Mappings of finite distortion: A new proof for discreteness and openness

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Abstract

We give a new and elementary proof of the known result: a mapping of finite distortion $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is discrete and open provided that its distortion function $K \in L^1_{\text{loc}}(\Omega)$ if $n = 2$; $K \in L^p_{\text{loc}}(\Omega)$ for some $p > n - 1$ if $n \geq 3$.

1 Introduction

Let $\Omega$ be a connected open set in $\mathbb{R}^n$, $n \geq 2$, and $f : \Omega \to \mathbb{R}^n$ be a mapping in the Sobolev class $W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$. The mapping has finite distortion provided that there is a measurable function $K = K(x) \geq 1$, such that $f$ satisfies the distortion inequality

$$|Df(x)|^n \leq K(x) J(x, f) = K(x) \det Df(x) \quad \text{a.e.}$$

Here and what follows, $|Df(x)|$ stands for the operator norm of the differential matrix

$$Df(x) = \begin{bmatrix} \frac{\partial f_i}{\partial x_j} \end{bmatrix}_{i,j=1,\ldots,n} \in \mathbb{R}^{n \times n}.$$

Reshetnyak [13] proved a remarkable result that if $K$ is bounded then mappings of finite distortion (called mappings of bounded distortion or quasiregular mappings) are continuous, open and discrete. Openness of a continuous mapping $f$ requires that it maps open sets to open sets and discreteness that the pre-image of any point in $\mathbb{R}^n$ is a set of isolated points in $\Omega$. For the exposition of the theory of mappings of bounded distortion, we refer the reader to the monographs by Reshetnyak [14], Rickman [15] and Iwaniec and Martin [9].

Since then, there has been great interest to generalize Reshetnyak’s result to mappings of finite distortion. Gol’dstein and Vodop’yanov showed in [4] that mappings of finite distortion in the Sobolev class $W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ are continuous.

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With regards to the discreteness and openness, let us mention the example by Ball [2]. In 1981, he constructed a nonconstant Lipschitz mapping of finite $L^p$-integrable distortion for all $p < n - 1$ that maps a line segment to a point. The mapping is a homeomorphism outside the line segment and as such, is neither discrete nor open.

In 1993, Iwaniec and Šverák [10] established discreteness and openness for mappings $f \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^n)$ with $L^1$-integrable distortion. They also conjectured that nonconstant mappings $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ whose distortion function lies on $L^{n-1}$ are open and discrete. Their two dimensional result was based on the solvability of the Beltrami equation, a method which is not available in higher dimensions.

Subsequently, Heinonen and Koskela [6] proved in higher dimensions that a quasi-light mapping $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ of finite distortion $K \in L^p(\Omega)$, where $p = 1$ if $n = 2$, and for some $p > n - 1$ if $n \geq 3$, is open and discrete. The quasi-lightness means that the components of the preimage of each $y \in \mathbb{R}^n$ are compact. Manfredi and Villamor showed in 1995 in [12] that the quasi-lightness assumption can be removed. In 2002, Björn [3] extended this result by showing that if the distortion function belongs to a certain Orlicz space, then $f$ is open and discrete. This Orlicz space is larger than $\bigcup_{p > n - 1} L^p_{\text{loc}}(\Omega)$ but smaller than $L^{n-1}_{\text{loc}}(\Omega)$.

In 2002, Hencl and Malý proved that a quasi-light mapping in $W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ with finite distortion function in $L^{n-1}(\Omega)$ is discrete and open. Even today, it is still not known if one can remove the quasi-lightness assumption. For further results on the discreteness and openness of quasi-light mappings of finite distortion, we refer to [7].

Our purpose of this note is to give a different proof for the result of Iwaniec and Šverák in the plane and for the result of Manfredi and Villamor in higher dimensions. Our unified approach does not even use the partial differential equations. We hope that this new approach also gives a better understanding of the problem of discreteness and openness for mappings of finite distortion.

**THEOREM 1.1.** Suppose that a non-constant mapping $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ has a finite distortion $K \in L^p(\Omega)$, where $p = 1$ if $n = 2$ and $p > n - 1$ if $n \geq 3$. Then $f$ is open and discrete.

The idea of our proof is the following. Let $f$ be a mapping satisfying the assumptions of Theorem 1.1. We consider the function

$$u(x) = \log^+ \log^+ \frac{1}{|f(x)|}$$

where the notation $\log^+ y$ stands for the maximum of $\log y$ and 0. The key observation is that $u \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^n)$ for $q = 1$ if $n = 2$, and for some $q > n - 1$ if $n \geq 3$. This follows from the integrability assumption on the distortion function of $f$. Then we apply the following lemma to conclude that $\mathcal{H}^1(f^{-1}(0)) = 0$, i.e., the pre-image set of $f$ at the origin has 1-Hausdorff measure zero. This shows that $f$ is light, and Theorem 1.1 follows by invoking Titus-Young theorem [17].
Lemma 1.1. (i) Let $B_R$ be a ball in $\mathbb{R}^n, n \geq 2$, and let $u \in W_0^{1,p}(B_R), n - 1 < p \leq n$. Then

$$\mathcal{H}_\infty^1(\{x \in B_R: |u(x)| > t\}) \leq \frac{c(n,p)}{t^p} R^{p-n+1} \int_{B_R} |\nabla u|^p \, dx$$

for all $t > 0$, where $c(n,p) > 0$.

