

# A NOTE ON THE MUCKENHOUPHT WEIGHTS

STEPHEN KEITH AND XIAO ZHONG

ABSTRACT. We present a weighted inequality for the distribution of the Hardy-Littlewood maximal functions, from which follows the open ended property of the Muckenhoupt weights.

## 1. INTRODUCTION

In this note, we consider the weight functions  $\omega$  in  $\mathbb{R}^n$  for which the Hardy-Littlewood operator is bounded on  $L^p(\omega(x)dx)$ , that is

$$(1) \quad \int_{\mathbb{R}^n} (Mf(x))^p \omega(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx$$

for all  $f \in L^p(\omega(x)dx)$ . Here  $p > 1$  is a number and  $Mf$  is the Hardy-Littlewood maximal function of  $f$ , defined as

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes  $Q$ , and  $|Q|$  denotes the Lebesgue measure of  $Q$ . The problem of identifying those weights for inequality (1) to hold was settled in 1972. The pioneering work of Muckenhoupt [8] showed that inequality (1) holds if and only if  $\omega$  is in the class  $A_p$ , nowadays known as a Muckenhoupt class. The class  $A_p$  consists of weights  $\omega$  with the property that there is a constant  $K \geq 1$  such that

$$(2) \quad \sup_Q \left( \frac{1}{|Q|} \int_Q \omega dx \right) \left( \frac{1}{|Q|} \int_Q \omega^{-\frac{1}{p-1}} dx \right)^{p-1} \leq K,$$

where the supremum is taken over all cubes  $Q$ .

That (1) implies (2) is immediate; substituting  $f(x) = w(x)^{-1/(p-1)} \chi_Q(x)$  in (1), we obtain (2). That (2) implies (1) is more involved. A key point of Muckenhoupt's proof, see also [4], is the open ended property of the  $A_p$  weights: if  $\omega \in A_p$ , then  $\omega \in A_{p-\varepsilon}$  for some  $\varepsilon = \varepsilon(n, p, K) > 0$ . We refer to [2], [3], [7], [8] for different proofs of the equivalence of (1) and (2) that do not involve the "reverse Hölder" inequality.

The purpose of this note is to establish the following inequality for the distribution of maximal functions in terms of the  $A_p$  weights. We use the notation  $\omega(E) = \int_E \omega(x) dx$ . For  $\lambda > 0$ , we denote the level set of  $Mf$  by

$$U_\lambda = \{x \in \mathbb{R}^n : Mf(x) > \lambda\}.$$

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**THEOREM 1.1.** *Let  $\omega$  be a weight in  $A_p$ ,  $p > 1$ . Then for every  $\delta > 0$ , there exists  $N_0 = N_0(n, p, K, \delta) > 0$  such that for every  $N > N_0$ , every  $\lambda > 0$  and all  $f \in C_0^\infty(\mathbb{R}^n)$ , we have*

$$(3) \quad \omega(U_\lambda) \leq \delta N^p \omega(U_{N\lambda}) + N^{2p} \omega(\{x \in \mathbb{R}^n : |f(x)| > N^{-2p}\lambda\}).$$

Inequality (3) is somehow similar to the so-called *good  $\lambda$  inequality*, see e.g. [5], [11]. It is, to our knowledge, new. It is also strong, in the sense that the open ended property of the  $A_p$  weights is an easy consequence. Indeed, let  $\delta = 1/4$  in (3), and  $N = N_0 + 1$ , where  $N_0$  is given by Theorem 1.1. Choose  $0 \leq \varepsilon < p - 1$  such that  $N^\varepsilon < 2$ . Now integrate both sides of (3) against the measure  $d\lambda^{p-\varepsilon}$  over  $(0, \infty)$  to obtain

$$\int_{\mathbb{R}^n} (Mf)^{p-\varepsilon} \omega dx \leq \frac{N^\varepsilon}{4} \int_{\mathbb{R}^n} (Mf)^{p-\varepsilon} \omega dx + C \int_{\mathbb{R}^n} |f|^{p-\varepsilon} \omega dx.$$

Therefore by the choice of  $\varepsilon$ , we have

$$\int_{\mathbb{R}^n} (Mf)^{p-\varepsilon} \omega dx \leq C \int_{\mathbb{R}^n} |f|^{p-\varepsilon} \omega dx,$$

from which follows that  $\omega \in A_{p-\varepsilon}$ .

