FYSA2042 osa B

Koe pe 28.8.2020. Kesto 4 tuntia. Kaavakokoelma lopussa.

Exam Friday August 28th, 2020. Duration: 4 hours. Questions in English and a collection of formulae at the end of the sheet

1. Vastaa lyhyesti seuraaviin kysymyksiin:

   (a) (1p) Milloin suurkanoninen joukko antaa samat tulokset kuin kanoninen joukko?

   (b) (1p) Milloin klassinen Maxwell-Boltzmann statistiikka on kyllin tarkka identtisten hiukkasten kuvaukseen?

   (c) (1p) Mitä tarkoittaa Bose-Einstein kondensaatio?

   (d) (2p) Hahmottele ideaalisen bosonikaasun faasidiagrammat (P, V)-tasossa (muutamia isotermejä), sekä (P, T)-tasossa.

   (e) (1p) Mitä Wienin siirtymälaki sanoo?

   (f) (2p) Miksi Debyen malli toimii paremmin kuin Einsteinin malli, erityisesti matalissa lämpötiloissa?

   (g) (2p) Mitä Boltzmannin teoriassa kuvaa törmäysintegraali?

2. (a) (5p) Osoita, että N hiukkasen kanonisen joukon partitiofunktion Z_N ja suurkanonisen joukon partitiofunktion $\mathcal{Z}$ välillä on yhteys

   $$\mathcal{Z} = \sum_N e^{\beta\mu N} Z_N \ .$$

   (b) (4p) Kirjoita Z_N tilatiheyden $g(\epsilon) = \sum_i \delta(\epsilon - \epsilon_i)$ avulla.

3. (9p) Mustan kappaleen säteilyn energiatiheys on $u(T) = aT^4$, $a = \frac{\pi^2 k_B^4}{15\hbar^3 c^3}$. Osoita, että mustan kappaleen säteilyn fotonien keskimääräinen energia on n. $2,7 \times k_B T$.

4. Tarkastellaan N:ää vuorovaikuttamatonta bosonia L^2 kokoisessa neliössä (2D kaasu) termodynaamisella rajalla. Energiaspektri on $\epsilon_k = \hbar^2 k^2/(2m)$.

   (a) (6p) Osoita, että niiden tilojen lukumäärä, joiden energia on alle ϵ, on

   $$N(\epsilon) = \sum_n \theta(\epsilon - \epsilon_n) = \frac{gL^2}{\pi}\left(\frac{2m\epsilon}{\hbar^2}\right) \ ,$$

   missä θ on steppifunktio ja summa on yli mikrotilojen.

   (b) (1p) Osoita, että tilatiheys on

   $$g(\epsilon) = \sum_n \delta(\epsilon - \epsilon_n) = \frac{dN(\epsilon)}{d\epsilon} = \frac{gL^2}{\pi}\left(\frac{2m}{\hbar^2}\right) \ .$$

   (c) (3p) Osoita tilatiheyden avulla, ettei 2-ulotteisessa ideaalisessa bosonikaasussa ole Bose-Einstein kondensaatiota.
   Vihje: Kondensaatiolämpötilassa T_c on kemiallinen potentiaali $\mu = 0^-$, mutta silloin $N = \langle N \rangle$ on integraali, joka ei suppene.

5. (a) (6p) Eristetyn laatikon tilavuus V ja sen sisäseinien lämpötila on T. Osoita, että termodynaamisessa tasapainossa laatikossa on keskimäärin

$$N = V \left(\frac{k_B T}{\hbar c} \right)^3 \times a$$

fotonia, missä a on dimensioton luku. Fotonikaasun tilatiheys on $g = V\epsilon^2/(\pi^2\hbar^3 c^3)$.

(b) (4p) Osoita edellisen tuloksen avulla, että fotonikaasun ominaislämpö on

$$c_V \propto T^3 \ .$$

1. Give brief answers to following questions:

   (a) (1p) When do the grand canonical ensemble and the canonical ensemble give equal results?

   (b) (1p) When does the classical Maxwell-Boltzmann statistics describe identical particles accurately?

   (c) (1p) What is Bose-Einstein condensation?

   (d) (2p) Draw a schetch of ideal boson gas phase diagrams in (P, V)-plane (a few isotherms), and in (P, T)-plane.

   (e) (1p) What does the Wien displacement law say?

   (f) (2p) Why does the Debye model work better than the Einstein model, especially at low temperatures?

   (g) (2p) What does the collision terms in Boltzmann theory represent?