(ii) If $u \in W^{1,1}(\mathbb{R}^2)$, then

$$\mathcal{H}_\infty^1(\{x \in \mathbb{R}^2: |u(x)| > t\}) \leq C \int_{\mathbb{R}^2} |\nabla u(x)| \, dx$$

(1)

for all $t > 0$, where $C > 0$.

Here $\mathcal{H}_\infty^1(E)$ denotes the 1-content of a set $E$. We refer to [5, Lemma 2.31] for the definition and for the proof of the first part (i) of Lemma 1.1. The second part (ii) of Lemma 1.1 is a consequence of the stronger estimate for functions $u$ in the Sobolev class $W^{1,1}(\mathbb{R}^n)$ (see [1]):

$$\int_0^\infty \mathcal{H}_\infty^{n-1}(\{x \in \mathbb{R}^n: M|u(x)| > t\}) \, dt \leq C(n) \int_{\mathbb{R}^n} |\nabla u| \, dx,$$

(2)

where $M$ stands for the standard Hardy-Littlewood maximal operator.

2 Proof of Theorem 1.1

Let $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ be a mapping with distortion function $K(x) \in L_{loc}^p(\Omega)$, where $p(n)$ is a fixed number such that $p(2) = 1$ and $p(n) > n - 1$ if $n \geq 3$. We will show that $f$ is discrete and open. Resnetnyak proved, [14], that a mapping of finite distortion belonging to the Sobolev class $W^{1,n}(\Omega, \mathbb{R}^n)$ has positive topological index (sense-preserving in the topological sense). Continuous sense-preserving light mappings (i.e. $f^{-1}(y)$ is totally disconnected for all $y \in \mathbb{R}^n$, that is, $f^{-1}(y)$ does not contain an arc) are open and discrete as shown in [17]. Therefore it is enough to show that

$$\mathcal{H}^1(f^{-1}(y)) = 0 \quad \text{for all } y \in \mathbb{R}^n.$$

(3)

Using the translation $f - y$, we see it suffices to verify this with $y = 0$ i.e. to show that

$$\mathcal{H}^1(f^{-1}(0)) = 0.$$

(4)

Now we define a function on $\Omega$: $u(x) = \log \log(1/|f(x)|)$ when $|f(x)| \leq 1/e$, and otherwise $u(x) = 0$. For simplicity, we assume in the following that $|f(x)| \leq 1/e$. We will show that $|\nabla u(x)| \in L_q^\Omega(\Omega)$ for $q = np(n)/(p(n) + 1)$. To this end, we fix a bounded open set $\Omega' \subset \Omega$ so that the closure of $\Omega'$ lies in $\Omega$. The distortion inequality $|Df|^n \leq K(x)J(x,f)$ together with Hölder’s inequality implies that

$$\int_{\Omega'} \frac{|Df|^q}{|f|^n \log^q(1/|f|)} \leq \left( \int_{\Omega'} \frac{J(x,f)}{|f|^n \log^q(1/|f|)} \right)^{q/n} \left( \int_{\Omega'} K(x)^{p(n)} \right)^{(n-q)/n},$$

(5)
where \( q = np(n) / (p(n) + 1) \). This inequality leads us to study integrals which have a special form. Namely, the integrals with integrands which are of the form a function depending on the norm of the mapping times the Jacobian determinant. In this kind of study, the finite distortion assumption is not a factor. This allows us to assume that our mapping is smooth i.e. we can apply the standard approximation argument.

