

**Sobolev -avaruudet ja moderni osittaisdifferentiaaliyhtälöiden teoria**  
**Demo 3**

1. Assume  $0 < \beta < \gamma \leq 1$ . Prove the interpolation inequality

$$\|u\|_{C^{0,\gamma}(\Omega)} \leq C \|u\|_{C^{0,\beta}(\Omega)}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C^{0,1}(\Omega)}^{\frac{\gamma-\beta}{1-\beta}}.$$

2. Prove that if  $u \in W^{1,p}((0,1))$  for some  $1 \leq p < \infty$ , then  $u$  is equal *a.e.* to an absolutely continuous function (that is, for any  $\varepsilon > 0$ , there is  $\delta > 0$  such that for any sequence of pairwise disjoint sub-intervals  $[a_i, b_i] \subset (0,1)$  satisfying

$$\sum_i |b_i - a_i| < \delta,$$

we have

$$\sum_i |u(b_i) - u(a_i)| < \varepsilon.)$$

and  $u'$  (which exists *a.e.*) belongs to  $L^p((0,1))$ .

3. Suppose that  $\Omega$  is a domain in  $\mathbb{R}^n$  and  $u \in W^{1,p}(\Omega)$  satisfies

$$Du = 0 \quad \textit{a.e. in } \Omega.$$

Prove that  $u$  is a constant *a.e.* in  $\Omega$ .

4. Assume that  $F : \mathbb{R} \rightarrow \mathbb{R}$  is  $C^1$ , with  $F'$  bounded. Suppose that  $\Omega \subset \mathbb{R}^n$  is open, bounded, and  $u \in W^{1,p}(\Omega)$  for some  $1 < p < \infty$ . Let  $v = F(u)$ . Show that

$$v_{x_i} = F'(u)u_{x_i} \quad \textit{and } F(u) \in W^{1,p}(\Omega).$$

5.

(i) Prove that if  $u \in W^{1,p}(\Omega)$ , then  $|u| \in W^{1,p}(\Omega)$ .

(ii) Prove that if  $u \in W^{1,p}(\Omega)$ , then  $u^+ = \max(u, 0)$ ,  $u^- \in W^{1,p}(\Omega)$  and

$$Du^+ = \begin{cases} Du & \textit{a.e. on } \{u > 0\}; \\ 0 & \textit{a.e. on } \{u < 0\}. \end{cases}$$

(iii) Prove

$$Du = 0 \quad \textit{a.e. on } \{u = 0\}.$$