2. (a) (5p) Show, that the relation between the N particle canonical partition function Z_N and the grand canonical partition function $\mathcal{Z}$ is

$$\mathcal{Z} = \sum_N e^{\beta\mu N} Z_N \ .$$

   (b) (4p) Write Z_N in terms of the density of states $g(\epsilon) = \sum_i \delta(\epsilon - \epsilon_i)$.

3. Black body radiation has energy density $u(T) = aT^4$, $a = \frac{\pi^2 k_B^4}{15\hbar^3 c^3}$. Show that the average energy of photons is approximately $2.7 \times k_B T$.

4. We have N noninteracting bosons in a 2D square L^2 (2D gas) in the thermodynamical limit. The energy spectrum is $\epsilon_k = \hbar^2 k^2/(2m)$.

   (a) (6p) Show, that the number of states below energy ϵ is

$$N(\epsilon) = \sum_n \theta(\epsilon - \epsilon_n) = \frac{gL^2}{(2\pi)^2} 4\pi \left(\frac{2m\epsilon}{\hbar^2} \right) ,$$

   where θ is the step function and the sum is over microstates.

   (b) (1p) Show, that the density of states is

$$g(\epsilon) = \frac{dN(\epsilon)}{d\epsilon} = \frac{gL^2}{\pi} \left(\frac{2m}{\hbar^2} \right) .$$

   (c) (3p) Based on the density of states, show that there is no Bose-Einstein condensation in 2-dimensional ideal boson gas.
   Hint: At the condensation temperature T_c the chemical potential is $\mu = 0^-$, but then $N = \langle N \rangle$ is a divergent integral.

5. (a) (6p) An isolated box has volume v and the inner wall temperature is T. Show, that in thermodynamical equilibrium there are on average

$$N = V \left(\frac{k_B T}{\hbar c} \right)^3 \times a$$

   photons in the box, where a is a dimensionless constant. The density of states of photon gas is $g = V\epsilon^2/(\pi^2 \hbar^3 c^3)$.

(b) (4p) Using the previous result, show that the spefific heat of photon gas is

$$c_V \propto T^3 \ .$$

Mahdollisesti hyödyllisiä tietoja / potentially useful information

$k_B = 1.3805 \times 10^{-23}\,\mathrm{J/K}$, $R = k_B N_A = 8.3143\,\mathrm{J/(mol\,K)}$, $N_A = 6.022 \times 10^{23}\,\mathrm{1/mol}$

$k_B \times 11600$ K ≈ 1 eV , $0\,^\circ\mathrm{C} = 273.15\,\mathrm{K}$, 1 atm $= 1.013 \times 10^5\,\mathrm{Pa}$, $g = 9.82\,\mathrm{m/s^2}$, $c = 2.998 \times 10^8\,\mathrm{m/s}$

$\hbar = 1.054 \times 10^{-34}$ Js , $1\,\mathrm{J} = 1.602 \times 10^{-19}\,\mathrm{eV}$, $m_e c^2 = 511\,\mathrm{keV}$ $1u = 1.66 \times 10^{-27}\,\mathrm{kg}$

$dE = \delta Q + \delta W \overset{\mathrm{rev.}}{=} TdS - PdV$, $dE = TdS - PdV + \mu dN$

$F = E - TS$, $G = E - TS + PV$, $H = E + PV$, $\Omega = E - TS - \mu N = -PV$

$S = k_B \ln \Omega$, $F = -k_B T \ln Z$, $E = -\frac{\partial}{\partial \beta} \ln Z$, $ln(n!) \approx n \ln n - n$, $\binom{N}{n} \equiv \frac{N!}{n!(N-n)!}$

$C_V = T\left(\frac{\partial S}{\partial T}\right)_{V,N} = \left(\frac{\partial E}{\partial T}\right)_{V,N}$, $C_P = T\left(\frac{\partial S}{\partial T}\right)_{P,N} = \left(\frac{\partial H}{\partial T}\right)_{P,N}$, $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,N}$, $\kappa_S = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{S,N}$

$\left(\frac{\partial x}{\partial y}\right)_z = \left[\left(\frac{\partial y}{\partial x}\right)_z\right]^{-1}$, $\left(\frac{\partial x}{\partial y}\right)_z = \left(\frac{\partial x}{\partial w}\right)_z \left(\frac{\partial w}{\partial y}\right)_z$, $\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$

$S = -k_B \sum_i p_i \ln p_i$, $p_i = \frac{1}{Z} e^{-\beta E_i}$, $Z = \sum_i e^{-\beta E_i}$, $\beta \equiv \frac{1}{k_B T}$

$\Omega_G(T, V, \mu) = -k_B T \ln \mathcal{Z}$, $p_i = \frac{1}{\mathcal{Z}} e^{-\beta(E_i - \mu N_i)}$, $\mathcal{Z} = \sum_i e^{-\beta(E_i - \mu N_i)}$

