On Lie group structures of homogeneous metric spaces

Ville Kivioja

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These slides are available at my homepage
http://users.jyu.fi/~vikakivi/
1. Introduction

2. Existence of Lie group structures

3. Finding all Lie group structures: CMI

4. Real shadow
A metric space is said to be *homogeneous*, if the action of its isometry group is transitive, i.e., given any two points on the manifold, there exists an isometry mapping first to the second.

Examples

- Euclidean spaces and Carnot groups with CC-distances
- Spheres \( S^n \)
- Rank-one symmetric spaces of non-cpt type \( H_R^n, H_C^n, H_Q^n, H_{Ca}^2 \)
- Flat torus
- Cartesian products of these

Non-examples

- Physical torus
- Ellipsoids
- Saddle surface
Definition

A metric space is said to be \textit{homogeneous}, if the action of its isometry group is transitive, i.e., given any two points on the manifold, there exists an isometry mapping first to the second.

Important class: \textbf{Lie groups with left-invariant distances}:

\[ G^L \subseteq \text{Isome}(G, d) \quad G^L \curvearrowright G \text{ transitively} \]

Examples are plentiful:

\[ \langle \cdot, \cdot \rangle \text{ on } T_e G \quad \rightsquigarrow \quad \rho_g(X, Y) := \langle dL_{g^{-1}}X, dL_{g^{-1}}Y \rangle \]

- Euclidean spaces and Carnot groups with CC-dist, \textbf{Lie groups}
- Spheres $\mathbb{S}^n$, \textbf{NOT Lie groups} (except $n \in \{1, 3\}$)
- Rank-one symmetric spaces $\mathbb{H}^n_\mathbb{R}, \mathbb{H}^n_\mathbb{C}, \mathbb{H}^n_\mathbb{Q}, \mathbb{H}^2_\mathbb{C}$, \textbf{Lie groups}
- Flat torus, \textbf{Lie group}
Metric space \((M, d)\) admits a structure of a Lie group, if any of the following equivalent conditions hold.

- \(M\) is isometric to a Lie group with a left-invariant admissible distance.
- There is a Lie group acting on \(M\) simply transitively by isometries.
- There is a group operation \(* : M \times M \rightarrow M\) that makes \(M\) a Lie group and the distance \(d\) left-invariant.
Existence, uniqueness and classification of Lie group structures on metric space.

Given a homogeneous metric space $M$ we wish to
- know if there exists a Lie group structure on $M$, and if yes
- know if there are more than one Lie group structure on $M$, and if yes,
- know what are all the possible Lie group structures on $M$.
- know what can happen after perturbations retaining the large scale structure (quasi-isometries)?

That is:
How, and how well, do the Lie groups model homogeneous metric spaces?
Existence, uniqueness and classification of Lie group structures on metric space.

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Examples of these phenomena:

The metric space $(\mathbb{S}^2, d_E)$ does not have a Lie group structure
Existence, uniqueness and classification of Lie group structures on metric space.

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Metric space:

$$\text{span}(\partial_x - y\partial_z, \partial_y + x\partial_z) \text{ subbundle in } \mathbb{R}^3 \rightarrow \text{CC-distance}$$

The only Lie group structure: Heisenberg group
Existence, uniqueness and classification of Lie group structures on metric space.

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The Euclidean space $\mathbb{R}^3$ admits exactly two Lie group structures: the Abelian one, and the one of $\widetilde{\mathbb{S}E}(2)$, i.e., the universal cover of the isometry group of the plane, i.e., $\mathbb{R}^2 \rtimes S^1$. 
Up to arbitrarily small perturbation, every locally compact homogeneous metric space is a Lie group:

**Theorem (CKLNO/Cornulier–De La Harpe/folklore)**

Let \((M, d)\) be a locally compact and connected homogeneous metric space. Then there is a Lie group \(G\), and for all \(\varepsilon > 0\) there is a left-invariant distance \(d_G\) on \(G\), and a surjective map \(F : G \to M\), so that

\[
d_G(g, h) - \varepsilon \leq d(F(g), F(h)) \leq d_G(g, h)
\]

In particular, \((M, d)\) is \((1, \varepsilon)\)-quasi-isometric to \((G, d_G)\).
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Note: \(S^2\) . . .
Up to arbitrarily small perturbation, every locally compact homogeneous metric space is a Lie group:

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In particular, $(M, d)$ is $(1, \varepsilon)$-quasi-isometric to $(G, d_G)$.

