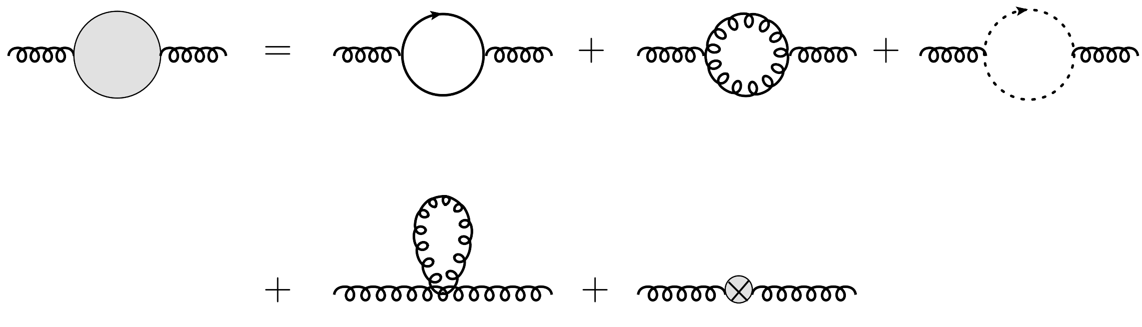
3.7.2 Gluon self energy

The gluon 2-point function involves four diagrams + the counter-term contribution:

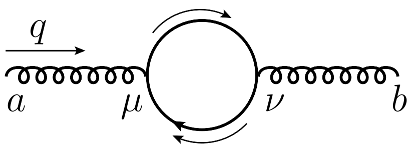


The quark loop

We can obtain the contribution of the quark loop from the QED result in Eq. (2.105) by supplementing the colour factor

$$\mathrm{Tr}(T^aT^b) = \delta^{ab}T_{\mathrm{R}}\,, \quad T_{\mathrm{R}} \equiv \frac{1}{2}\,, \tag{3.195}$$

and summing over all quark flavors that can circulate in the loop. In terms of the QCD coupling $\alpha_s \equiv g^2/4\pi$,



$$= i\Pi^{(1)}_{ab,\mu\nu}(q)$$

$$i\Pi^{(1)}_{ab,\mu\nu}(q) = \delta^{ab}\left[q^2g^{\mu\nu} - q^\mu q^\nu\right] \times i\Pi^{(1)}(q^2) \tag{3.196}$$

$$i\Pi(q^2) = \frac{-2i\alpha_{\mathrm{s}}}{\pi}T_R\int_0^1 dx x(1-x)\sum_f\left[\frac{2}{\epsilon} - \gamma_E + \log\frac{4\pi\mu^2}{\Delta_f}\right]$$

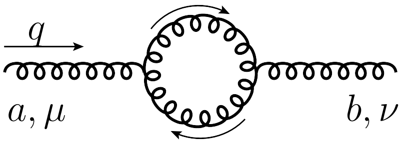
$$\Delta = m_f^2 - x(1-x)q^2\,.$$

If the number of quark species is n_f, then the UV divergent part is just,

$$i\Pi^{(1)}(q^2) = \frac{-i\alpha_{\rm s}}{3\pi} T_R n_f \left[\frac{2}{\epsilon} - \gamma_E + \log 4\pi\right] + \text{finite terms}$$

The gluon loop

Despite the gluon being massless, the gluon loop



$$= i\Pi^{(2)}_{ab,\mu\nu}(q)$$

has no infrared (IR) divergence so no infrared regulator is required. In the Feynman gauge, the UV-divergent contribution is (Ex.),

$$\begin{aligned} i\Pi^{(2)}_{ab,\mu\nu}(q) &= \frac{i\alpha_{\rm s}}{8\pi} C_{\rm G} \delta^{ab} \left(\frac{2}{\epsilon} - \gamma_E + \log 4\pi\right) \\ &\times \left[\left(\frac{41}{12}\right)\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) - \left(\frac{1}{4}\right)\left(q^2 g^{\mu\nu} + q^\mu q^\nu\right)\right] \\ &+ \text{finite terms}\,, \end{aligned} \tag{3.197}$$

where the colour factor C_G is

$$f^{acd} f^{bcd} = \delta^{ab} C_{\rm G}\,, \quad C_{\rm G} \equiv 3\,. \tag{3.198}$$

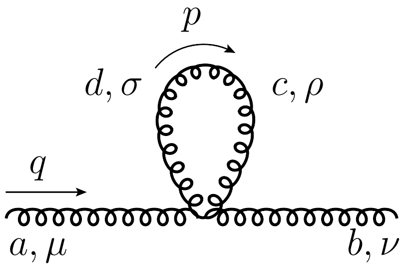
Notice that unlike the quark-loop contribution, the gluon in Eq. (3.197) is not transversal, i.e.

$$q^\mu \Pi^{(2)}_{ab,\mu\nu}(q) \neq 0\,. \tag{3.199}$$

It is of course the second term in Eq. (3.197) that breaks the transversality.

The gluon flower

The next diagram is the one in which we decorate the gluon propagator with a gluon "flower",



$$= i\Pi^{(3)}_{ab,\mu\nu}(q)$$

By using the QCD Feynman rules,

$$
\begin{aligned}
i\Pi^{(3)}_{ab,\mu\nu}(q) = \frac{1}{2}\int \frac{d^D p}{(2\pi)^D}\left[\frac{-ig_{\sigma\rho}\delta^{cd}}{p^2+i\epsilon}\right](-ig^2\mu^\epsilon) \quad (3.200) \\
\left[\underbrace{f^{abe}f^{cde}}_{\to 0}(g^{\mu\sigma}g^{\nu\rho}-g^{\mu\rho}g^{\nu\sigma})+\underbrace{f^{ace}f^{bde}}_{\to\delta^{ab}C_{\mathrm{G}}}(g^{\mu\nu}g^{\sigma\rho}-g^{\mu\rho}g^{\nu\sigma})\right. \\
\left.+\underbrace{f^{ade}f^{bce}}_{\to\delta^{ab}C_{\mathrm{G}}}(g^{\mu\nu}g^{\sigma\rho}-g^{\mu\sigma}g^{\nu\rho})\right] \\
= -g^2\mu^\epsilon\delta^{ab}C_{\mathrm{G}}(D-1)g^{\mu\nu}\times\int\frac{d^Dp}{(2\pi)^D}\frac{1}{p^2+i\epsilon}\,.
\end{aligned}
$$

The treatment of the integral here is a delicate issue. Let us first do the p^0 integral,

$$\int\frac{dp^0}{p^2+i\epsilon}=\int\frac{dp^0}{(p^0-|\mathbf{p}|+i\epsilon)\,(p^0+|\mathbf{p}|-i\epsilon)}=\frac{-\pi i}{|\mathbf{p}|}\,.$$

Thus,

$$\int\frac{d^Dp}{(2\pi)^D}\frac{1}{p^2+i\epsilon}=\frac{-i\pi}{(2\pi)^D}\int d\Omega_D\int_0^\infty d|\mathbf{p}||\mathbf{p}|^{D-3}\,. \quad (3.201)$$

In 4 space-time dimensions the integral diverges quadratically and in order to tame the UV divergence we need to lower the dimesion down to $D<2$. In this case, however, the integral becomes IR divergent, so the above integral does not exist in any (real) spacetime dimension!

We can regulate the IR divergence by giving the gluon a fictitious mass m_g. In this case, we can use our standard formulae to write,

$$\int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^2 - m_g^2 + i\epsilon} = \frac{-i}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) \left(m_g^2\right)^{1-\epsilon/2} . \qquad (3.202)$$

The integral is now defined for $D < 2$ and by analytic continuation the right-hand side is also valid in $D = 4 - \epsilon$ dimensions with ϵ being small. We see that in 4 dimensions the result is proportional to the squared regulator mass m_g^2 so if we take the $\epsilon, m_g \to 0$ limit in such a way that

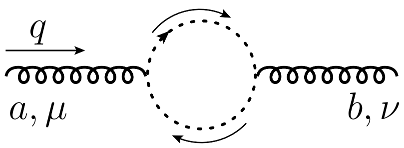
$$\Gamma\left(1 - \frac{D}{2}\right) m_g^2 = m_g^2 \left(\frac{2}{\epsilon} + \cdots\right) \xrightarrow{\epsilon \to 0,\, m_g \to 0} 0\,, \qquad (3.203)$$

(e.g. taking $m_g = \epsilon^2$) the integral vanishes.

More generally, the integral $\int d^4p/(p^2 + i\epsilon)$ has a dimension of mass squared, but yet there is no mass/momentum scale in the integral that could give it the correct dimension. Objects like this are often called **scaleless integrals**, and in dimensional regularization they are usually **defined** to be zero (e.g. by using the prescription above).

The ghost loop

The fourth contribution to the gluon self energy is the diagram with a ghost loop,



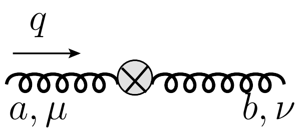
$$= i\Pi^{(4)}_{ab,\mu\nu}(q)$$

The UV-divergent contribution is (Ex.),

$$
\begin{aligned}
i\Pi^{(4)}_{ab,\mu\nu}(q) = \frac{i\alpha_{\rm s}}{8\pi} C_{\rm G} \delta^{ab} \left(\frac{2}{\epsilon} - \gamma_E + \log 4\pi \right) \\
\times \left[\left(\frac{-1}{12} \right) \left(q^2 g^{\mu\nu} - q^\mu q^\nu \right) + \left(\frac{1}{4} \right) \left(q^2 g^{\mu\nu} + q^\mu q^\nu \right) \right] \\
+ \text{finite terms}\,,
\end{aligned}
\tag{3.204}
$$

As in Eq. (3.197), two Lorentz structures appear. The latter does not fulfill the expectation of transversality, but the coefficient of this term is exactly the same as in Eq. (3.197) so they will eventually cancel.

