**FYSA2042 osa B**

Koe 08.04.2022 klo 12:00-16:00.
Tehtäviä on 5 kappaletta.
Viimeisellä sivulla on joukko kaavoja, joista voi olla hyötyä.

Exam 08.04.2022 at 12:00-16:00.
Questions in English are at the end.
There are 5 questions.
The last sheet has a collection of potentially useful formulae.

1. Vastaa seuraaviin kysymyksiin. Perustele lyhyesti, käytä kaavoja täsmentämään vastausta.

   (a) (1p) Miksi ideaalisten bosonien kemiallinen potentiaali on negatiivinen kaikissa lämpötiloissa?

   (b) (1p) Mitä Wienin siirtymälaki tarkoittaa? Kaava?

   (c) (2p) Mitä on degeneroitunut elektronikaasu? Miksi Chandrasekharin raja (1,4 Auringon massaa) saadaan käyttämällä nimenomaan *ultrarelativistisen* degeneroituneen elektronikaasun painetta?

   (d) (2p) Hahmottele muutamia ideaalisen bosonikaasun $(P, V)$-tason isotermejä ja $(P, T)$-tason kuvaaja.

   (e) (2p) Mitä Boltzmannin teoriassa kuvaa törmäysintegraali $\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}$?

   (f) (1p) Miksi fotonikaasun kemiallinen potentiaali $\mu = 0$?

   (g) (1p) Mitä tarkoittaa Bose-Einstein kondensaatio?

2. (9p) Ideaalisten bosonien ja fermionien yksihiukkastilan $i$ suurkanoninen partitiofunktio on

$$\mathcal{Z}_i = \sum_n e^{-\beta(\epsilon_i - \mu)n} \ , \tag{1}$$

   missä $\epsilon_i$ on yksihiukkastilan energia, ja $n$ käy yli yksihiukkastilan kaikkien mahdollisten miehitysten. Laske summa $\mathcal{Z}_i$ bosoneille ja fermioneille. Suurkanoninen partitiofunktio on tulo $\mathcal{Z} = \prod_i \mathcal{Z}_i$. Kirjoita suuren potentiaalin $\Omega$ lauseke bosoneille ja fermioneille, ja laske sen avulla yksihiukkastilan $i$ miehityksen odotusarvo $\langle n_i \rangle$ bosoneille ja fermioneille. Vihje: $\langle n_i \rangle = \frac{\partial \Omega}{\partial \epsilon_i}$.

3. (a) (5p) Tarkastellaan hiukkasia laatikossa, jonka tilavuus on $V = L^3$, missä $L$ on laatikon särmän pituus. Hiukkasten energiaspektri on

$$\epsilon = \frac{\hbar^2 k^2}{2m} \ , \tag{2}$$

   missä $k = |\mathbf{k}|$, ja kvanttimekaniikan mukaan sallitut $\mathbf{k}$:n arvot ovat

$$\mathbf{k}_i = (i_1, i_2, i_3)\frac{\pi}{L} \quad , \ i_1, i_2, i_3 = 1, 2, 3, \dots \ . \tag{3}$$

   Laske energia-avaruuden tilatiheys $g(\epsilon)$ haluamallasi tavalla.

(b) (4p) Kanoninen partitiofunktio on

$$Z = \frac{1}{N!}\left(\frac{V}{\lambda_T^3}\right)^N \ . \tag{4}$$

Millaisia hiukkasia partitiofunktio kuvaa, ja mikä on aineen tilayhtälö?

4. (a) (2p) Mitä merkitystä on Bohr-van Leeuwen teoreemalla?

   (b) (4p) Mitä ovat Landaun tasot? Matemaattinen kaava ja sen sanallinen kuvaus vaaditaan täysiin pisteisiin.

   (c) (4p) Debyen malli kidevärähtelyjen (fononien) ominaislämmölle perustuu tilatiheyteen

$$g(\omega) = \begin{cases} \frac{9N}{\omega_D^3}\omega^2 & \text{, jos } \omega < \omega_D \\ 0 & \text{muulloin .} \end{cases} \tag{5}$$

   Laske tilatiheyden avulla hilan värähtelymoodien lukumäärä. Miten värähtelymoodien lukumäärä liittyy siihen, että tilatiheys katkaistaan Debye-taajuuden $\omega_D$ kohdalta? Mitä malli olettaa fononien dispersiosta?