Note: $S^2$ is not $(1, \varepsilon)$-QI to singleton. It is $(1, \varepsilon)$-QI to $SO(3)$ with a certain distance.
Up to arbitrarily small perturbation, every locally compact homogeneous metric space is a Lie group:

**Theorem (CKLNO/Cornulier–De La Harpe/folklore)**

Let \((M, d)\) be a locally compact and connected homogeneous metric space. Then there is a Lie group \(G\), and for all \(\varepsilon > 0\) there is a left-invariant distance \(d_G\) on \(G\), and a surjective map \(F: G \to M\), so that

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In particular, \((M, d)\) is \((1, \varepsilon)\)-quasi-isometric to \((G, d_G)\).

Here \(G = \text{Iso}(M, d)\) and \(F\) is the orbit map \(F(\varphi) = \varphi(o)\). The interesting question is, who is \(d_G\)?
Assume \((M, d)\) loc cpt, homog, \(\varepsilon > 0\), \(o \in M\), \(G := \text{Iso}(M, d)\)

Claim \(\exists d_G\) s.t. \(F : (G, d_G) \to (M, d), F(\varphi) = \varphi(o)\) is \((1, \varepsilon)\)-QI.

Proof Define Busemann gauge

\[
\rho(o, p) := \ell \min\{n \in \mathbb{N} \mid p \in V_n(o, \ell)\}
\]

where \(\bar{B}(o, 2\ell)\) is compact and

\[
V_n(o, \ell) := \bigcup_{p \in V_{n-1}(o, \ell)} \bar{B}(p, \ell)
\]
Assume \((M, d)\) loc cpt, homog, \(\varepsilon > 0\), \(o \in M\), \(G := \text{Iso}(M, d)\)

Claim \(\exists d_G\) s.t. \(F : (G, d_G) \rightarrow (M, d), F(\varphi) = \varphi(o)\) is \((1, \varepsilon)\)-QI.

Proof Define Busemann gauge

\[\rho(o, p) := \ell \min\{n \in \mathbb{N} | p \in V_n(o, \ell)\}\]

The distance that does the job is

\[d_G(\varphi, \psi) = \sup_{q \in M} d(\varphi(q), \psi(q)) e^{-\rho(o, q)/\varepsilon}\]
"Many of homogeneous metric spaces admit Lie group structures:"

**Theorem (CKLNO)**

Let \((M, d)\) be a locally compact and connected homogeneous metric space. Assume \(M\) admits a dilation, i.e., there is a bijection \(\delta: M \rightarrow M\) and a number \(\lambda > 1\) so that

\[
d(\delta(x), \delta(y)) = \lambda d(x, y) \quad \forall x, y \in M
\]

Then \((M, d)\) admits a structure of a nilpotent (gradable) Lie group, so that \(d\) becomes left-invariant and \(\delta\) becomes an automorphism.
”Many of homogeneous metric spaces admit Lie group structures:”

**Theorem (Heintze)**

Let $(M, g)$ be homogeneous Riemannian manifold, with strictly negative sectional curvatures. Then $M$ admits a structure of a solvable Lie group.

**Conjecture (Alekseevskii)**

Let $(M, g)$ be homogeneous Riemannian manifold, that is non-compact, non-flat and satisfies

\[ \text{Ric}_g = \lambda g \]  

(Einstein)

for some $\lambda \in \mathbb{R}$, then $M$ admits a structure of a solvable Lie group.
If a Lie group structure exist, how to find all the other ones? A bit too hard, but we take a bit different, related, approach:

**Definition**

We say two Lie groups $G$ and $H$ *can be made isometric*, write $G \overset{\text{CMI}}{\sim} H$, if there are left-invariant distance functions $d_G$ on $G$ and $d_H$ on $H$, inducing the respective manifold topologies, such that

$$ (G, d_G) \overset{\text{isome}}{\simeq} (H, d_H) $$

The following are equivalent with the condition $G \overset{\text{CMI}}{\sim} H$

- There is a metric space $M$ and two simply transitive subgroups of $\text{Iso}(M)$ isomorphic to $G$ and $H$ respectively.
- There is a left-invariant admissible distance function $d_G$ on $G$ and a simply transitive action $H \ltimes G$ by isometries.
If a Lie group structure exist, how to find all the other ones? A bit too hard, but we take a bit different, related, approach:

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$$(G, d_G) \overset{\text{isome}}{\cong} (H, d_H)$$

Approach is now

- Suppose a metric space is given, and a Lie group structure is found.
- We aim to find all the possible other Lie groups, that can be made isometric to the given Lie group, even if not with the given distance.
If a Lie group structure exist, how to find all the other ones? A bit too hard, but we take a bit different, related, approach:

**Definition**

We say two Lie groups $G$ and $H$ can be made isometric, write $G \overset{\text{CMI}}{\sim} H$, if there are left-invariant distance functions $d_G$ on $G$ and $d_H$ on $H$, inducing the respective manifold topologies, such that

$$(G, d_G) \overset{\text{isome}}{\simeq} (H, d_H)$$

Sometimes we can prove the uniqueness in this stronger way:

**Theorem (K.–Le Donne 2017)**

*Connected nilpotent Lie groups can be made isometric only if they are isomorphic.*
Proof: If two nilpotent Riemannian Lie groups are isometric, then they are isomorphic (Wolf, 60s). Then use the following:

**Theorem (K.–Le Donne)**

*If* $(G, d)$ *is a connected Lie group with left-invariant distance* $d$ *inducing the manifold topology, then*

- Isome($G, d$) *is a Lie group acting smoothly.*
- *Stabilisers of* Isome($G, d$) *are compact.*
- *There is a left-invariant Riemannian metric* $\rho$ *on* $G$ *such that* Isome($G, d$) $<$ Isome($G, \rho$).
In particular, suppose $F : (G, d_G) \rightarrow (H, d_H)$ is an isometry. Take $\rho$ Riemannian metric in $G$ so that

$$\text{Isome}(G, d) < \text{Isome}(G, \rho)$$

Now $F_*\rho$ is a Riemannian metric on $H$. Need to show it is left-invariant:

$$\text{Isome}(G, \rho) \xrightarrow{C_F} \text{Isome}(H, F_*\rho)$$

$$\text{Isome}(G, d_G) \xrightarrow{C_F} \text{Isome}(H, d_H)$$

$\rightarrow F$ is an isometry of Riemannian Lie groups $(G, \rho)$ and $(H, F_*\rho)$, so an isomorphism.
Corollary

If two Lie groups can be made isometric, they can be made isometric using Riemannian distances. In particular, they are quasi-isometric.

Conjecture (folklore)

If two nilpotent simply connected Lie groups are quasi-isometric, they are isomorphic.

→ **Inverse question:** Which of the pairs of Lie groups that are quasi-isometric, can indeed be made isometric?
Lie groups tend to have ‘good representatives’ in terms of isometry-relation:

Theorem (Breuillard)

Any connected Lie group of polynomial growth can be made isometric to a nilpotent group, namely its nilshadow.

Theorem (Alekseevski)

Any Heintze group can be made isometric to a purely real Heintze group.

Definition

A Lie group of the form $N \rtimes_A \mathbb{R}$ is a Heintze group, if $N$ is nilpotent simply connected Lie group and the action (in the Lie algebra level) is given by $A \in \text{der}(n)$ so that any root of the characteristic polynomial of $A$ has strictly positive real part. A Heintze group is purely real if those roots are real numbers.
Lie groups tend to have 'good representatives' in terms of isometry-relation:

**Theorem (Breuillard)**

*Any connected Lie group of polynomial growth can be made isometric to a nilpotent group, namely its nilshadow.*

**Theorem (Alekseevski)**

*Any Heintze group can be made isometric to a purely real Heintze group.*

\[ G_1 \overset{\text{CMI}}{\sim} G_2 \] and both are \( \left\{ \begin{array}{l} \text{nilpotent} \\ \text{purely real Heintze} \end{array} \right\} \Rightarrow G_1 \cong G_2 \]
Lie groups tend to have 'good representatives' in terms of isometry-relation:

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**Theorem (Alekseevski)**

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**Theorem (Gordon–Wilson + K.–Le Donne)**

If two completely solvable Lie groups can be made isometric, they are isomorphic.
Theorem (CKLNO)