Finally, we add the contribution of the counter-term vertex,



$$
\begin{aligned}
&= i\Pi^{\rm c.t.}_{ab,\mu\nu}(q) \\
&= -i\Big[\left(q^2 g^{\mu\nu} - q^\mu q^\nu \right) \delta_3 + \delta_\xi q^\mu q^\nu \Big] \delta^{ab}\,.
\end{aligned}
$$

We are now ready to combine all the pieces to find the final 1-loop gluon self-energy correction,

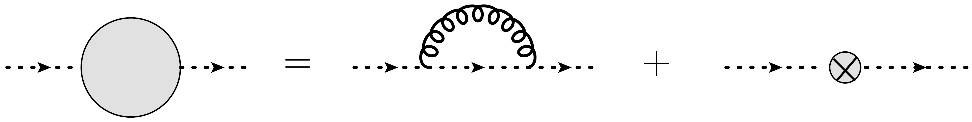
$$
\begin{aligned}
i\Pi_{ab,\mu\nu}(q) &= i\Pi^{(1)}_{ab,\mu\nu}(q) + i\Pi^{(2)}_{ab,\mu\nu}(q) + i\Pi^{(3)}_{ab,\mu\nu}(q) + i\Pi^{(4)}_{ab,\mu\nu}(q) + i\Pi^{\rm c.t.}_{ab,\mu\nu}(q) \\
&= \frac{-i\alpha_s}{4\pi} \left(q^2 g^{\mu\nu} - q^\mu q^\nu \right) \delta^{ab} \left[\frac{4}{3} T_R n_f - C_{\rm G} \frac{5}{3} \right] \left[\frac{2}{\epsilon} - \gamma_E + \log 4\pi \right] \\
&- i\Big[\left(k^2 g^{\mu\nu} - k^\mu k^\nu \right) \delta_3 + \delta_\xi k^\mu k^\nu \Big] \delta^{ab}\,.
\end{aligned}
\tag{3.205}
$$

We see that we can choose $\delta_\xi = 0$ in the Feynman gauge, but since the loop contributions to $i\Pi_{ab,\mu\nu}(q)$ are always transverse (see Ex.11), this is true in other gauges as well. We read off the counter term δ_3,

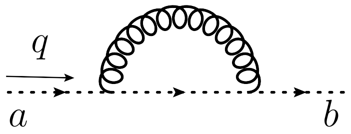
$$
\boxed{\delta_3 = \frac{\alpha_s}{4\pi} \left[\frac{5}{3} C_{\rm G} - \frac{4}{3} T_R n_f \right] \frac{2}{\epsilon} + f_3^{\rm scheme}\,.}
\tag{3.206}
$$

3.7.3 Ghost self energy

The ghost 2-point function involves only one diagram + the counter-term contribution:



The above loop diagram gives,



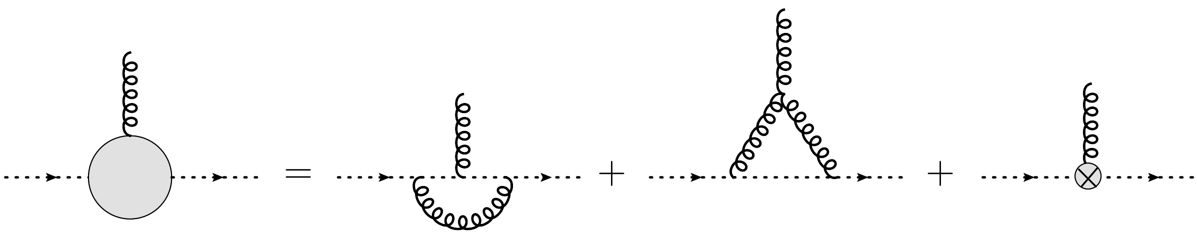
$$
\begin{aligned}
&= i\Pi_{ab}^{\text{ghost}}(q) \\
&= \frac{-i\alpha_s}{4\pi} C_{\text{G}} \delta^{ab} \frac{q^2}{2} \left[\frac{2}{\epsilon} - \gamma_E + \log 4\pi \right] \\
&+ \text{finite terms}\,, \qquad (3.207)
\end{aligned}
$$

and since the counter-term vertex is just $i\delta^{ab}\delta_2^c q^2$, we see that

$$
\delta_2^c = \frac{\alpha_s}{4\pi} \left(\frac{C_{\text{G}}}{2} \right) \times \frac{2}{\epsilon} + f_{2c}^{\text{scheme}}\,. \qquad (3.208)
$$

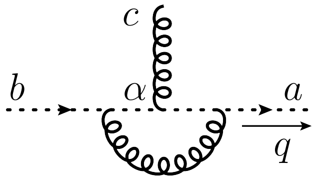
3.7.4 Ghost-gluon vertex

The 1-loop correction to the ghost-gluon vertex involves two diagrams:



The contributions of the individual diagrams are,

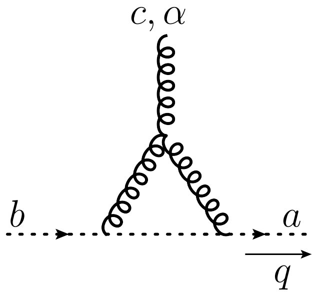
$$i\Lambda^{\alpha,(1)}_{acb}(q) =$$



$$= -g\mu^{\epsilon/2} q^\alpha f^{acb} \times \frac{\alpha_s}{4\pi}\frac{C_{\mathrm{G}}}{8}\left[\frac{2}{\epsilon} - \gamma_{\mathrm{E}} + \log(4\pi)\right] \tag{3.209}$$

$$+ \text{finite terms}$$

$$i\Lambda^{\alpha,(2)}_{acb}(q) =$$



$$= g\mu^{\epsilon/2} q^\alpha f^{acb} \times \frac{\alpha_s}{4\pi}\frac{-3C_{\mathrm{G}}}{8}\left[\frac{2}{\epsilon} - \gamma_{\mathrm{E}} + \log(4\pi)\right] \tag{3.210}$$

$$+ \text{finite terms}$$

In total,

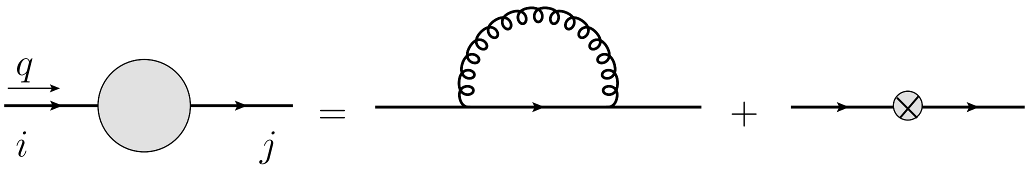
$$\begin{aligned} i\Lambda^{\alpha}_{acb}(q) &= i\Lambda^{\alpha,(1)}_{acb}(q) + i\Lambda^{\alpha,(2)}_{acb}(q) - g\mu^{\epsilon/2} q^\alpha \delta^c_1 f^{acb} \\ &= g\mu^{\epsilon/2} q^\alpha f^{acb} \times \left\{ -\frac{\alpha_s}{4\pi}\frac{C_{\mathrm{G}}}{2}\left[\frac{2}{\epsilon} - \gamma_{\mathrm{E}} + \log(4\pi)\right] - \delta^c_1 \right\}, \end{aligned} \tag{3.211}$$

and we can read off the value of the counter term δ^c_1,

$$\boxed{\delta^c_1 = -\frac{\alpha_s}{4\pi}\left(\frac{C_{\mathrm{G}}}{2}\right) \times \frac{2}{\epsilon} + f^{\mathrm{scheme}}_{1c}\,.} \tag{3.212}$$

3.7.5 Quark self energy

The 1-loop calculation of the quark self energy $-i\Sigma^{ij}(q)$,



is, in practice, identical to the QED case in Section 2.3.2. The only difference is that we need to multiply the QED result with a colour factor,

$$T^a_{jk}T^a_{ki} \overset{(3.31)}{=} \frac{1}{2}\left[\delta_{ji}\delta_{kk} - \frac{1}{3}\delta_{jk}\delta_{ki}\right] = \delta_{ij}C_{\mathrm{F}} \tag{3.213}$$

$$C_{\mathrm{F}} = \frac{4}{3}\,, \tag{3.214}$$

and change $e^2 \to g^2$. By doing these two operations, the quark 2-point function is of the form,

$$\begin{aligned} -i\Sigma^{ij}(q) = \frac{-i\alpha_s}{4\pi}C_{\mathrm{F}}\delta^{ij}\Bigg\{ \left(4m - \not{q}\right)\left[\frac{2}{\epsilon} - \gamma_{\mathrm{E}} + \log\left(\frac{4\pi\mu^2}{m^2}\right)\right]\Bigg\} \\ + i\left(\delta_2\not{q} - \delta_m\right)\delta^{ij} + \text{finite terms}\,, \end{aligned} \tag{3.215}$$

so we can identify the counter terms,

$$\delta_2 = \frac{-\alpha_s}{4\pi}C_{\mathrm{F}} \times \frac{2}{\epsilon} + f_2^{\text{scheme}} \tag{3.216}$$

$$\delta_m = \frac{-\alpha_s m}{4\pi}4C_{\mathrm{F}} \times \frac{2}{\epsilon} + f_m^{\text{scheme}}\,. \tag{3.217}$$

3.7.6 Slavnov-Taylor identities go quantum

We have now explicitly calculated 5 QCD counter terms at 1-loop level:

$$\delta_3 = \frac{\alpha_s}{4\pi}\left[\frac{5}{3}C_{\rm G} - \frac{4}{3}T_R n_f\right]\frac{2}{\epsilon} + f_3^{\rm scheme} \qquad \text{(gluon field)}$$

$$\delta_2^c = \frac{\alpha_s}{4\pi}\left(\frac{C_{\rm G}}{2}\right) \times \frac{2}{\epsilon} + f_{2c}^{\rm scheme} \qquad \text{(ghost field)}$$

$$\delta_2 = \frac{-\alpha_s}{4\pi}C_{\rm F} \times \frac{2}{\epsilon} + f_2^{\rm scheme} \qquad \text{(quark field)}$$

$$\delta_m = \frac{-\alpha_s m}{4\pi}4C_{\rm F} \times \frac{2}{\epsilon} + f_m^{\rm scheme} \qquad \text{(quark mass)}$$

$$\delta_1^c = -\frac{\alpha_s}{4\pi}\left(\frac{C_{\rm G}}{2}\right) \times \frac{2}{\epsilon} + f_{1c}^{\rm scheme} \qquad \text{(ghost} - \text{gluon vertex)}$$