The proof of Theorem 1.1 uses an argument similar to that appearing in [6], where we studied the open ended property of the Poincaré inequality. The argument can be useful for obtaining a stronger inequality from a given one. We hope that the method can be applied to other problems from harmonic analysis. This is one of our motivations for writing this note.

## 2. PROOF OF THEOREM 1.1

Fix a non-negative function  $f \in C_0^\infty(\mathbb{R}^n)$ . We apply the Calderón-Zygmund lemma to  $f$ . For fixed  $t > 0$ , we obtain a disjoint family of dyadic cubes  $\{Q_j^t\}$  such that

$$(4) \quad t < \frac{1}{|Q_j^t|} \int_{Q_j^t} f(x) dx \leq 2^n t$$

and  $f(x) \leq t$  if  $x \notin \cup_j Q_j^t$ . See [1], [10] for the details. This decomposition can be carried out simultaneously for all values of  $t$ . If  $t_1 > t_2$ , the cubes  $\{Q_j^{t_1}\}$  are then sub-cubes of the cubes in  $\{Q_j^{t_2}\}$ . We denote by  $U_t^* = \cup_j Q_j^t$ . We have the following lemma, from which Theorem 1.1 follows easily.

**Lemma 2.1.** *Let  $\omega$  be a weight in  $A_p$ ,  $p > 1$ . Then for any integer  $\alpha$ , there exists an integer  $k_1$  that depends only on  $n, p, K$  and  $\alpha$  such that for all integers  $k \geq k_1$ , every  $\lambda > 0$  and all  $f \in C_0^\infty(\mathbb{R}^n)$ , we have*

$$(5) \quad \omega(U_\lambda^*) \leq 2^{kp-\alpha} \omega(U_{2^k\lambda}^*) + 3^{kp} \omega(\{x \in \mathbb{R}^n : |f(x)| > 3^{-k}\lambda\}).$$

We now prove Theorem 1.1 by Lemma 2.1. It is easy to show that the sets  $U_\lambda$  and  $U_\lambda^*$  are comparable, in the sense that

$$U_\lambda^* \subset U_\lambda, \text{ and } U_{c\lambda} \subset U_\lambda^* \text{ for } c = c(n) \geq 1.$$

This with (5) of Lemma 2.1 shows that

$$\omega(U_{c\lambda}) \leq 2^{kp-\alpha} \omega(U_{2^k\lambda}) + 3^{kp} \omega(\{x \in \mathbb{R}^n : |f(x)| > 3^{-k}\lambda\}),$$

which proves Theorem 1.1 by choosing suitable  $\alpha$  and  $k$ .

It remains to prove Lemma 2.1. The proof makes repeated use of the next inequality, which follows from Hölder's inequality and (2). For every cube  $Q$  and every non-negative function  $g \in L^p(\omega(x)dx)$ , we have

$$(6) \quad \begin{aligned} \frac{1}{|Q|} \int_Q g \, dx &\leq \frac{1}{|Q|} \left( \int_Q \omega^{-\frac{1}{p-1}} \, dx \right)^{\frac{p-1}{p}} \left( \int_Q g^p \omega \, dx \right)^{\frac{1}{p}} \\ &\leq K^{\frac{1}{p}} \left( \frac{1}{\omega(Q)} \int_Q g^p \omega \, dx \right)^{\frac{1}{p}}. \end{aligned}$$

This requirement on  $\omega$  characterizes the  $A_p$  class, see [11].

*Proof of Lemma 2.1.* We fix a cube  $Q_0 = Q_j^\lambda$  from  $U_\lambda^*$ . We will show that

$$(7) \quad \omega(Q_0) \leq 2^{kp-\alpha} \omega(U_{2^k\lambda}^* \cap Q_0) + 3^{kp} \omega(\{x \in Q_0 : |f(x)| > 3^{-k}\lambda\}),$$

from which (5) follows, by summing over all such cubes. Inequality (7) is proved over the remainder of this paper. Let  $\alpha, k$  be integers. Suppose, in order to achieve a contradiction, that (7) does not hold. The assumed negation of (7) implies that

$$(8) \quad \begin{aligned} \omega(U_{2^k\lambda}^* \cap Q_0) &< 2^{-kp+\alpha} \omega(Q_0), \quad \text{and} \\ \omega(\{x \in Q_0 : |f(x)| > 3^{-k}\lambda\}) &< 3^{-kp} \omega(Q_0). \end{aligned}$$

At the end of the proof, a lower bound will be specified for  $k$ . This bound depends only on  $n, p, \alpha$  and the constant  $K$  in (2). To achieve the contradiction, and thereby prove (7), we take  $k_1$  to be this lower bound.