5. (a) (2p) Einstein-Smoluchowski relaatio on (1D tapaus)

$$\langle x^2 \rangle = 2Dt \ . \tag{6}$$

   Mitä relaatio kuvaa? Miksei $\langle x^2 \rangle$ ole verrannollinen $t$:n neliöön?

   (b) (2p) Mitä tarkoitetaan fluktuaatio-dissipaatio-teoreemalla? Kuvaile sanallisesti tai anna esimerkki.

   (c) (6p) Eristetyn laatikon tilavuus $V$ ja sen sisäseinien lämpötila on $T$. Osoita, että termodynaamisessa tasapainossa laatikossa on keskimäärin

$$N = V\left(\frac{k_B T}{\hbar c}\right)^3 \times a \tag{7}$$

   fotonia, missä $a$ on dimensioton luku. Fotonikaasun tilatiheys on $g(\epsilon) = V\epsilon^2/(\pi^2\hbar^3c^3)$.

**QUESTIONS IN ENGLISH**

1. Answer the following questions. Justify briefly. Using formulas is encouraged.
   (a) (1p) Why is the chemical potential of ideal bosons negative at all temperatures?
   (b) (1p) What is the Wien displacement law?
   (c) (2p) What is degenerate electron gas? Why is the Chandrasekhar limit (1.4 Solar masses) obtained using the pressure of *ultrarelativistic* degenerate electron gas?
   (d) (2p) Draw a schetch of ideal boson gas phase diagrams in $(P, V)$-plane (a few isotherms), and in $(P, T)$-plane.
   (e) (2p) What does the collision term $\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}$ in Boltzmann theory represent?
   (f) (1p) Why is the photon gas chemical potential $\mu = 0$?
   (g) (1p) What is Bose-Einstein condensation?

2. (9p) The grand canonical partition function of ideal bosons and fermions for a single particle state $i$ is

$$\mathcal{Z}_i = \sum_n e^{-\beta(\epsilon_i - \mu)n} \ , \tag{8}$$

   where $\epsilon_i$ is the energy and $n$ is the occupation of the single particle state. Evaluate $\mathcal{Z}_i$ (calculate the sum) for bosons and fermions. The grand canonical partition function is the product $\mathcal{Z} = \prod_i \mathcal{Z}_i$. Write down the grand potential $\Omega$ for bosons and fermions, and use it to get the expectation values of the occupations $\langle n_i \rangle$ of the single particle state $i$ for bosons ja fermions.
   Hint: $\langle n_i \rangle = \frac{\partial \Omega}{\partial \epsilon_i}$.

3. (a) (5p) Let's examine particles in a box. The volume is $V = L^3$, where the side lengths are $L$. The energy spectrum of the particles is

$$\epsilon = \frac{\hbar^2 k^2}{2m} \ , \tag{9}$$

   where $k = |\mathbf{k}|$ and, according to quantum mechanics, the possible $\mathbf{k}$ values are

$$\mathbf{k}_i = (i_1, i_2, i_3)\frac{\pi}{L} \quad , \ i_1, i_2, i_3 = 1, 2, 3, \ldots \ . \tag{10}$$

   Calculate the density of states $g(\epsilon)$ with a method of your own choice.
   (b) (4p) The canonical partition function is

$$Z = \frac{1}{N!}\left(\frac{V}{\lambda_T^3}\right)^N \ . \tag{11}$$

   What kind of particles does this represent and what is the equation of state?

4. (a) (2p) What is the significance of the Bohr-van Leeuwen teorem?
   (b) (4p) What are Landau levels? A mathematical expression and it's explanation in words is required for full points.

(c) (4p) The Debye model of lattice vibrations is based on the density of states

$$g(\epsilon) = \begin{cases} \frac{9N}{\omega_D^3}\omega^2 & , \text{ if } \omega < \omega_D \\ 0 & \text{else} . \end{cases} \tag{12}$$

Using $g(\epsilon)$, calculate how many vibrational modes there are. How is the number of vibrational modes related to the fact, that the density of states has a cut off at the Debye-frequency $\omega_D$? What does the model assume about the phonon dispersion?