The above 5 counter terms + Slavnov-Taylor identities in Eq. (3.194) specify the renormalized QCD Lagangian completely at 1 loop. Even we did not compute e.g, the 3-gluon function explicitly, it will now automatically be finite i.e. the Slavnov-Taylor identities correctly set the 1-loop counter term. To argue how this comes about, let us consider the following BRS identity,

$$\delta_{\rm BRS}\langle\Omega|A^a_{0\mu}(x)A^b_{0\nu}(y)\overline{c}^c_0(z)|\Omega\rangle = 0\,, \tag{3.218}$$

where we have put the subscripts "0" to stress that these are bare fields (remember our BRS transformations apply on bare fields). We execute the BRS transformation:

$$\begin{aligned} &\langle\Omega|\left[D^{ad}_{0\mu}(x)c^d_0(x)\delta\xi\right]A^b_{0\nu}(y)\overline{c}^c_0(z)|\Omega\rangle \\ &+ \langle\Omega|A^a_{0\mu}(x)\left[D^{bd}_{0\nu}(y)c^d_0(y)\delta\xi\right]\overline{c}^c_0(z)|\Omega\rangle \\ &+ \langle\Omega|A^a_{0\mu}(x)A^b_{0\nu}(y)\left[\frac{1}{\xi_0}\partial\cdot A^c_0(z)\right]\delta\xi|\Omega\rangle = 0\,, \end{aligned} \tag{3.219}$$

where

$$D^{ab}_{0\mu}(x) = \delta^{ab}\partial_\mu + g_0 A^c_{0\mu}(x) f^{acb}\,. \tag{3.220}$$

Eliminating the Grassmann factor $\delta\xi$,

$$\begin{aligned}
&\frac{1}{\xi_0}\langle\Omega| A^a_{0\mu}(x) A^b_{0\nu}(y)\left[\partial\cdot A^c_0(z)\right]|\Omega\rangle \qquad (3.221)\\
&= \langle\Omega|\Big[\partial_\mu c^a_0(x) + g_0 f^{ahd} A^h_{0\mu}(x) c^d_0(x)\Big] A^b_{0\nu}(y)\overline{c}^c_0(z)|\Omega\rangle\\
&+ \langle\Omega| A^a_{0\mu}(x)\Big[\partial_\nu c^b_0(y) + g_0 f^{bhd} A^h_{0\nu}(y) c^d_0(y)\Big]\overline{c}^c_0(z)|\Omega\rangle\,.
\end{aligned}$$

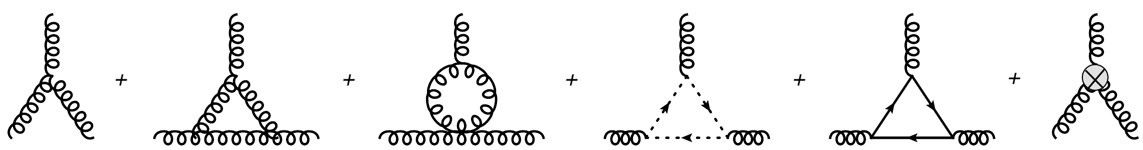
Trading the bare fields and parameters with the renormalized ones by using the relations,

$$\begin{aligned}
A^a_{0\mu} &= \sqrt{Z_3}A^a_\mu\,,\\
c^a_0 &= \sqrt{Z^c_2}c^a\,,\quad \overline{c}^a_0 = \overline{c}^a\sqrt{Z^c_2}\,,\\
g_0 &= Z_g g\mu^{\epsilon/2} = \frac{Z^c_1}{Z^c_2\sqrt{Z_3}} g\mu^{\epsilon/2}\\
\xi_0 &= Z_3\xi\,.
\end{aligned}$$

we get,

$$\begin{aligned}
&\frac{1}{\xi}\langle\Omega| A^a_\mu(x) A^b_\nu(y)\left[\partial\cdot A^c(z)\right]|\Omega\rangle \qquad (3.222)\\
&= \langle\Omega|\Big[Z^c_2\,\partial_\mu c^a(x) + g Z^c_1 f^{ahd} A^h_\mu(x) c^d(x)\Big] A^b_\nu(y)\overline{c}^c(z)|\Omega\rangle\\
&+ \langle\Omega| A^a_\mu(x)\Big[Z^c_2\,\partial_\nu c^b(y) + g Z^c_1 f^{bhd} A^h_\nu(y) c^d(y)\Big]\overline{c}^c(z)|\Omega\rangle\,.
\end{aligned}$$

The left-hand side contains the gluonic 3-point function,



multiplied by the gluon propagators. The gluon propagators are now finite (as the counter term δ_3 has been set), so if we can show that the right-hand side of Eq. (3.222) is finite, we know that the 3-gluon counter term is the correct one.

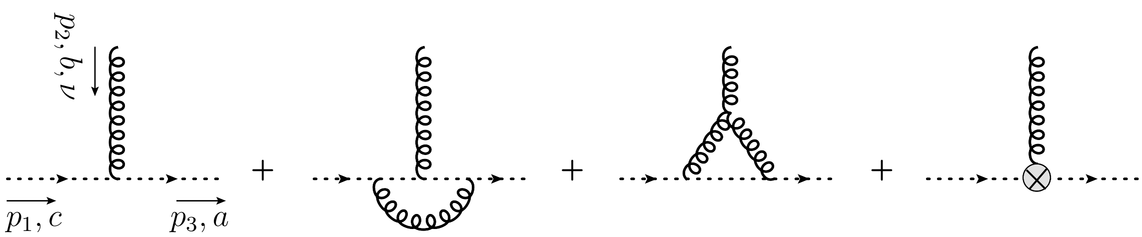
As the first term on the right-hand side of Eq. (3.222) we have,

$$Z_2^c \, \partial_\mu^x \langle \Omega | T c^a(x) A_\nu^b(y) \overline{c}^c(z) | \Omega \rangle \,. \tag{3.223}$$

The ground-state expectation value here is just a ghost-gluon vertex (multiplied with the appropriate propagators) and it's finite. Taking the Fourier transform,

$$\int d^4x e^{ip_3 \cdot x} \int d^4y e^{-ip_2 \cdot y} \int d^4z e^{-ip_1 \cdot z} \left[\partial_\mu^x \langle \Omega | T c^a(x) A_\nu^b(y) \overline{c}^c(z) | \Omega \rangle \right]$$

$$= (-ip_\mu^3) K(p_1) K(p_3) D_{\nu\eta}(p_2) \, \times \tag{3.224}$$



$$= (-ip_\mu^3) K(p_1) K(p_3) D_{\nu\eta}(p_2) \; \times \left[- g f^{abc} p_3^\eta + \cdots \right]$$

The loop diagrams cancel against the counter term (we have explicitly tuned the counter term like this). However, the factor $Z_2^c = 1 + \delta_2^c$ in front of this term is contains a divergence, so at this order the remaining divergent term is

$$(-ip_\mu^3) K(p_1) K(p_3) D_{\nu\eta}(p_2) \; \times \left[- g f^{abc} p_3^\eta \right] \times \delta_2^c \,. \tag{3.225}$$

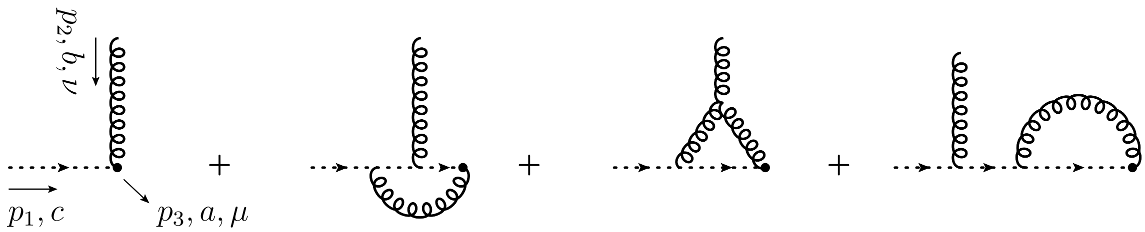
Let's then check the second term on the right-hand of Eq. (3.222). It reads,

$$Z_1^c g f^{ahd} \langle \Omega | T A_\mu^h(x) c^d(x) A_\nu^b(y) \overline{c}^c(z) | \Omega \rangle \tag{3.226}$$

By taking the same Fourier transform as above,

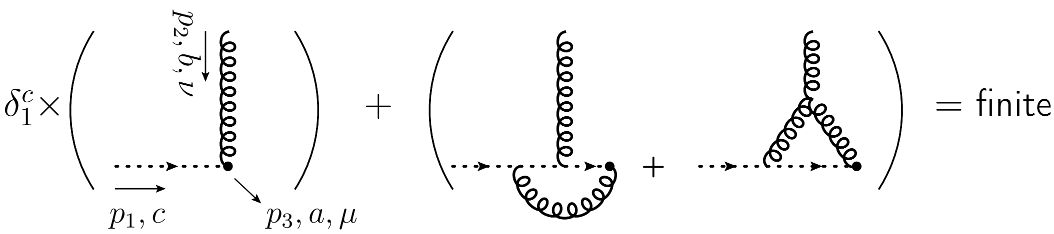
$$\int d^4x e^{ip_3 \cdot x} \int d^4y e^{-ip_2 \cdot y} \int d^4z e^{-ip_1 \cdot z} \left[g f^{ahd} \langle \Omega | T A_\mu^h(x) c^d(x) A_\nu^b(y) \overline{c}^c(z) | \Omega \rangle \right]$$

$$= K(p_1) D_{\nu\eta}(p_2) \, \times \tag{3.227}$$



$$= K(p_1) D_{\nu\eta}(p_2) \; \times \left[g f^{abc} g^{\eta}_{\mu} + \cdots \right]$$

The first three diagrams are essentially the same as in the ghost-gluon vertex function, but just the outgoing ghost line erased. The structure of the diagrams is thus the same and the factor $Z_1^c = 1 + \delta_1^c$ cancels the UV divergences from the two first loop diagrams,



The remaining diagram contains a ghost-gluon 3-point function with a loop correction in the outgoing ghost line. This divergence cancels with the one in Eq. (3.225).

In this way, we can argue why the gluonic 3-point function will be finite without actually computing it explicitly.

