

**Sobolev -avaruudet ja moderni osittaisdifferentiaaliyhtälöiden teoria**  
**Demo 2**

1. Let  $\alpha$  be a constant such that  $0 < \alpha < 1$ . Define the function  $u : B(0, 1) = \{y \in \mathbb{R}^2 : |y| < 1\} \rightarrow \mathbb{R}$  as

$$u(x) = |x|^{\alpha-1}x_1, \quad \text{for } x = (x_1, x_2) \in B_1.$$

For which  $1 \leq q \leq \infty$  does  $u$  belong to  $W^{1,q}(B(0, 1))$ ?

2. Prove that if  $u \in W^{1,p}((0, 1))$  for some  $1 \leq p < \infty$ , then

$$|u(x) - u(y)| \leq |x - y|^{1-\frac{1}{p}} \left( \int_0^1 |u'|^p dt \right)^{\frac{1}{p}}.$$

3. Denote by  $U$  the open square  $\{x \in \mathbb{R}^2 : |x_1| < 1, |x_2| < 1\}$ . Define

$$u(x) = \begin{cases} 1 - x_1 & x_1 > 0, |x_2| < x_1; \\ 1 + x_1 & x_1 < 0, |x_2| < -x_1; \\ 1 - x_2 & x_2 > 0, |x_1| < x_2; \\ 1 + x_2 & x_2 < 0, |x_1| < -x_2. \end{cases}$$

For which  $1 \leq q \leq \infty$  does  $u$  belong to  $W^{1,q}(U)$ ?

4. Integrate by parts to prove

$$\int_{\Omega} |Du|^2 dx \leq \left( \int_{\Omega} u^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |D^2u|^2 dx \right)^{\frac{1}{2}}$$

for all  $u \in C_0^\infty(\Omega)$ .

5. Integrate by parts to prove

$$\int_{\Omega} |Du|^p dx \leq C \left( \int_{\Omega} u^p dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |D^2u|^p dx \right)^{\frac{1}{2}}$$

for  $2 \leq p < \infty$  and all  $u \in C_0^\infty(\Omega)$ , where  $C = C(n, p) > 0$ .

6. Verify that

$$u(x) = \log \log \left( 1 + \frac{1}{|x|} \right)$$

belongs to  $W^{1,n}(B(0, 1))$ ,  $B(0, 1) \subset \mathbb{R}^n$ .