Sobolev -avaruudet ja moderni osittaisdifferentiaaliyhtälöiden teoria
Demo 5

1. Let $\Omega \subset \mathbb{R}^n$ be a domain and $u \in W_0^{1,2}(\Omega)$. For a constant $k > 0$, define the function $T_k u$ as

$$T_k u(x) = \begin{cases} k, & \text{if } \; u(x) > k; \\ u(x), & \text{if } \; -k \leq u(x) \leq k; \\ -k, & \text{if } \; u(x) < -k. \end{cases}$$

Show that $T_k u \in W_0^{1,2}(\Omega)$ and that $(T_k u)^2 \in W_0^{1,2}(\Omega)$.

2. Suppose that $u \in W^{1,2}(\Omega)$ and $\eta \in C_0^\infty(\Omega)$. Prove that $\varphi = \Delta_i^{-h}((\Delta_i^h u)\eta) \in W_0^{1,2}(\Omega), i = 1, 2, ..., n,$ for all h small enough.

3. Let u be a function in $C^1(B_R(0)), B_R(0) \subset \mathbb{R}^2$, and write for $0 < r < R$,

$$\omega(r) = \operatorname{osc}_{\partial B_r(0)} u = \max_{\partial B_r(0)} u - \min_{\partial B_r(0)} u, \quad D(r) = \int_{B_r(0)} |Du|^2 \, dx.$$

If ω is non-decreasing, show that for $0 < r < R$,

$$\omega(r) \leq \pi D(R)^{1/2} \log^{-1/2}(R/r).$$

4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that $A(x) : \Omega \to \mathbb{S}^{n \times n}$ satisfies for $0 < \lambda \leq \Lambda < \infty$

$$\lambda |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \Lambda |\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and $a.e.\ x \in \Omega$. Let ϕ be from $W^{1,2}(\Omega)$. Prove that the equation

$$\begin{cases} \operatorname{div}(A(x)Du) = 0, & \text{in } \Omega; \\ u - \phi \in W_0^{1,2}(\Omega) \end{cases}$$

has a unique weak solution.

5. Let α be a constant such that $0 < \alpha < 1$. Define the function $u : B_1 = \{y \in \mathbb{R}^2 : |y| < 1\} \to \mathbb{R}$ as

$$u(x) = |x|^{\alpha - 1} x_1, \quad \text{for } \; x = (x_1, x_2) \in B_1.$$

Define

$$A(x) = \begin{pmatrix} \frac{x_1^2 + \alpha^2 x_2^2}{|x|^2} & (1 - \alpha^2)\frac{x_1 x_2}{|x|^2} \\ (1 - \alpha^2)\frac{x_1 x_2}{|x|^2} & \frac{\alpha^2 x_1^2 + x_2^2}{|x|^2} \end{pmatrix}.$$

Show that

i) $\alpha^2 |\xi|^2 \leq (A(x)\xi, \xi) \leq |\xi|^2, \quad \forall x \in B_1, \xi \in \mathbb{R}^2.$

ii) u is a weak solution of

$$\operatorname{div}(A(x)\nabla u) = 0 \quad \text{in } B_1.$$

6. Let $u : B_1 = \{y \in \mathbb{R}^n : |y| < 1\} \to \mathbb{R}$ be defined as

$$u(x) = \begin{cases} |x|^{-n+2}, & \text{if } \; n \geq 3; \\ \log \frac{1}{|x|}, & \text{if } \; n = 2. \end{cases}$$

Show that

i) $u \in W^{1,p}(\Omega)$, for all $p < n/(n-1)$.

ii) For all $\varphi \in C_0^\infty(B_1)$, it holds

$$\int_{B_1} (\nabla u(x), \nabla \varphi(x))\, dx = c(n)\varphi(0),$$

where $c(n)$ is a positive constant depending on n.