3.7.7 Asymptotic freedom of QCD

Having now worked out the counter terms, we can assemble the β function of QCD. From the Slavnov-Taylor identity Eq. (3.194) we see that there are several equivalent ways to find Z_g. Using the ghost-gluon vertex as an example,

$$
\begin{aligned}
Z_g &= \frac{Z_1^c}{Z_2^c\sqrt{Z_3}} = \frac{1+\delta_1^c}{(1+\delta_2^c)\sqrt{1+\delta_3}} \qquad (3.228)\\
&= 1 + \delta_1^c - \delta_2^c - \frac{1}{2}\delta_3 + \mathcal{O}(\alpha_s^2)\\
&= 1 + \frac{\alpha_s}{4\pi}\Bigg[- \left(\frac{C_{\rm G}}{2}\right) \times \frac{2}{\epsilon} + f_{1c}^{\rm scheme} - \left(\frac{C_{\rm G}}{2}\right) \times \frac{2}{\epsilon} - f_{2c}^{\rm scheme}\\
&\qquad - \frac{1}{2}\left[\frac{5}{3}C_{\rm G} - \frac{4}{3}T_R n_f\right]\frac{2}{\epsilon} - \frac{1}{2}f_3^{\rm scheme}\Bigg]\\
&= 1 + \frac{\alpha_s}{4\pi}\left[\frac{-11C_{\rm G}+4T_{\rm R}n_{\rm f}}{3}\right]\frac{1}{\epsilon} + f_{1c}^{\rm scheme} - f_{2c}^{\rm scheme} - \frac{1}{2}f_3^{\rm scheme} .
\end{aligned}
$$

Remember form Eq. (2.157) that in the MS scheme the β function can be obtained from Z_g, via

$$
\beta(g) = -\epsilon\left(\frac{g}{2}\right) + \frac{g^2}{2}\frac{dZ_{g,1}}{dg} , \qquad (3.229)
$$

where $Z_{g,1}$ is the coefficient of the $1/\epsilon$ pole. The 1-loop QCD β function is thus,

$$
\boxed{\beta(g) = -\frac{g^3}{(4\pi)^2}\left[\frac{11C_{\rm G}-4T_{\rm R}n_{\rm f}}{3}\right]} \qquad (3.230)
$$

Provided that

$$
n_{\rm f} < \frac{11C_{\rm G}}{4T_{\rm R}} = \frac{33}{2} ,
$$

i.e. that the number of different quark species is 16 at most, the β function is negative, and the theory is thereby asymptotically free. The origin of this property is in the non-Abelianity of QCD ($\hat{=}$ gluon self interactions) as in QED the structure constants f^{abc} – and thereby also the factor $C_{\rm G}$ – is zero.

The fact that non-Abelian gauge theories can have a negative β function was first discovered by Gross, Wilczek, and Politzer in 1973. This earned them a Nobel Prize in 2004.

The scale-dependent coupling can be obtained from Eq. (2.171),

$$\frac{\partial \overline{g}(g,t)}{\partial t} = \beta\left(\overline{g}(g,t)\right), \quad \overline{g}(g,0) = g\,, \quad t = \log(\mu'/\mu) \tag{3.231}$$

It follows that,

$$\overline{g}^2(g,t) = \frac{g^2}{1 + 2g^2\beta_0 t} \tag{3.232}$$

$$\beta_0 = \frac{1}{(4\pi)^2}\left[\frac{11C_{\mathrm{G}} - 4T_{\mathrm{R}}n_{\mathrm{f}}}{3}\right]. \tag{3.233}$$

If $\beta_0 > 0$ we clearly see that the QCD coupling gets weaker as the scale μ' increases.

It is conventional to define the **QCD scale parameter** Λ_{QCD} by,

$$\Lambda_{\mathrm{QCD}} \equiv \mu \exp\left[\frac{-1}{2g^2\beta_0}\right], \tag{3.234}$$

which corresponds to the scale at which the running coupling $\overline{g}$ diverges. Eliminating the coupling g in favour of Λ_{QCD} gives,

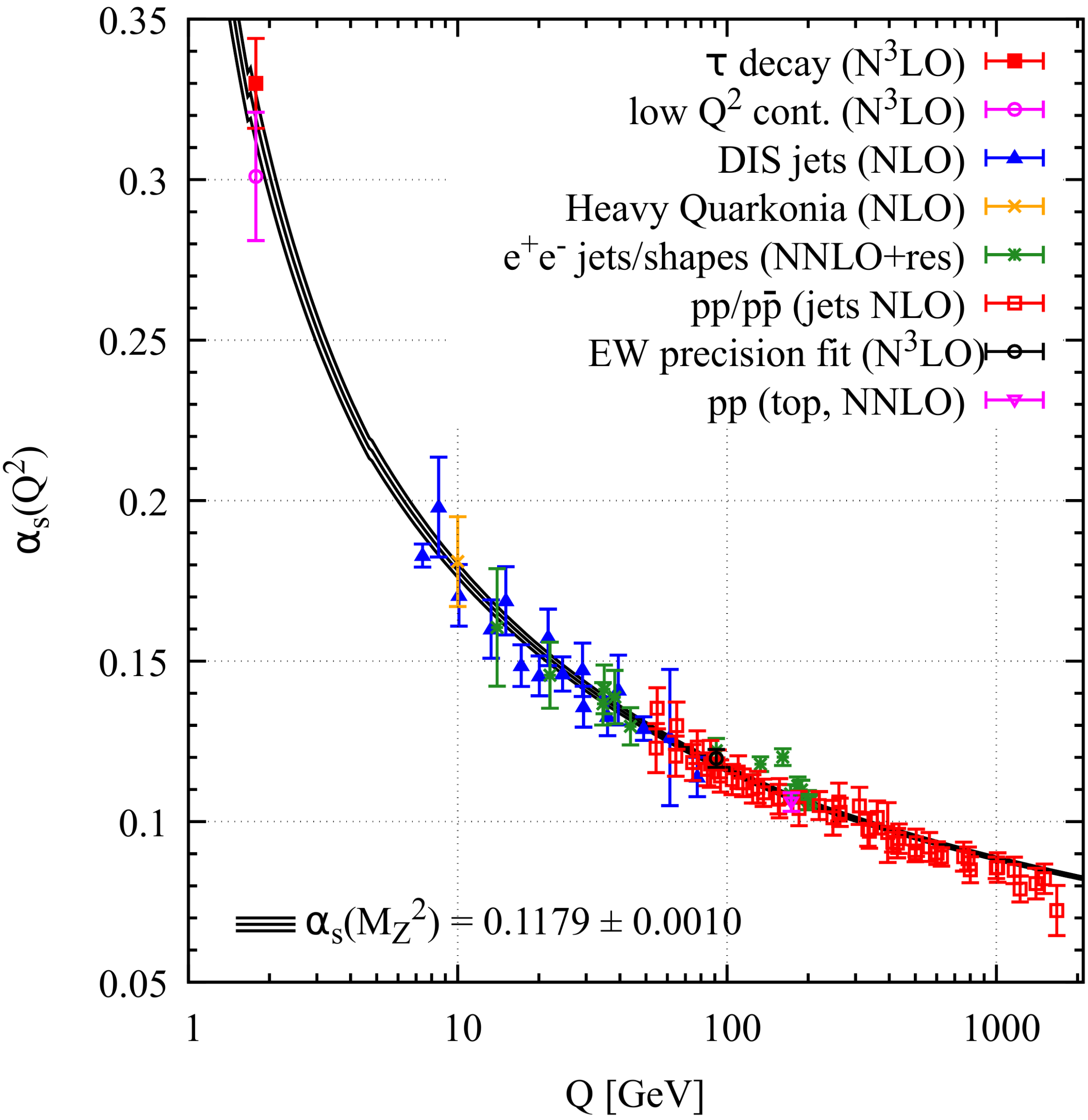
$$\overline{g}^2(g,t) = \frac{1}{\beta_0 \log\left(\mu'^2/\ \Lambda^2_{\mathrm{QCD}}\right)}. \tag{3.235}$$

Replacing $\mu'^2 \to Q^2$ and writing in terms of $\alpha_s = g^2/4\pi$,

$$\alpha_s(Q^2) = \frac{1}{4\pi\beta_0 \log\left(Q^2/\ \Lambda^2_{\mathrm{QCD}}\right)}. \tag{3.236}$$

The above process in which a dimensionless parameter g is replaced by a dimensionful one (Λ_{QCD}) is an example of **dimensional transmutation**. The value of Λ_{QCD} must be obtained by comparing calculated cross sections with experimental data. This results with $\Lambda_{\mathrm{QCD}} \sim 200\,\mathrm{MeV}$. Today, the value of strong coupling is usually given by its value at the renormalization scale $Q^2 = M_Z^2$, where M_Z denotes the mass of the Z boson. The world average is $\alpha_s(M_Z^2) \approx 0.118$.

[Figure from Prog. Theor. Exp. Phys. 2020, 083C01]

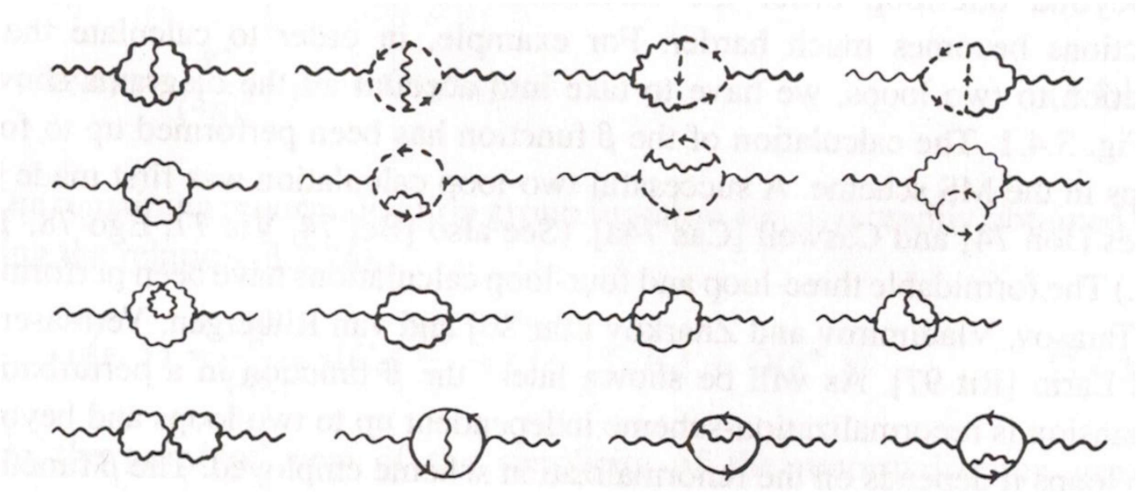


The asymptotic freedom is a necessary property for a theory to describe the nuclear force. Indeed, in the late 60's the first experimental data on inelastic electron-nucleon scattering was approximately consistent with the assumption that the nucleus is made of almost free particles (partons). That is, the interactions between these particles had to become weak at short distances (=high momentum scale). However, at that time all the known renormalizable theories behaved just in the opposite way, i.e. the coupling constant became stronger at short distances until hitting the Landau pole.