5. (a) (2p) The Einstein-Smoluchowski relation is (1D case)

$$\langle x^2 \rangle = 2Dt \ . \tag{13}$$

What is the significance of rthe relation? Why isn't $\langle x^2 \rangle$ proportional to the square of $t$?

(b) (2p) What is the fluctuation-dissipation theorem? In words, or give an example.

(c) (6p) An isolated box has volume $v$ and the inner wall temperature is $T$. Show, that in thermodynamical equilibrium there are on average

$$N = V \left( \frac{k_B T}{\hbar c} \right)^3 \times a \tag{14}$$

photons in the box, where $a$ is a dimensionless constant. You may use the fact, that the density of states of photon gas is $g(\epsilon) = V\epsilon^2/(\pi^2\hbar^3 c^3)$.

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## Mahdollisesti hyödyllisiä tietoja / potentially useful information

$k_B = 1.3805 \times 10^{-23}\,\mathrm{J/K}$ , $R = k_B N_A = 8.3143\,\mathrm{J/(mol\,K)}$ , $N_A = 6.022 \times 10^{23}\,\mathrm{1/mol}$

$k_B \times 11600$ K $\approx 1$ eV , $0\,^\circ\mathrm{C} = 273.15\,\mathrm{K}$ , 1 atm $= 1.013 \times 10^5\,\mathrm{Pa}$ , $g = 9.82\,\mathrm{m/s^2}$ , $c = 2.998 \times 10^8\,\mathrm{m/s}$

$\hbar = 1.054 \times 10^{-34}$ Js , $1\,\mathrm{J} = 1.602 \times 10^{-19}$ eV , $m_e c^2 = 511\,\mathrm{keV}$ $1u = 1.66 \times 10^{-27}\,\mathrm{kg}$

$$dU = \delta Q + \delta W \overset{\text{rev}}{=} TdS - PdV \ , \quad dU = TdS - PdV + \mu dN$$

$$F = U - TS \ , \quad G = U - TS + PV \ , \quad H = U + PV \ , \quad \Omega = U - TS - \mu N = -PV$$

$$S = k_B \ln \Omega \ , \quad F = -k_B T \ln Z \ , \quad U = -\frac{\partial}{\partial \beta} \ln Z \ , \quad ln(n!) \approx n \ln n - n \ , \quad \binom{N}{n} \equiv \frac{N!}{n!(N-n)!}$$

$$C_V = T \left(\frac{\partial S}{\partial T}\right)_{V,N} = \left(\frac{\partial U}{\partial T}\right)_{V,N} \ , \quad C_P = T \left(\frac{\partial S}{\partial T}\right)_{P,N} = \left(\frac{\partial H}{\partial T}\right)_{P,N} \ , \quad \kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,N} \ , \quad \kappa_S = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{S,N}$$

$$\left(\frac{\partial x}{\partial y}\right)_z = \left[\left(\frac{\partial y}{\partial x}\right)_z\right]^{-1} \ , \quad \left(\frac{\partial x}{\partial y}\right)_z = \left(\frac{\partial x}{\partial w}\right)_z \left(\frac{\partial w}{\partial y}\right)_z \ , \quad \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

$$S = -k_B \sum_i p_i \ln p_i \ , \quad p_i = \frac{1}{Z} e^{-\beta E_i} \ , \quad Z = \sum_i e^{-\beta E_i} \ , \quad \beta \equiv \frac{1}{k_B T}$$

$$\Omega(T,V,\mu) = -k_B T \ln \mathcal{Z} \ , \quad p_i = \frac{1}{\mathcal{Z}} e^{-\beta(E_i - \mu N_i)} \ , \quad \mathcal{Z} = \sum_i e^{-\beta(E_i - \mu N_i)}$$

$$\lambda_T = \left(\frac{2\pi\hbar^2}{m k_B T}\right)^{1/2} \ , \quad Z_1(T,V) = V \lambda_T^{-3} \ , \quad Z_N(T,V) = \frac{1}{N!}\left[Z_1(T,V)\right]^N$$

$$PV = N k_B T = nRT \ , \quad U = \frac{3}{2} N k_B T \ , \quad \left(\frac{dP}{dT}\right)_{\text{cx}} = \frac{\Delta S}{\Delta V} = \frac{L_{1\to2}}{T \Delta V_{1\to2}}$$

