\ 80\ 50]^\top$ A/m^2 and the metal-electrode voltage $V_1 = 0.120$ V.

For parts (a)–(b), use the current injection $\mathbf{I}$ given above. For part (c), suppose eight voltage-change measurements collected from several current-injection experiments are stacked in the vector $\Delta\mathbf{V}$, while $\Delta\boldsymbol{\sigma}$ describes the changes in conductivity values of five pixels in a difference image. In scaled units, these quantities satisfy the linearized model $\Delta\mathbf{V} \approx J(\boldsymbol{\sigma}_0)\Delta\boldsymbol{\sigma}$. The necessary data are defined below.

After running the code below, write the MATLAB commands and report the results needed to answer the following questions.

```
% Inputs to CEM
I  = [5; -3; -2; 0];                      % Applied currents in mA
A1 = 1e-4;                                % Electrode area in m^2
z1 = 1e-3;                                % Contact impedance in ohm*m^2

% Outputs from CEM
j1 = [20; 30; 40; 50; 60; 70; 80; 50];    % Current density in A/m^2
V1 = 0.120;                               % Electrode voltage in V

% Measured data (voltage-change measurements from several experiments)
dV = [0.10; -0.25; -0.20; 0.10; 0.45; -0.10; -0.45; 0.25];

% Jacobian corresponding to reference conductivity sigma0
J = [ 2  0 -1  0  1;
      1  1  0 -1  0;
      0  2  1  0 -1;
     -1  0  2  1  0;
      0 -1  0  2  1;
      1  0 -1  0  2;
      0  1  0 -1  1;
      1 -1  1  0  0];

% % Proposed changes in the conductivity values of 5 pixels
dsigma = [0.10; -0.15; 0; 0.20; -0.10];
```

(a) Verify the applied currents $\mathbf{I}$ satisfy conservation of charge. Approximate $I_1 = \int_{e_1} j_1\, dS$ by treating j_1 as constant on each patch. Verify that the result agrees with the specified value of I_1.

(b) Compute the tissue-side potential on each patch using $u = V_1 - z_1 j_1$. Compute $R_{\text{contact}} = z_1/A_1$ and verify numerically that

$$V_1 - \mathrm{mean}(u) = I_1 R_{\text{contact}},$$

using I_1 in amperes.

(c) Compute $J(\boldsymbol{\sigma}_0)\Delta\boldsymbol{\sigma}$ and verify that it agrees with $\Delta\mathbf{V}$. In words, what does the fourth column of $J(\boldsymbol{\sigma}_0)$ describe?