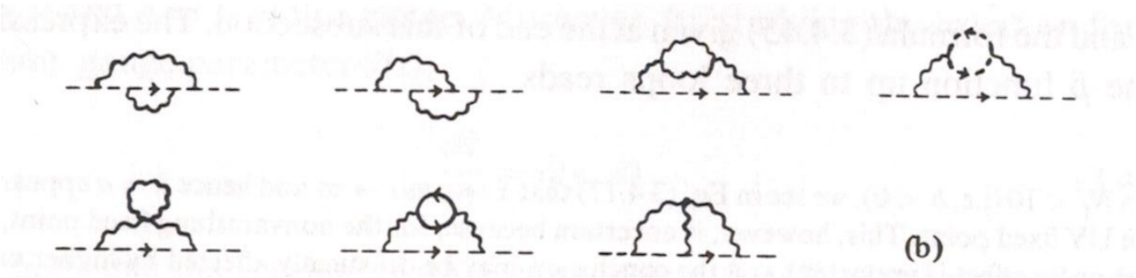
Given the fundamental importance of the asymptotic freedom, it is crucial to know whether this property of QCD stands also at higher orders. To this end, the QCD β function has been calculated up to 5-loop accuracy. Each new loop involves two interaction vertices so the every second power of g appears in the expansion of the QCD β function,

$$\beta(g) = -\beta_0 g^3 - \beta_1 g^5 - \beta_2 g^7 - \beta_3 g^9 + \cdots \qquad (3.237)$$

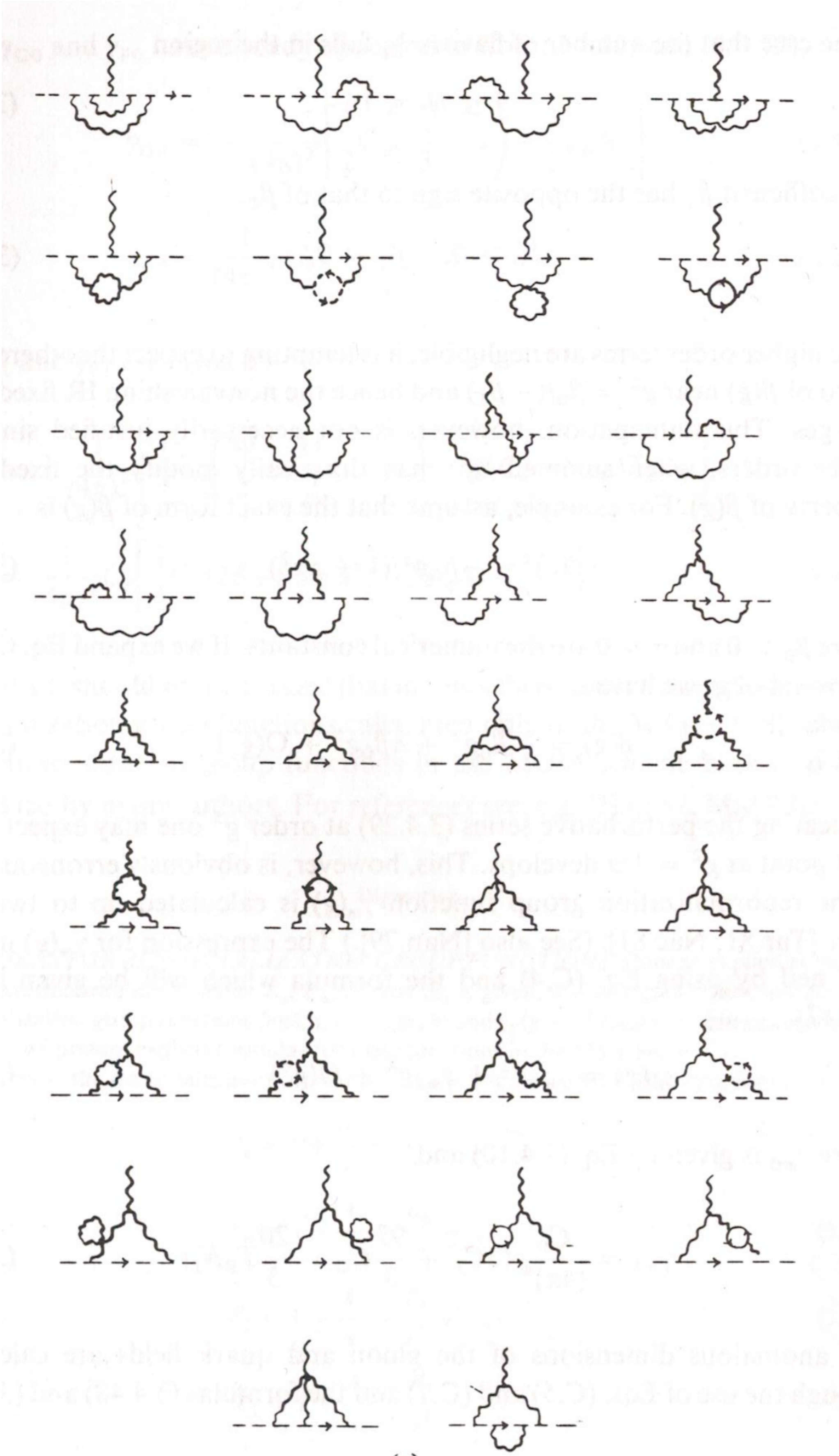
At two-loop accuracy the easiest way to obtain the coefficient b_1 is to calculate Z_g from $Z_g = Z_1^c/(Z_2^c\sqrt{Z_3})$. The diagrams for the 2-loop gluon self-energy corrections are [Figures from Muta]:



The ghost-field renormalization factor Z_2^c involves diagrams:



Finally, the ghost-gluon vertex function involves the following 34 diagrams:



(c)

The full set of known coefficients (in the $\overline{\mathrm{MS}}$ scheme) is [see Phys.Rev.Lett. 118 (2017) 8, 082002] :

$$\beta_0 = \frac{1}{(4\pi)^2}\left[11 - \frac{2n_\mathrm{f}}{3}\right], \tag{3.238}$$

$$\beta_1 = \frac{1}{(4\pi)^4}\left[102 - \frac{38n_\mathrm{f}}{3}\right], \tag{3.239}$$

$$\beta_2 = \frac{1}{(4\pi)^6}\left[\frac{2857}{2} - \frac{5033n_\mathrm{f}}{18} + \frac{325n_\mathrm{f}^2}{54}\right], \tag{3.240}$$

$$\begin{aligned}\beta_3 = \frac{1}{(4\pi)^8}\Bigg\{&\frac{149753}{6} + 3564\zeta_3 - \left[\frac{1078361}{162} + \frac{6508\zeta_3}{27}\right] n_\mathrm{f} \\ &+ \left[\frac{50065}{162} + \frac{6472\zeta_3}{81}\right] n_\mathrm{f}^2 + \frac{1093}{729} n_\mathrm{f}^3\Bigg\},\end{aligned} \tag{3.241}$$

$$\begin{aligned}\beta_4 = \frac{1}{(4\pi)^{10}}\Bigg\{&\frac{8157455}{16} + \frac{621885\zeta_3}{2} - \frac{88209\zeta_4}{2} - 288090\zeta_5 \\ &+ \left[-\frac{336460813}{1944} - \frac{4811164\zeta_3}{81} + \frac{33935\zeta_4}{6} + \frac{1358995\zeta_5}{27}\right] n_\mathrm{f} \\ &+ \left[\frac{25960913}{1944} + \frac{698531\zeta_3}{81} - \frac{10526\zeta_4}{9} - \frac{381760\zeta_5}{81}\right] n_\mathrm{f}^2 \\ &+ \left[-\frac{630559}{5832} - \frac{48722\zeta_3}{243} + \frac{1618\zeta_4}{27} + \frac{460\zeta_5}{9}\right] n_\mathrm{f}^3 \\ &+ \left[\frac{1205}{2916} - \frac{152\zeta_3}{81}\right] n_\mathrm{f}^4\Bigg\},\end{aligned} \tag{3.242}$$

where ζ_s refers to the Riemann zeta function,

$$\zeta_s \equiv \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1\,. \tag{3.243}$$

From the expressions for β_0, β_1, β_2, and β_3 one can easily check that if the number of flavours n_f is small enough, these coefficients remain positive and thus the β function in Eq. (3.237) stays negative. The same is true for β_4 though one will need to know the numerical values of the zeta function to see this. For the important special case $n_\mathrm{f} = 6$,

$$\beta \overset{(n_\mathrm{f}=6)}{\approx} -\beta_0 g^3 \left[1 + 0.93\left(\frac{\alpha_s}{\pi}\right) - 0.29\left(\frac{\alpha_s}{\pi}\right)^2 + 5.52\left(\frac{\alpha_s}{\pi}\right)^3 + 0.15\left(\frac{\alpha_s}{\pi}\right)^4\right].$$

This is positive definite for all values of α_s, and as the coefficients in front of the factors $(\alpha_s/\pi)^n$ are not particularly large, the perturbative series seems to converge very well.

