Pair 1: Boundary Currents and Voltages

Choose exactly one: Problem A or Problem B.

0.1 Problem 1A (Hand Computations) [2 points]

Consider an EIT system with $L = 8$ electrodes. We use the convention that positive current enters the domain and negative current leaves the domain.

(a) Determine which of the following current patterns, given in mA, are physically admissible. Explain briefly.

$$\mathbf{I}^{(1)} = \begin{bmatrix}1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^\top,$$

$$\mathbf{I}^{(2)} = \begin{bmatrix}4 & 0 & -4 & 0 & 4 & 0 & -4 & 0\end{bmatrix}^\top, \qquad \mathbf{I}^{(3)} = \begin{bmatrix}-1 & 2 & 0 & 0 & -1 & 2 & 0 & 0\end{bmatrix}^\top,$$

$$\mathbf{I}^{(4)} = \begin{bmatrix}\cos 0 & \cos\frac{3\pi}{4} & \cos\frac{6\pi}{4} & \cos\frac{9\pi}{4} & \cos\frac{12\pi}{4} & \cos\frac{15\pi}{4} & \cos\frac{18\pi}{4} & \cos\frac{21\pi}{4}\end{bmatrix}^\top.$$

(b) Let $\mathcal{I}$ denote the space of admissible current patterns. If $\mathbf{e}_j$ is the jth standard basis vector in $\mathbb{R}^8$, show that the vectors $\{\mathbf{e}_1 - \mathbf{e}_8,\ \mathbf{e}_2 - \mathbf{e}_8,\ \ldots,\ \mathbf{e}_7 - \mathbf{e}_8\}$ form a basis for $\mathcal{I}$. What is $\dim(\mathcal{I})$?

(c) Suppose the measured electrode voltages are $\mathbf{V} = \begin{bmatrix}2.4 & 1.1 & -0.3 & -1.2 & -0.8 & 0.5 & 1.7 & 0.6\end{bmatrix}^\top$ mV. Let $\mathbf{1} = [1\ 1\ \ldots 1]^\top \in \mathbb{R}^8$. Compute the voltage vectors obtained using the following two choices of voltage reference (also called gauge normalizations): $\mathbf{V}^{(0)} = \mathbf{V} - \frac{1}{8}(\mathbf{1}^\top\mathbf{V})\mathbf{1}$ and $\mathbf{V}^{(8)} = \mathbf{V} - V_8\mathbf{1}$. Verify that $\mathbf{V}^{(0)}$ has mean zero and that the eighth entry of $\mathbf{V}^{(8)}$ is zero.

(d) Show algebraically that, for any $i, j \in \{1, 2, \ldots, 8\}$,

$$(V^{(0)})_i - (V^{(0)})_j = \left(V^{(8)}\right)_i - \left(V^{(8)}\right)_j = V_i - V_j.$$

Briefly explain why $\mathbf{V}$, $\mathbf{V}^{(0)}$, and $\mathbf{V}^{(8)}$ therefore encode the same physical voltage information.

Problem 1B (MATLAB) [2 points]

Consider an EIT system with $L = 16$ electrodes. We use the convention that positive current enters the domain and negative current leaves the domain.

(a) Run the following MATLAB code, which defines trial current pattern matrices `I1, I2, I3, I4`, in mA. Determine which of these are physically admissible. Explain briefly. Keep in mind that in floating-point arithmetic, values near machine precision should be interpreted as zero.

```
L = 16;
E = eye(L);

I1 = E - circshift(E,1,1);
I2 = E - circshift(E,8,1);
I3 = E - 0.5*circshift(E,1,1);

theta = (0:L-1)'*2*pi/L;
I4 = zeros(L,L-1);
for n = 1:L/2-1
    I4(:,2*n-1) = cos(n*theta);
    I4(:,2*n)   = sin(n*theta);
end
I4(:,L-1) = cos((L/2)*theta);
```

(b) Compute the rank of each of `I1`, `I2`, `I3`, `I4`. The space $\mathcal{I} = \left\{\mathbf{I} \in \mathbb{R}^{16} : \mathbf{1}^\top \mathbf{I} = 0\right\}$ of admissible current patterns has dimension 15 (where $\mathbf{1} = [1\ 1\ \dots 1]^\top \in \mathbb{R}^{16}$). Use these results combined with your answer to (a) to answer the following questions:

- Which matrices have column space exactly equal to $\mathcal{I}$?
- Which matrices have linearly independent columns?
- Which matrices, or subsets of their columns, form a basis for $\mathcal{I}$?

Briefly explain why having high rank does not by itself guarantee that a collection of current patterns is physically admissible.

(c) Suppose the measured electrode voltages (in mV) are

```
V = [2.4; 1.1; -0.3; -1.2; -0.8; 0.5; 1.7; 0.6; ...
     2.0; 0.9; -0.5; -1.4; -1.0; 0.3; 1.5; 0.2];
```

Compute the voltage vectors obtained using the following two choices of voltage reference (also called gauge normalizations): $\mathbf{V}^{(0)} = \mathbf{V} - \frac{1}{16}(\mathbf{1}^\top \mathbf{V})\mathbf{1}$ and $\mathbf{V}^{(16)} = \mathbf{V} - V_{16}\mathbf{1}$, storing the results as `V0` and `V16`. Verify that `V0` has mean zero and that the sixteenth entry of `V16` is zero (interpret values near machine precision as zero).

(d) Run the following code, where `V0` and `V16` denote the vectors computed in part (c):

```
D   = V   - V.';
D0  = V0  - V0.';
D16 = V16 - V16.';
err0  = max(abs(D(:)-D0(:)));
err16 = max(abs(D(:)-D16(:)));
```

Explain what the entries of `D`, `D0`, and `D16` represent. Report `err0` and `err16`, and explain why the results show that the three voltage vectors encode the same physical voltage information (interpret values near machine precision as zero).