$$\langle n_i \rangle_{\text{FD}} = \frac{1}{e^{\beta(\epsilon_i - \mu)} + 1} \overset{T \to 0}{\to} \theta(\epsilon_F - \epsilon_i) \ , \quad \frac{\partial \langle n_i \rangle_{\text{FD}}}{\partial \epsilon_i} \overset{T \to 0}{\to} -\delta(\epsilon_i - \epsilon_F) \ , \quad \langle n_i \rangle_{\text{BE}} = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1} \ , \quad \langle n_i \rangle_{\text{MB}} = e^{-\beta(\epsilon_i - \mu)}$$

$$\Omega_{\text{FD}} = -k_B T \sum_i \ln[1 + e^{-\beta(\epsilon - \mu)}] \overset{T \to 0}{\to} -\sum_i (\mu - \epsilon_i)\theta(\mu - \epsilon_i) \ , \quad \Omega_{\text{BE}} = k_B T \sum_i \ln[1 - e^{-\beta(\epsilon - \mu)}]$$

$$\sum_i f(k_i) \approx \frac{gV}{2\pi^2} \int_0^\infty dk k^2 f(k) \ , \quad \epsilon(k) = \frac{\hbar^2 k^2}{2m} \ (NR) \ , \quad \epsilon(k) = \hbar c k \ (UR)$$

$$u(\omega, T) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1} \ , \quad u(T) = aT^4 \ , \quad I(T) = \sigma T^4 \ , \quad a = \frac{\pi^2 k_B^4}{15 \hbar^3 c^3} \ , \quad \sigma = \frac{ac}{4}$$

$$\int_0^\infty dx \frac{x^p}{e^x - 1} = p!\zeta(p+1) \ , \quad \int_0^\infty dx \frac{x^p}{e^x + 1} = (1 - 2^{-p})\zeta(p+1)$$

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} \ , \quad \zeta(\frac{3}{2}) \approx 2.61 \ , \quad \zeta(2) = \frac{\pi^2}{6} \ , \quad \zeta(\frac{5}{2}) \approx 1.34 \ , \quad \zeta(3) \approx 1.20 \ , \quad \zeta(\frac{7}{2}) \approx 1.13 \ , \quad \zeta(4) = \frac{\pi^4}{90}$$

$$\Gamma(p+1) = p! \ , \quad \Gamma(z+1) = z\Gamma(z) \ , \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$\int_0^\infty dx e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}} \ , \quad \int_0^\infty dx x^{2n} e^{-ax^2} = (-1)^n \left[\frac{d^n}{da^n} \frac{1}{2}\sqrt{\frac{\pi}{a}}\right]_{a=1}$$

$$\int_0^\infty dx x e^{-ax^2} = \frac{1}{2a} \ , \quad \int_0^\infty dx x^{2n+1} e^{-ax^2} = (-1)^n \left[\frac{d^n}{da^n} \frac{1}{2a}\right]_{a=1}$$

$$\int d\epsilon \frac{f(\epsilon)}{e^{\beta(\epsilon - \mu)} + 1} \approx \int_0^\mu d\epsilon f(\epsilon) + \frac{\pi^2}{6}(k_B T)^2 \left.\frac{df(\epsilon)}{d\epsilon}\right|_\mu + \mathcal{O}(T^4)$$

$$\sum_{n=0}^\infty a x^n = \frac{a}{1-x}, |x| < 1, \ , \quad \sum_{n=1}^\infty \frac{x^n}{n} = -\ln(1-x), \ |x| < 1 \ , \quad e^x = \sum_{n=0}^\infty \frac{x^n}{n!} \ , \quad (a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

$$\frac{\partial f}{\partial t} + \dot{\mathbf{r}} \cdot \boldsymbol{\nabla} f + \dot{\mathbf{p}} \cdot \boldsymbol{\nabla}_{\mathbf{p}} f = -\frac{1}{\tau}(f - f_0) \ , \quad \dot{\mathbf{r}} = \mathbf{v} \ , \quad \dot{\mathbf{p}} = \mathbf{F}$$

If $\dfrac{\partial f}{\partial t} = 0$ and $f = f_0 + f'$ and $f' \ll f_0$, then$f' \approx -\tau(\dot{\mathbf{r}} \cdot \boldsymbol{\nabla} f_0 + \dot{\mathbf{p}} \cdot \boldsymbol{\nabla}_{\mathbf{p}} f_0)$