Beyond the 1-loop approximation, the expression for the running coupling cannot be expressed in a closed form but approximate formulas can be derived. We write first the evolution equation Eq. (2.171) in a form,

$$\int_{t(\Lambda)}^{t(Q)} = \log\left(\frac{Q}{\Lambda}\right) = -\int_{g(\Lambda)}^{g(Q)} \frac{d\lambda}{\lambda^3}\frac{1}{\beta_0 + \beta_1\lambda^2 + \beta_2\lambda^4 + \cdots}. \tag{3.244}$$

If only the two first coefficients of the β function are kept, the above integral gives an exact result,

$$\log\left(\frac{Q^2}{\Lambda^2}\right) = \frac{1}{\beta_0}\left[\!\!\left[\frac{1}{g^2(Q)} - \frac{1}{g^2(\Lambda)} + \frac{\beta_1}{\beta_0}\log\frac{g^2(Q)\left[\beta_0 + \beta_1 g^2(\Lambda)\right]}{g^2(\Lambda)\left[\beta_0 + \beta_1 g^2(Q)\right]}\right]\!\!\right], \tag{3.245}$$

which, however, cannot be solved for $g^2(Q)$ in an analytic way. Beyond the 2-loop accuracy, the integral in Eq. (3.244) can be performed (Ex.) by expanding the integrand in powers of λ^2, assuming it's $\ll 1$. Keeping terms up to β_3,

$$\log\left(\frac{Q^2}{\Lambda^2}\right) = \frac{1}{\beta_0}\left[\!\!\left[\frac{1}{g^2(Q^2)} + \frac{2\beta_1}{\beta_0}\log\left[g(Q^2)\right] + \left(\frac{\beta_2}{\beta_0} - \frac{\beta_1^2}{\beta_0^2}\right)g^2(Q^2)\right.\right.$$
$$\left.\left. + \left(\frac{\beta_3}{2\beta_0} - \frac{\beta_1\beta_2}{\beta_0^2} + \frac{\beta_1^3}{2\beta_0^3}\right)g^4(Q^2) + \mathcal{O}(g^6)\right]\!\!\right] + C\,. \tag{3.246}$$

By defining the constant C, we essentially define the strength of the coupling at the reference scale Λ (but not yet the value of Λ itself). It is conventional to adopt the definition

$$C = \frac{\beta_1}{\beta_0^2} \log \beta_0 \,. \tag{3.247}$$

By solving Eq. (3.246) iteratively (Ex.), we find an approximate expression for the scale-dependent strong coupling $\alpha_s(Q^2)$,

$$\begin{aligned}\alpha_s(Q^2) = \frac{1}{4\pi\beta_0 L} \Bigg[\!\!\Bigg[1 - \frac{b_1 \log L}{\beta_0 L} + \frac{1}{(\beta_0 L)^2} \left[b_1^2 \left(\log^2 L - \log L - 1 \right) + b_2 \right] \\ + \frac{1}{(\beta_0 L)^3} \left[b_1^3 \left(-\log^3 L + \frac{5}{2} \log^2 L + 2 \log L - \frac{1}{2} \right) - 3 b_1 b_2 \log L + \frac{b_3}{2} \right] \\ + \mathcal{O}(1/L^4) \Bigg]\!\!\Bigg] \,, \end{aligned} \tag{3.248}$$

where $L = \log(Q^2/\Lambda^2)$ and $b_k = \beta_k/\beta_0$. One can now tune the parameter Λ such that e.g. $\alpha_s(M_Z^2) \approx 0.118$.

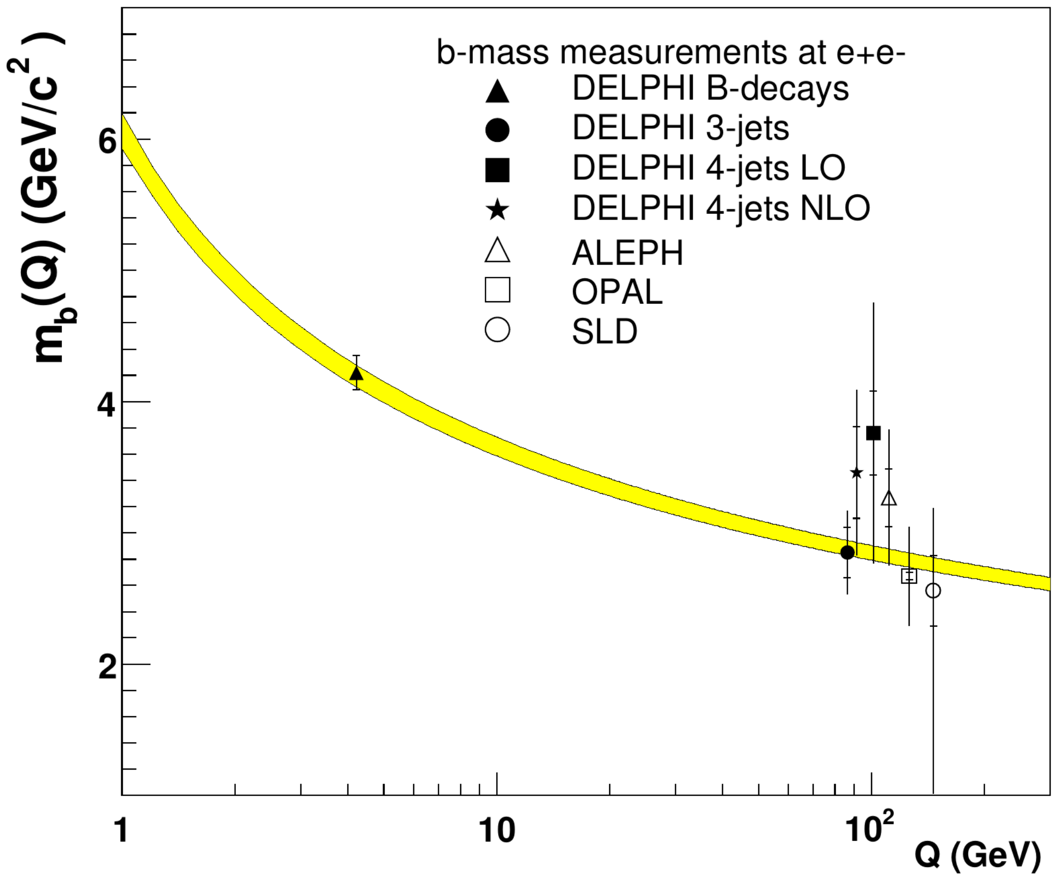
Using the result of this and Chapter 2, we can also work out the 1-loop scale-dependence of the quark masses. In the $\overline{\mathrm{MS}}$ scheme,

$$\gamma_m(\alpha_s) = \left(\frac{\alpha_s}{4\pi} \right) 6 C_F \tag{3.249}$$

and it follows that (Ex.),

$$m_{\overline{\mathrm{MS}}}(Q^2) = m_{\overline{\mathrm{MS}}}(m_{\overline{\mathrm{MS}}}^2) \left[1 - \frac{\alpha_s(Q^2)}{\pi} \log \left(\frac{Q^2}{m_{\overline{\mathrm{MS}}}^2} \right) \right] . \tag{3.250}$$

By using here the explicit 1-loop expression for $\alpha_s(Q^2)$ from Eq. (3.236), we see that the latter term becomes a constant at large Q^2 so the quark masses cease to run towards asymptotically large Q^2. The running of the quark masses has been experimentally "confirmed" and the figure below shows an example in the case of bottom/beauty quark [Figure from Eur.Phys.J.C 55 (2008) 525-538] :



The constant γ_m has been calculated up to 4 loops. The first three coefficients are [PDG, Quark Masses] ,

$$\gamma_m^{\overline{\mathrm{MS}}}(\alpha_s) = 8\left(\frac{\alpha_s}{4\pi}\right) + 2\left(\frac{202}{3} - \frac{20n_\mathrm{f}}{9}\right)\left(\frac{\alpha_s}{4\pi}\right)^2 \tag{3.251}$$
$$+ 2\left[1249 + \left(-\frac{2216}{27} - \frac{160\zeta(3)}{3}\right)n_\mathrm{f} - \frac{140n_\mathrm{f}^2}{81}\right]\left(\frac{\alpha_s}{4\pi}\right)^3 ,$$

where n_f is the number of quark flavours.

The relation between the pole mass m_{P} and the $\overline{MS}$ mass $m_{\overline{\mathrm{MS}}}$ is known up to 3 loops [see Phys.Lett.B 482 (2000) 99-108] ,

$$m_{\mathrm{P}} = m_{\overline{\mathrm{MS}}}(m_{\overline{\mathrm{MS}}}) \left[\!\!\left[1 + \frac{4}{3}\left(\frac{\alpha_s(m_{\overline{\mathrm{MS}}})}{\pi}\right) \right.\right. \tag{3.252}$$

$$+ \left[-1.0414 N_{\mathrm{L}} + 13.4434 \right] \left(\frac{\alpha_s(m_{\overline{\mathrm{MS}}})}{\pi}\right)^2$$

$$\left.\left. + \left[0.6527 N_{\mathrm{L}}^2 - 26.655 N_{\mathrm{L}} + 190.595 \right] \left(\frac{\alpha_s(m_{\overline{\mathrm{MS}}})}{\pi}\right)^3 \right]\!\!\right]$$

Here N_{L} is the number massless quark flavours. In the case of bottom/beauty quark $m_{\overline{\mathrm{MS}},\mathrm{bottom}}(m_{\overline{\mathrm{MS}},\mathrm{bottom}}) \sim 4\,\mathrm{GeV}$ and at this scale $\alpha_s(m_{\overline{\mathrm{MS}},\mathrm{bottom}}) \sim 0.24$. This gives,

$$m_{\mathrm{P},\mathrm{bottom}} \approx m_{\overline{\mathrm{MS}},\mathrm{bottom}}(m_{\overline{\mathrm{MS}},\mathrm{bottom}}) \left[1 + 0.10 + 0.05 + 0.03 \right] \tag{3.253}$$

Although the higher-order corrections may appear small at first sight, the loop corrections are all of the same order of magnitude, so the perturbative expansion seems to converge rather badly. This is called a **renormalon problem**.

4 Weak interactions

The phenomenology of weak interactions is discussed in the Particle Physics course. Here, we will first remind ourselves of the ingredients of the Glashow-Weinberg-Salam (GWS) electroweak Lagrangian and then see how to quantize it.

4.1 Electroweak Lagrangian [Peskin 20, 11.1]

The GWS theory of weak interactions is based on imposing a local $\mathrm{U}(1) \times \mathrm{SU}(2)$ invariance. In other words, the Lagrangian should remain intact when the matter fields of type k undergo a transformation,

$$\boxed{\psi_k(x) \to \psi'_k(x) = \exp\left[iY_k\chi(x)\right] \exp\left[i\theta^i(x)\frac{\sigma^i}{2}\right]\psi_k(x)\,,} \tag{4.1}$$

where the parameters Y_k are called **hypercharges**, and σ_i are the usual Pauli spin matrices (= the generators of $\mathrm{SU}(2)$ transformations),

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.2}$$

which fulfill the commutation relation

$$\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = i\epsilon^{ijk}\left(\frac{\sigma^k}{2}\right), \tag{4.3}$$

$$\epsilon^{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of } 123 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123 \\ \;\;0 & \text{otherwize} \end{cases}. \tag{4.4}$$

So here ϵ^{ijk} plays the role of structure constants f^{abc} in QCD. Repeating the steps of Sect. 3.2 we find that in this case we can express the covariant derivative in the form,

$$D_\mu \psi_k(x) = \left[\partial_\mu - igA^i_\mu(x)\frac{\sigma^i}{2} - ig'Y_k B_\mu(x)\right]\psi_k(x)\,, \tag{4.5}$$

where g and g' are real parameters and the vector fields $A^i_\mu(x)$ and $B_\mu(x)$ transform according to,

$$B_\mu(x) \to B_\mu + \frac{1}{g'}\partial_\mu \chi\,, \tag{4.6}$$

$$A^i_\mu(x)\frac{\sigma^i}{2} \to U(x)\left[A^i_\mu(x)\frac{\sigma^i}{2} + \frac{i}{g}\partial_\mu\right]U^\dagger(x)\,, \tag{4.7}$$

$$U(x) = \exp[i\theta^i(x)\frac{\sigma^i}{2}]$$

or for an infinitesimal $\theta^i(x)$,

$$A^i_\mu(x) \to A^i_\mu + \frac{1}{g}\partial_\mu\theta^i + \epsilon^{ijk}A^j_\mu\theta^k\,, \tag{4.8}$$

As in Sect. 3.2 the covariant derivatives are objects which transform as the field on which they act, so terms

$$i\overline{\psi}_k\gamma^\mu D_\mu\psi_k \tag{4.9}$$

are invariant under the transformation specified by Eqs. (4.1) and (4.6).