$\lambda_T = \left(\frac{2\pi\hbar^2}{m k_B T}\right)^{1/2}$, $Z_1(T, V) = V \lambda_T^{-3}$, $Z_N(T, V) = \frac{1}{N!}\left[Z_1(T, V)\right]^N$

$PV = N k_B T = nRT$, $E = \frac{3}{2} N k_B T$, $\left(\frac{dP}{dT}\right)_{\mathrm{cx}} = \frac{\Delta S}{\Delta V} = \frac{L_{1\to 2}}{T \Delta V_{1\to 2}}$

$\langle n_i \rangle_{\mathrm{FD}} = \frac{1}{e^{\beta(\epsilon_i - \mu)} + 1} \overset{T \to 0}{\to} \theta(\epsilon_F - \epsilon_i)$, $\frac{\partial \langle n_i \rangle_{\mathrm{FD}}}{\partial \epsilon_i} \overset{T \to 0}{\to} -\delta(\epsilon_i - \epsilon_F)$, $\langle n_i \rangle_{\mathrm{BE}} = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}$, $\langle n_i \rangle_{\mathrm{MB}} = e^{-\beta(\epsilon_i - \mu)}$

$\Omega_{\mathrm{FD}} = -k_B T \sum_i \ln[1 + e^{-\beta(\epsilon - \mu)}] \overset{T \to 0}{\to} -\sum_i (\mu - \epsilon_i)\theta(\mu - \epsilon_i)$, $\Omega_{\mathrm{BE}} = k_B T \sum_i \ln[1 - e^{-\beta(\epsilon - \mu)}]$

$\sum_i f(k_i) \approx \frac{gV}{2\pi^2} \int_0^\infty dk k^2 f(k)$, $\epsilon(k) = \frac{\hbar^2 k^2}{2m}$ (NR) , $\epsilon(k) = \hbar c k$ (UR)

$u(\omega, T) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}$, $u(T) = aT^4$, $I(T) = \sigma T^4$, $a = \frac{\pi^2 k_B^4}{15\hbar^3 c^3}$, $\sigma = \frac{ac}{4}$

$\int_0^\infty dx \frac{x^p}{e^x - 1} = p!\zeta(p+1)$, $\int_0^\infty dx \frac{x^p}{e^x + 1} = (1 - 2^{-p})\zeta(p+1)$

$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$, $\zeta(\frac{3}{2}) \approx 2.61$, $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(\frac{5}{2}) \approx 1.34$, $\zeta(3) \approx 1.20$, $\zeta(\frac{7}{2}) \approx 1.13$, $\zeta(4) = \frac{\pi^4}{90}$

$\Gamma(p+1) = p!$, $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\int_0^\infty dx e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}$, $\int_0^\infty dx x^{2n} e^{-ax^2} = (-1)^n \left[\frac{d^n}{da^n} \frac{1}{2}\sqrt{\frac{\pi}{a}}\right]_{a=1}$

$\int_0^\infty dx x e^{-ax^2} = \frac{1}{2a}$, $\int_0^\infty dx x^{2n+1} e^{-ax^2} = (-1)^n \left[\frac{d^n}{da^n} \frac{1}{2a}\right]_{a=1}$

$\int d\epsilon \frac{f(\epsilon)}{e^{\beta(\epsilon - \mu)} + 1} \approx \int_0^\mu d\epsilon f(\epsilon) + \frac{\pi^2}{6}(k_B T)^2 \left.\frac{df(\epsilon)}{d\epsilon}\right|_\mu + \mathcal{O}(T^4)$

$\sum_{n=0}^\infty a x^n = \frac{a}{1-x}, |x| < 1,$, $\sum_{n=1}^\infty \frac{x^n}{n} = -\ln(1-x),\ |x| < 1$, $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$, $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$

$\frac{\partial f}{\partial t} + \dot{\mathbf{r}} \cdot \boldsymbol{\nabla} f + \dot{\mathbf{p}} \cdot \boldsymbol{\nabla}_{\mathbf{p}} f = -\frac{1}{\tau}(f - f_0)$, $\dot{\mathbf{r}} = \mathbf{v}$, $\dot{\mathbf{p}} = \mathbf{F}$

Jos $\frac{\partial f}{\partial t} = 0$ ja $f = f_0 + f'$ ja $f' \ll f_0$, niin $f' \approx -\tau(\dot{\mathbf{r}} \cdot \boldsymbol{\nabla} f_0 + \dot{\mathbf{p}} \cdot \boldsymbol{\nabla}_{\mathbf{p}} f_0)$