Let \( \mathfrak{g} \) be a solvable Lie algebra. Choose a vector subspace \( \mathfrak{a} \subseteq \mathfrak{g} \) satisfying \( \mathfrak{n} \oplus \mathfrak{a} = \mathfrak{g} \) and \( \text{ad}_s(\mathfrak{a})(\mathfrak{a}) = \{0\} \). Define

\[
\varphi : \mathfrak{g} \rightarrow \text{der}(\mathfrak{g}) \quad \varphi(X) = -\text{ad}_{s_i}(\pi_a(X))
\]

Then

i) \( \text{Gr}(\varphi) := \{(X, \varphi(X) | X \in \mathfrak{g})\} \) is a completely solvable ideal of \( \mathfrak{g} \rtimes \text{der}(\mathfrak{g}) \),

ii) When the vector space \( \mathfrak{g} \) is equipped with the operation

\[
[X, Y]_\mathbb{R} := [X, Y] + \varphi(X)(Y) - \varphi(Y)(X)
\]

then the map \( \tau(X) = (X, \varphi(X)) \) is a Lie algebra isomorphism from \((\mathfrak{g}, [\cdot, \cdot]_\mathbb{R})\) to \(\text{Gr}(\varphi)\), the real shadow of \( \mathfrak{g} \).
\[ n \oplus a = g \quad \text{ad}_s(a)(a) = \{0\} \]

\[ \varphi : g \to \text{der}(g) \quad \varphi(X) = -\text{ad}_{\text{si}}(\pi_a(X)) \]

\[ [X, Y]_{\text{rrad}} := [X, Y] + \varphi(X)(Y) - \varphi(Y)(X) \]

Observations:

- If \( g \) is completely solvable (e.g. nilpotent), then \( \varphi \equiv 0 \) and the real shadow is itself.

- If \( g \) has polynomial growth, then \( \text{ad}_{\text{si}}(X) = \text{ad}_s(X) \) for all \( X \in g \), so the real shadow equals to what is called \textit{nilshadow}.
Example: $\widetilde{SE}(2)$. Basis $\{X, Y, \Theta\}$ with

$$[X, Y] = 0 \quad [\Theta, X] = Y \quad [\Theta, Y] = -X$$

$\rightarrow$ solvable and polynomial growth.

$$n = \text{span}(X, Y) \quad \text{so} \quad a := \text{span}(\Theta) \quad \rightarrow \quad \begin{cases} g = n \oplus a \\ \text{ad}_s(a)(a) = \{0\} \end{cases}$$

Now

$$\text{ad}(\Theta) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \rightarrow \quad \text{ad}_{si}(\Theta) = \text{ad}(\Theta)$$

$$\varphi(\Theta)(X)$$

so

$$[\Theta, X]_{\mathbb{R}} = [\Theta, X] - \text{ad}_{si}(\pi_a(\Theta))(X) + \text{ad}_{si}(\pi_a(X))(\Theta)$$

$$= Y - \text{ad}(\Theta)(X) + 0 = Y - Y = 0$$

$$[\Theta, Y]_{\mathbb{R}} = 0 \quad \text{and} \quad [X, Y]_{\mathbb{R}} = [X, Y] = 0$$

So the real shadow is $\mathbb{R}^3$. 
Theorem (Breuillard + CKLNO, in the thesis of Sebastiano Nicolussi Golo)

Let $G$ and $N$ be solvable simply connected Lie groups, and assume $N$ is nilpotent. Then the following are equivalent:

- $G$ can be made isometric to $N$,
- $G$ has polynomial growth and $N$ is the nilshadow of $G$.

Left open the following questions:

- If groups $G_1$ and $G_2$ of polynomial growth can be made isometric, do they have the same nilshadow?
- If groups $G_1$ and $G_2$ of polynomial growth have the same nilshadow, can they be made isometric to themselves?
- What about other solvable groups?

→ Attempt to make 'can be made isometric' to purely algebraically checkable relation (at least for solvable groups)!
Left open the following questions:

- If groups $G_1$ and $G_2$ of polynomial growth can be made isometric, do they have the same nilshadow?
- If groups $G_1$ and $G_2$ of polynomial growth have the same nilshadow, can they be made isometric to themselves?
- What about other solvable groups?

**Solution:**

**Conjecture (CKLNO)**

Let $G_1$ and $G_2$ be simply connected solvable Lie groups. Then the following are equivalent:

- $G_1$ can be made isometric to $G_2$,
- $G_1$ and $G_2$ have isomorphic real shadows.