The construction of the field-strength tensors is also identical to the case of QCD and by defining (see Eq. (3.56)),

$$B_{\mu\nu} \equiv \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x)\,, \tag{4.10}$$

$$A^i_{\mu\nu} \equiv \partial_\mu A^i_\nu(x) - \partial_\nu A^i_\mu(x) + g\epsilon^{ijk}A^j_\mu(x)A^k_\nu(x)\,, \tag{4.11}$$

we know that the combinations

$$-\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}A^{i}_{\mu\nu}A^{i,\mu\nu} \tag{4.12}$$

remain invariant in transformations of Eq. (4.6) and Eq. (4.7).

4.1.1 Matter content

The essence of the GWS theory is to treat the left- and right-chiral fermion fields on a different footing. To see why this is even possible, we first recall from QFT I that the left- and right-handed components of a Dirac field are defined as

$$\text{Left-handed component: } \psi_L = P_L\psi = \left(\frac{1-\gamma^5}{2}\right)\psi \tag{4.13}$$

$$\text{Right-handed component: } \psi_R = P_R\psi = \left(\frac{1+\gamma^5}{2}\right)\psi\,. \tag{4.14}$$

Now, since $\psi = \psi_L + \psi_R$, the gauge invariant terms $i\overline{\psi}\gamma^\mu D_\mu\psi$ can be written as

$$\begin{aligned} i\overline{\psi}\gamma^\mu D_\mu\psi &= i\overline{\psi}_L\gamma^\mu D_\mu\psi_L + i\overline{\psi}_R\gamma^\mu D_\mu\psi_R \\ &\quad + i\overline{\psi}_L\gamma^\mu D_\mu\psi_R + i\overline{\psi}_R\gamma^\mu D_\mu\psi_L\,. \end{aligned} \tag{4.15}$$

However, due to the properties of γ^5, the mixed terms will give zero. For example,

$$\begin{aligned} i\overline{\psi}_L\gamma^\mu D_\mu\psi_R &= i\psi^\dagger\left(\frac{1-\gamma^5}{2}\right)\gamma^0\gamma^\mu D_\mu\left(\frac{1+\gamma^5}{2}\right)\psi \\ &= i\overline{\psi}\left(\frac{1+\gamma^5}{2}\right)\gamma^\mu D_\mu\left(\frac{1+\gamma^5}{2}\right)\psi \\ &= i\overline{\psi}\left(\frac{1+\gamma^5}{2}\right)\left(\frac{1-\gamma^5}{2}\right)\gamma^\mu D_\mu\psi = 0\,. \end{aligned} \tag{4.16}$$

The gauge invariant terms $i\overline{\psi}\gamma^\mu D_\mu\psi$ thus split into two parts,

$$i\overline{\psi}\gamma^\mu D_\mu\psi = i\overline{\psi}_L\gamma^\mu D_\mu\psi_L + i\overline{\psi}_R\gamma^\mu D_\mu\psi_R\,, \tag{4.17}$$

i.e. the kinetic and interaction terms of the left- and right-chiral components decouple. For this reason the hypercharges Y_k can be different between ψ_R and ψ_L. We can also assign the two chiral components of fermion fields into different representations of the gauge group: The left-chiral fermions go to doublets [which transform under SU(2)], while the right-chiral fermions go to singlets [which do not transform under SU(2)]. For the three families of leptons and quarks,

$$\begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix}, \quad \begin{pmatrix} \nu_L^\mu \\ \mu_L \end{pmatrix}, \quad \begin{pmatrix} \nu_L^\tau \\ \tau_L \end{pmatrix}, \quad (e_R)\,, \quad (\mu_R)\,, \quad (\tau_R)$$

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \quad \begin{pmatrix} t_L \\ b_L \end{pmatrix}, \quad \begin{matrix} (u_R) \\ (d_R) \end{matrix}\,, \quad \begin{matrix} (c_R) \\ (s_R) \end{matrix}\,, \quad \begin{matrix} (t_R) \\ (b_R) \end{matrix}$$

Each symbol here denotes a Dirac field, e.g. $e_L = P_L\psi_e$. In the standard GWS theory there are no right-chiral neutrinos.

For example, the left-handed neutrino-electron doublet transforms as

$$\begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix} \to \begin{pmatrix} \nu_L^{e\prime} \\ e_L' \end{pmatrix} = \exp\left[iY_{\nu e}\chi(x)\right] \exp\left[i\theta^i(x)\frac{\sigma^i}{2}\right] \begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix}, \tag{4.18}$$

whereas e.g. the right-handed electron singlet transforms as

$$e_R \to e_R' = \exp\left[iY_e\chi(x)\right] e_R\,. \tag{4.19}$$

The drawback of assigning left- and right-handed fields into different representations is that the usual fermion mass terms $-m\overline{\psi}\psi$ are forbidden. There are again 4 terms,

$$-m\overline{\psi}\psi = -m\overline{\psi}_L\psi_L - m\overline{\psi}_R\psi_R - m\overline{\psi}_L\psi_R - m\overline{\psi}_R\psi_L\,, \tag{4.20}$$

but now the "diagonal" ones are zero since e.g.,

$$\overline{\psi}_L \psi_L = \psi^\dagger \left(\frac{1-\gamma^5}{2}\right) \gamma^0 \left(\frac{1-\gamma^5}{2}\right) \psi \tag{4.21}$$

$$= \overline{\psi} \left(\frac{1+\gamma^5}{2}\right) \left(\frac{1-\gamma^5}{2}\right) \psi = 0\,.$$

Thus, in the mass terms the left- and right-handed components mix,

$$-m\overline{\psi}\psi = -m\overline{\psi}_L \psi_R - m\overline{\psi}_R \psi_L\,. \tag{4.22}$$

These terms are not gauge invariant as e.g. the transformations of the left-handed fields mix the two components in the doublet, e.g.

$$e_L \longrightarrow e'_L = A e_L + B \nu^e_L\,.$$

In addition, the hypercharges of ψ_L and ψ_R may not be the same.

4.1.2 Interactions of the matter and gauge fields

Having assigned the fermion fields into specific representations, the covariant derivatives now uniquely fix the interactions between the fermion fields and the gauge bosons. The covariant derivative in Eq. (4.9) involves an interaction terms,

$$g\overline{\psi}\gamma^\mu \Big[A^i_\mu(x)\frac{\sigma^i}{2} + g' Y_k B_\mu(x)\Big]\psi\,. \tag{4.23}$$

We write $A^i_\mu(x)\sigma^i/2$ explicitly,

$$A^i_\mu(x)\frac{\sigma^i}{2} = \frac{A^1_\mu}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{A^2_\mu}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A^3_\mu}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.24}$$

$$= \frac{1}{2}\begin{pmatrix} A^3_\mu & A^1_\mu - iA^2_\mu \\ A^1_\mu + iA^2_\mu & -A^3_\mu \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} A^3_\mu/\sqrt{2} & W^\dagger_\mu \\ W_\mu & -A^3_\mu/\sqrt{2} \end{pmatrix},$$

where in the last step we defined a complex vector field

$$W_\mu \equiv (A^1_\mu + iA^2_\mu)/\sqrt{2}\,. \tag{4.25}$$

To see what kind of interaction term is the one that involves W_μ, we can use e.g. the electron-neutrino doublet as an example,

$$\begin{aligned}
&\frac{g}{\sqrt{2}}\begin{pmatrix} \overline{\nu}^e_L & \overline{e}_L \end{pmatrix}\gamma^\mu \begin{pmatrix} 0 & W^\dagger_\mu \\ W_\mu & 0 \end{pmatrix}\begin{pmatrix} \nu^e_L \\ e_L \end{pmatrix} = \frac{g}{\sqrt{2}}\overline{\nu}^e_L \not{W}^\dagger_\mu e_L + \frac{g}{\sqrt{2}}\overline{e}_L \not{W}_\mu \nu^e_L \\
&= \frac{g}{\sqrt{2}}\overline{\nu}^e \left(\frac{1+\gamma^5}{2}\right)\not{W}^\dagger_\mu \left(\frac{1-\gamma^5}{2}\right) e + \text{h.c.} \qquad (4.26)\\
&= \frac{g}{\sqrt{2}}\overline{\nu}^e \not{W}^\dagger_\mu \left(\frac{1-\gamma^5}{2}\right)\left(\frac{1-\gamma^5}{2}\right) e + \text{h.c.} \\
&= \frac{g}{2\sqrt{2}}\overline{\nu}^e_L \not{W}^\dagger_\mu \left(1-\gamma^5\right) e + \text{h.c.} \qquad\qquad (\text{h.c.} = \text{Hermitian conjugation})
\end{aligned}$$

We see that there are interactions between the two fields in the left-handed doublets (e.g. between the electron and its neutrino). The complex vector field W_μ represents the $W^\pm$ boson. The interaction above is called the **charged-current interaction**.

Let us then group together the remaining diagonal interaction terms.

