Return by tue 23.11. by 2 pm.

1. *Master Equation of Photonic Crystals.* (See the lectures given by Timo). Start from the Maxwell's equations

$$\begin{array}{ll} \nabla \cdot \mathbf{E} = 0 & \nabla \times \mathbf{E} + \mu_0 \frac{\partial \mathbf{H}}{\partial t} = 0 \\ \nabla \times \mathbf{H} - \epsilon \frac{\partial \mathbf{E}}{\partial t} = 0 & \nabla \cdot \mathbf{H} = 0. \end{array}$$

Assume that the time-dependence of the fields $\mathbf{E}$ and $\mathbf{H}$ is of the form $\exp(i\omega t)$ and show that then $\mathbf{H}$ satisfies a Hermitian eigenvalue problem (master equation)

$$M\mathbf{H} =: \nabla \times \left(\frac{1}{\mu_0 \epsilon} \nabla \times \mathbf{H} \right) = \omega^2 \mathbf{H}.$$

Finally show that the operator M is indeed Hermitian, thus $\langle M\mathbf{H}|\mathbf{H}\rangle = \langle \mathbf{H}|M\mathbf{H}\rangle$
Hint: Become convinced that for well-behaved (physical) fields $\mathbf{A}$ and $\mathbf{B}$ you get $\int (\nabla \times \mathbf{A}) \cdot \mathbf{B} \, \mathrm{d}^3 r = \int \mathbf{A} \cdot (\nabla \times \mathbf{B}) \, \mathrm{d}^3 r$ by using partial integration. **(2 point)**

2. Read the article about photonic crystals and aswer briefly to the following questions:

   1. How do the photonic crystals exist in nature?
   2. What is the difference between the photonic crystal optic fibres and the wave guides discussed on the lecture?
   3. What is the difference between the photonic crystal band gap and the solid state (electronic) band gap?

   (3 points)

3. *Curie's law of paramagnetism.* (See also Ashcroft & Mermin: Solid State Physics problem 31.9) The partition function (Elliot eq. (7.167)) can be written as

$$Z = e^{-\beta F} = \sum_m e^{-\beta E_m} = \sum_{m=-j}^{j} \langle jm | e^{-\beta \mathcal{H}} | jm \rangle = \mathrm{Tr}\{e^{-\beta \mathcal{H}}\}$$

where $\mathcal{H} = g_J \mu \mathbf{H} \cdot \mathbf{J}$ is the Hamiltonian with magnetic field $\mathbf{H}$ and angular momentum operator $\mathbf{J}$ and F is the Helmholtz free energy. The trace is carried out in the eigenbasis of J^2. Furthermore $\mu = \mu_0 \mu_B$ and $\beta = 1/k_B T$. The susceptibility is then defined as

$$\chi = -\frac{n}{\mu_0} \frac{\partial^2 F}{\partial H^2}$$

for a concentration n of atomic moments. The Curie's law can be deduced at high temperature without going through the algebra of Brillouin functions, for we can expand $\exp(-\beta\mathcal{H}) = 1 - \beta\mathcal{H} + (\beta\mathcal{H})^2/2 - \ldots$. Evaluate the free energy to second order in the field, using the fact that for the cartesian components of angular momentum operator (J_x, J_y, J_z) we have

$$\mathrm{Tr}\{J_i J_j\} = \frac{1}{3} \delta_{ij} \mathrm{Tr}\{J^2\}$$

and extract the high-temperature ($g_J \mu H \ll k_B T$) susceptibility (Elliot eq. (7.174)). **(2 points)**