Our starting point of the proof of (7) is to consider the so-called *good* part of the Calderón-Zygmund decomposition of the function  $f$  at the level  $2^k\lambda$ . Let  $\{Q_i\}_i$  denote all of the cubes from  $U_{2^k\lambda}^*$  that are sub-cubes of  $Q_0$ . We define a function  $u$  in  $Q_0$  as

$$u(x) = \begin{cases} f(x) & \text{if } x \in Q \setminus \cup_i Q_i; \\ \frac{1}{|Q_i|} \int_{Q_i} f(y) \, dy & \text{if } x \in Q_i. \end{cases}$$

By (4), we have  $u(x) \leq 2^{n+k}\lambda$  for all  $x \in Q_0$ . By (8), we have that

$$\int_{\cup Q_i} u^p \omega \, dx \leq 2^{(n+k)p} \lambda^p \omega(\cup Q_i) \leq 2^{np+\alpha} \lambda^p \omega(Q_0).$$

Note that  $u(x) = f(x)$  if  $x \in Q_0 \setminus \cup Q_i$ . By (8) again,

$$(9) \quad \begin{aligned} \int_{Q_0 \setminus \cup Q_i} u^p \omega \, dx &\leq 2^{(n+k)p} \lambda^p \omega(\{x \in Q_0 : f(x) > 3^{-k}\lambda\}) \\ &\quad + 3^{-kp} \omega(\{x \in Q_0 : f(x) \leq 3^{-k}\lambda\}) \\ &\leq (2^{(n+k)p} 3^{-kp} + 3^{-kp}) \lambda^p \omega(Q_0). \end{aligned}$$

Thus, the  $L^p(\omega)$ -norm of  $u$  has an upper bound that is independent of  $k$ ,

$$(10) \quad \int_{Q_0} u^p \omega \, dx \leq (2^{np+\alpha} + 2^{(n+k)p} 3^{-kp} + 3^{-kp}) \lambda^p \omega(Q_0) \leq 2^{2np+\alpha+1} \lambda^p \omega(Q_0).$$

We again apply the Calderón-Zygmund lemma, but this time to the function  $u$  at the level  $2^i\lambda$ , for each integer  $i$  satisfying  $[k/2] \leq i \leq k-1$ . Here  $[k/2]$  denotes the

biggest integer less than  $k/2$ . Fix one such  $i$ . We obtain a disjoint family of dyadic cubes  $\{\tilde{Q}_j^{2^i\lambda}\}_j$  inside  $Q_0$  such that

$$(11) \quad 2^i\lambda < \frac{1}{|\tilde{Q}_j^{2^i\lambda}|} \int_{\tilde{Q}_j^{2^i\lambda}} u(x) dx \leq 2^{n+i}\lambda$$

and  $u(x) \leq 2^i\lambda$  if  $x \notin \cup_j \tilde{Q}_j^{2^i\lambda}$ . We consider the *good* part of the Calderón-Zygmund decomposition of the function  $u$  at the level  $2^i\lambda$ , defined as

$$u_i(x) = \begin{cases} u(x) & \text{if } x \in Q \setminus \cup_j \tilde{Q}_j^{2^i\lambda}; \\ \frac{1}{|\tilde{Q}_j^{2^i\lambda}|} \int_{\tilde{Q}_j^{2^i\lambda}} u(y) dy & \text{if } x \in \tilde{Q}_j^{2^i\lambda}. \end{cases}$$

We observe from (11) and (6) that for each  $\tilde{Q}_j^{2^i\lambda}$ ,

$$2^i\lambda < \frac{1}{|\tilde{Q}_j^{2^i\lambda}|} \int_{\tilde{Q}_j^{2^i\lambda}} u(y) dy \leq K^{\frac{1}{p}} \left( \frac{1}{\omega(\tilde{Q}_j^{2^i\lambda})} \int_{\tilde{Q}_j^{2^i\lambda}} u^p \omega dy \right)^{\frac{1}{p}}.$$