For left-handed doublets:

$$\overline{\psi}_k\gamma^\mu \begin{pmatrix} g'Y_kB_\mu + gA^3_\mu/2 & 0 \\ 0 & g'Y_kB_\mu - gA^3_\mu/2 \end{pmatrix}\psi_k\,. \tag{4.27}$$

For right-handed singlets:

$$\overline{\psi}_k\gamma^\mu\left[g'Y_kB_\mu\right]\psi_k\,. \tag{4.28}$$

We would like to identify the photon field. It cannot be B_μ as that would imply that e.g. electron and its neutrino have the same strength of interaction, i.e. the electric charge. Nor can it be A^3_μ as that generates interaction only between the left-handed fields (in QED the interaction term is the same for both handednesses). The solution is to try a linear combination of the two which preserves the overall normalizations. Thus, we express the fields B_μ and A_μ^3 in terms of two other fields A_μ and Z_μ:

$$\begin{pmatrix} A_\mu^3 \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathrm{W}} & \sin\theta_{\mathrm{W}} \\ -\sin\theta_{\mathrm{W}} & \cos\theta_{\mathrm{W}} \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} . \tag{4.29}$$

Here the parameter θ_{W} is known as the **Weinberg weak-mixing angle**. Writing Eq. (4.27) and Eq. (4.28) in terms of A_μ and Z_μ gives,

<u>For left-handed doublets:</u>

$$\overline{\psi}_k \gamma^\mu \left[A_\mu \left(\frac{\sigma^3}{2} g \sin\theta_{\mathrm{W}} + Y_k g' \cos\theta_{\mathrm{W}} \right) + Z_\mu \left(\frac{\sigma^3}{2} g \cos\theta_{\mathrm{W}} - Y_k g' \sin\theta_{\mathrm{W}} \right) \right] \psi_k$$

<u>For right-handed singlets:</u>

$$\overline{\psi}_k \gamma^\mu \left[A_\mu \left(Y_k g' \cos\theta_{\mathrm{W}} \right) + Z_\mu \left(-Y_k g' \sin\theta_{\mathrm{W}} \right) \right] \psi_k$$

From the above terms we see that we can identify A_μ as the photon field if we choose the parameters g, g' and θ_{W} such that the elementary charge e is given by

$$e = g' \cos\theta_{\mathrm{W}} = g \sin\theta_{\mathrm{W}} \,, \tag{4.30}$$

and furthermore choose the hypercharges appropriately. For example, in the case of u and d quarks the fractional charges are $Q_u = 2/3$ and $Q_d = -1/3$, so we need

$$\begin{pmatrix} Q_u & 0 \\ 0 & Q_d \end{pmatrix} = \frac{\sigma_3}{2} + Y_{ud} = \begin{pmatrix} Y_{ud} + 1/2 & 0 \\ 0 & Y_{ud} - 1/2 \end{pmatrix}$$

$$Q_u = Y_u \tag{4.31}$$

$$Q_d = Y_d$$

which implies that

$$Y_{ud} = 1/6 \,, \quad Y_u = 2/3 \,, \quad Y_d = -1/3 \,. \tag{4.32}$$

to get the correct fractional charges. Similarly for other quark/lepton species. In this way, the the terms involving the A_μ field reduce to standard QED interaction terms,

$$\mathcal{L}_{\text{QEDint.}} = \sum_f eQ_f \overline{\psi}_f \not{A} \psi_f \,. \tag{4.33}$$

The remaining interaction terms involving Z_μ can be written as

$$\frac{e}{2\sin\theta_{\rm W}\cos\theta_{\rm W}} Z_\mu \left[\overline{\psi}_{kL}\gamma^\mu \left(T_k^3 - 2Q_k\sin^2\theta_{\rm W}\right)\psi_{kL} + \overline{\psi}_{kR}\gamma^\mu \left(-2Q_k\sin^2\theta_{\rm W}\right)\psi_{kR}\right],$$

where T_k^3 is the eigenvalue of the σ^3, i.e $+1$ for up-type quarks and neutrinos and -1 for down-type quarks and charged leptons:

$$\sigma^3 \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ 0 \end{pmatrix} = +1 \times \begin{pmatrix} u \\ 0 \end{pmatrix}$$

$$\sigma^3 \begin{pmatrix} 0 \\ d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ d \end{pmatrix} = -1 \times \begin{pmatrix} 0 \\ d \end{pmatrix}.$$

Since the field Z_μ is real, it represents a neutral gauge boson. Thus, the corresponding interaction term is called a **neutral-current interaction** and the field Z_μ is identified with the neutral Z boson. The neutral-current Lagrangian can be written compactly as

$$\mathcal{L}_{\rm NC} = g_Z \sum_k \overline{\psi}_k \gamma^\mu Z_\mu \left[L_k(1-\gamma^5) + R_k(1+\gamma^5)\right]\psi_k \,, \tag{4.34}$$

where

$$g_Z = \frac{e}{4\sin\theta_{\rm W}\cos\theta_{\rm W}} = \frac{g}{4\cos\theta_{\rm W}} \tag{4.35}$$

$$L_k = T_k^3 - 2Q_k\sin^2\theta_{\rm W} \tag{4.36}$$

$$R_k = -2Q_k\sin^2\theta_{\rm W} \tag{4.37}$$

The resulting values of L_k and R_k are:

Table 1: Neutral-current couplings of fermions

coupling	ν_e, ν_μ, ν_τ	e, μ, τ	u, c, t	d, s, b
L_f	1	$-1+2\sin^2\theta_{\rm W}$	$1-\frac{4}{3}\sin^2\theta_{\rm W}$	$-1+\frac{2}{3}\sin^2\theta_{\rm W}$
R_f	0	$2\sin^2\theta_{\rm W}$	$-\frac{4}{3}\sin^2\theta_{\rm W}$	$\frac{2}{3}\sin^2\theta_{\rm W}$

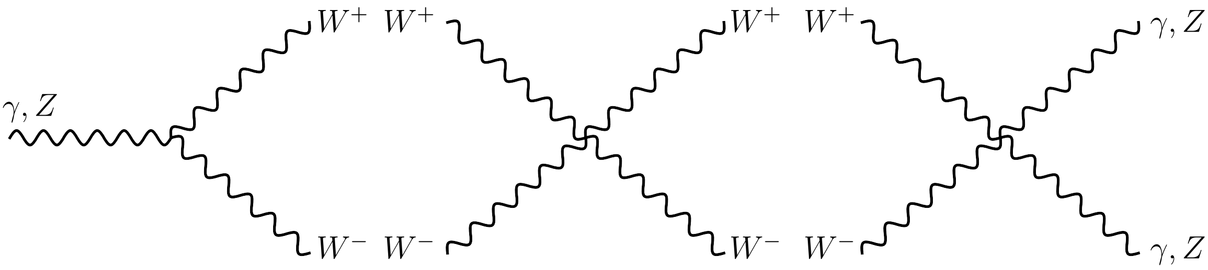
By taking the definitions of the A_μ and Z_μ fields from Eq. (4.29) and substituting them into Eq. (4.12), we find the quadratic kinetic terms,

$$\begin{aligned} &-\frac{1}{4}\left(\partial_\mu A_\nu-\partial_\nu A_\mu\right)\left(\partial^\mu A^\nu-\partial^\nu A^\mu\right) \\ &-\frac{1}{4}\left(\partial_\mu Z_\nu-\partial_\nu Z_\mu\right)\left(\partial^\mu Z^\nu-\partial^\nu Z^\mu\right) \\ &-\frac{1}{2}\left(\partial_\mu W^\dagger_\nu-\partial_\nu W^\dagger_\mu\right)\left(\partial^\mu W^\nu-\partial^\nu W^\mu\right) \end{aligned} \tag{4.38}$$

plus qubic and quartic interaction terms (Ex.),

$$\begin{aligned} \mathcal{L}_3 &= -ie\cot\theta_{\rm W}\left\{\left(\partial^\mu W^\nu-\partial^\nu W^\mu\right)W^\dagger_\mu Z_\nu-\left(\partial^\mu W^{\nu\dagger}-\partial^\nu W^{\mu\dagger}\right)W_\mu Z_\nu\right. \\ &\quad \left.+\left(\partial^\mu Z^\nu-\partial^\nu Z^\mu\right)W_\mu W^\dagger_\nu\right\} \\ &\quad -ie\left\{\left(\partial^\mu W^\nu-\partial^\nu W^\mu\right)W^\dagger_\mu A_\nu-\left(\partial^\mu W^{\nu\dagger}-\partial^\nu W^{\mu\dagger}\right)W_\mu A_\nu\right. \\ &\quad \left.+\left(\partial^\mu A^\nu-\partial^\nu A^\mu\right)W_\mu W^\dagger_\nu\right\} \\ \mathcal{L}_4 &= -\frac{e^2}{2\sin\theta_{\rm W}}\left\{(W^\dagger_\mu W^\mu)^2-W^\dagger_\mu W^{\mu\dagger}W_\nu W^\nu\right\} \\ &\quad -e^2\cot^2\theta_{\rm W}\left\{W^\dagger_\mu W^\mu Z_\nu Z^\nu-W^\dagger_\mu W^\nu Z_\nu Z^\mu\right\} \\ &\quad -e^2\left\{W^\dagger_\mu W^\mu A_\nu A^\nu-W^\dagger_\mu W^\nu A_\nu A^\mu\right\} \\ &\quad -e^2\cot\theta_{\rm W}\left\{2W^\dagger_\mu W^\mu A^\nu Z_\nu-W^\dagger_\mu W^\nu\left[A_\nu Z^\mu+Z_\nu A^\mu\right]\right\}. \end{aligned} \tag{4.39}$$

Terms $\mathcal{L}_3$ and $\mathcal{L}_4$ generate three types of interactions between the gauge bosons:



4.1.3 The Higgs mechanism

In the construction above, all particles are massless as the gauge invariance forbids all mass terms. To generate masses for fermions and gauge bosons, we will need to invoke the Higgs mechanism. This is done by introducing a complex doublet of scalar fields

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1(x) + i\phi_2(x) \\ \phi_3(x) + i\phi_4(x) \end{pmatrix} = \begin{pmatrix} \phi^+(x) \\ \phi^0(x) \end{pmatrix}, \tag{4.40}$$

where the designations "$+$" and "0" anticipate that we are going to arrange the hypercharges such that $\phi^+(x)$ carries electric charge $+1$ and $\phi^0(x)$ is neutral. We introduce the following ϕ-dependent terms in the Lagrangian,

$$\mathcal{L}_{\text{Higgs}} = (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi) \tag{4.41}$$

$$V(\phi) = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2\,, \tag{4.42}$$

where the covariant derivative is the same as in Eq. (4.5),

$$D_\mu\phi(x) = \left[\partial_\mu - igA^i_\mu(x)\frac{\sigma^i}{2} - ig'Y_\phi B_\mu(x)\right]\phi(x)\,. \tag{4.43}$$

Thanks to the covariant derivative, $\mathcal{L}_{\text{Higgs}}$ is invariant in gauge transformations. Since,

$$T^3_{\phi^+} = 1\,, \quad T^3_{\phi^0} = -1\,,$$