**Lemma 2.1.** Let \( f \) be a mapping in \( C^\infty(\Omega, \mathbb{R}^n) \) and \( \Psi \in C^1(\mathbb{R}_+, \mathbb{R}) \). Then there is a constant \( C = C(n) \) such that for every test-function \( \eta \in C^\infty_0(\Omega, [0, \infty)) \), we have

\[
\left| \int_\Omega \eta^n \left[ n\Psi(|f|^2) + 2|f|^2\Psi'(|f|^2) \right] J(\cdot, f) \right| \leq C(n) \int_\Omega \eta^{n-1} |\nabla \eta||f|\Psi(|f|^2) |Df|^{n-1}.
\]

(6)

**Proof.** Fix \( i \in \{1, ..., n\} \). By Stokes’ Theorem, we have

\[
\int_\Omega J(x, f_1, ..., f_{i-1}, \eta, f_i+1, ..., f_n) \, dx = 0.
\]

(7)

By the Product Rule, this is equivalent with the identity

\[
\int_\Omega \eta^n \left[ \Psi(|f|^2) + 2\Psi'(|f|^2) f_i^2 \right] J(x, f) =
\]

\[
= (-1)^i \int_\Omega \eta^{n-1} \Psi(|f|^2) f_i J(\cdot, f_1, ..., f_{i-1}, \eta, f_{i+1}, ..., f_n).
\]

(8)

Summing over \( i \)'s, we find that

\[
\int_\Omega \eta^n \left[ n\Psi(|f|^2) + 2|f|^2\Psi'(|f|^2) \right] J(x, f) =
\]

\[
= \sum_{i=1}^n (-1)^i \int_\Omega \eta^{n-1} \Psi(|f|^2) f_i J(\cdot, f_1, ..., f_{i-1}, \eta, f_{i+1}, ..., f_n).
\]

(9)

Thus, the claimed estimate (6) follows from the point-wise inequality

\[
|J(x, f_1, ..., f_{i-1}, \eta, f_{i+1}, ..., f_n)| \leq |\nabla \eta(x)||Df(x)|^{n-1}.
\]

(10)

Fix a positive \( \epsilon \). Applying Lemma 2.1 for the function

\[
\Psi(t) = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{\varphi_\epsilon(s)}{s \log^\theta(s^{-1})} \, ds,
\]

where \( \varphi_\epsilon(s) = \frac{1}{1 + \epsilon^2 s} \).
we have
\[
\int_{\Omega} n^{-1} J(\cdot, f) \frac{\partial \varphi_e}{\partial |j|^2} \leq C(n) \int_{\Omega} |\nabla \eta|^{n-1} |\nabla| \left( \frac{\|Df\|}{|f|} \right)^{-1} f_0 |f|^{s} \log^{n+1} |f| ds (s^{-1}) \tag{11}
\]
Employing the facts that the function $s \to \varphi_e(s)$ is increasing and less than or equal 1 give
\[
\int_{\Omega} n^{-1} J(\cdot, f) \frac{\partial \varphi_e}{\partial |j|^2} \leq C(n) \int_{\Omega} |\nabla \eta|^{n-1} |\nabla| \left( \frac{\|Df\|}{|f|} \right)^{-1} f_0 |f|^{s} \log^{n+1} |f| ds (s^{-1}) \tag{12}
\]
Now, the standard approximation argument shows that the estimate (12) is also valid for $f \in W^{1,n}_\text{loc}(\Omega, \mathbb{R}^n) \cap C(\Omega, \mathbb{R}^n)$. Under the assumptions of Theorem 1.1, we have
\[
\int_{\Omega} n^{-1} J(\cdot, f) \frac{\partial \varphi_e}{\partial |j|^2} \leq C(n) \left( \int_{\Omega} n^{-1} J(\cdot, f) \frac{\partial \varphi_e}{\partial |j|^2} \right)^{\frac{n-1}{n}} \left( \int_{\Omega} |\nabla \eta|^{n} K^{n-1} \right) \tag{13}
\]
and so
\[
\int_{\Omega} n^{-1} J(\cdot, f) \frac{\partial \varphi_e}{\partial |j|^2} \leq c(n) \int_{\Omega} |\nabla \eta|^{n} K^{n-1}. \tag{14}
\]
By the Monotone Convergence Theorem, we find that
\[
\left| \int_{\Omega} n^{-1} J(\cdot, f) \frac{\partial \varphi_e}{\partial |j|^2} \right| \leq c(n) \int_{\Omega} |\nabla \eta|^{n} K^{n-1}. \tag{15}
\]
It follows from the inequalities (5) and (15) that
\[
\int_{\Omega'} \frac{|Df|^q}{|f|^{q/n}} \leq c(n, q) \left( \int_{\Omega'} |\nabla \eta|^{n} K^{n-1} \right)^{q/n} \left( \int_{\Omega'} K(x)^{p(n)} \right)^{(n-q)/n}
\]
where $q = np(n)/(p(n) + 1)$ and $\eta$ is a smooth function so that $\eta = 1$ in $\Omega'$. Note that $q = 2$ if $n = 2$ and $q > n - 1$ if $n \geq 3$. The last inequality shows that $|\nabla \log(1/|f|)|^q$ is locally integrable. Now, for any $k > 0$ define $u_k = \eta \min(\log(1/|f|), k)$. Applying Lemma 1.1 with $u_k$, we obtain the claimed requirement (4) by letting $k \to \infty$. Therefore, Theorem 1.1 follows.
References


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