Sobolev -avaruudet ja moderni osittaisdifferentiaaliyhtälöiden teoria
Demo 6

Let $u \in W^{1,2}(\Omega), \Omega \subset \mathbb{R}^n, n \geq 2$, be a NON-NEGATIVE weak subsolution of equation

$$-\mathrm{div}(A(x)Du) \leq 0, \qquad \text{in } \Omega,$$

that is, for any NON-NEGATIVE $\varphi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega \langle A(x)Du, D\varphi\rangle \, dx \leq 0.$$

Here $A(x) \in \mathbb{S}^{n\times n}$ satisfies for $0 < \lambda \leq \Lambda < \infty$,

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^n, \forall\, a.e.\, x \in \Omega.$$

Prove that $u \in L^\infty_{\mathrm{loc}}(\Omega)$, and prove that for any $0 < q \leq 2$, there is $c = c(n, \lambda, \Lambda, q) > 0$ such that for any ball $B(x,r) \subset \Omega$ and for any $0 < \sigma < 1$, we have

$$||u||_{L^\infty(B(x,\sigma r))} \leq \frac{c}{(1-\sigma)^{\frac{n}{q}}} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} u^q \, dy \right)^{\frac{1}{q}}.$$