Thus, letting  $\tilde{U}_{2^i\lambda}^* = \cup_j \tilde{Q}_j^{2^i\lambda}$ , we have by (10) that

$$(2^i\lambda)^p \omega(\tilde{U}_{2^i\lambda}^*) \leq K \int_{\tilde{U}_{2^i\lambda}^*} u^p \omega dy \leq C\lambda^p \omega(Q_0),$$

and therefore that

$$(12) \quad \omega(\tilde{U}_{2^i\lambda}^*) \leq C2^{-pi}\omega(Q_0),$$

where  $C = C(n, p, \alpha, K) > 0$ .

The crucial point of the proof is to consider the function given by

$$h = \frac{1}{k - [k/2]} \sum_{i=[k/2]}^{k-1} u_i.$$

We note that the average of a function is the same as that of the *good* part of its Calderón-Zygmund decomposition. Thus,

$$\int_{Q_0} u_i dx = \int_{Q_0} u dx = \int_{Q_0} f dx,$$

and therefore,

$$\int_{Q_0} h dx = \int_{Q_0} f dx.$$

Recall that  $Q_0$  is a cube from the set  $U_\lambda^*$ . Thus (4) and then (6) gives us

$$(13) \quad \lambda < \frac{1}{|Q_0|} \int_{Q_0} f dx = \frac{1}{|Q_0|} \int_{Q_0} h dx \leq K^{\frac{1}{p}} \left( \frac{1}{|Q_0|} \int_{Q_0} h^p \omega dx \right)^{1/p}.$$

We now obtain an upper integral estimate for  $h$  that contradicts (13). Let the function  $g$  defined on  $Q_0$  be given by

$$g(x) = \frac{1}{k - [k/2]} \sum_{i=[k/2]}^{k-1} 2^{n+i}\lambda \chi_{|\tilde{U}_{2^i\lambda}^* \cup U_{2^k\lambda}^*|}(x),$$

for every  $x \in Q$ . Then by (11) we have

$$h(x) \leq u(x) \chi_{|Q_0 \setminus U_{2^k\lambda}^*|} + g(x)$$

for every  $x \in Q_0$ . Thus by (12) and (8),

$$\begin{aligned} \int_{Q_0} g^p \omega \, dx &\leq \frac{1}{(k - [k/2])^p} \int_{Q_0} \sum_{i=[k/2]}^{k-1} \left( \sum_{j=[k/2]}^i 2^{n+j} \lambda \right)^p \chi|_{\tilde{U}_{2^i \lambda}^* \cup U_{2^k \lambda}^*} \, dx \\ &\leq \frac{C}{(k - [k/2])^p} \sum_{i=[k/2]}^{k-1} 2^{(i+1)p} 2^{-ip} \lambda^p \omega(Q_0) \\ &= \frac{C}{(k - [k/2])^{p-1}} \lambda^p \omega(Q_0), \end{aligned}$$

where  $C = C(n, p, \alpha, K) > 0$ . This with (9) implies that

$$\begin{aligned} \int_{Q_0} h^p \omega \, dx &\leq 2^p \int_{Q_0 \setminus U_{2^k \lambda}^*} u^p \omega \, dx + 2^p \int_{Q_0} g^p \omega \, dx \\ &\leq 2^p \left( 2^{(n+k)p} 3^{-kp} + 3^{-kp} + \frac{C}{(k - [k/2])^{p-1}} \right) \lambda^p \omega(Q_0). \end{aligned}$$

This with (13) gives us

$$1 \leq K 2^p \left( 2^{(n+k)p} 3^{-kp} + 3^{-kp} + \frac{C}{(k - [k/2])^{p-1}} \right).$$

Since  $p > 1$ , we achieve a contradiction when  $k$  is sufficient large. This completes the proof of Lemma (2.1), and therefore that of Theorem 1.1.

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UNIVERSITY OF JYVÄSKYLÄ, JYVÄSKYLÄ, FINLAND.  
*E-mail address:* zhong@maths.jyu.fi