FYSA2042 osa B, kevät 2019

Koe pe 14.5.2019. Kesto 4 tuntia. Kaavakokoelma lopussa.

Exam Friday June 14th, 2019. Duration: 4 hours. Questions in English and a collection of formulae at the end of the sheet

1. Vastaa lyhyesti seuraaviin kysymyksiin:
   (a) (1p) Mitä tarkoittavat keskimääräinen vapaa matka ja keskimääräinen vapaa aika?
   (b) (1p) Milloin klassinen Maxwell-Boltzmann statistiikka on kyllin tarkka identtisten hiukkasten kuvaukseen?
   (c) (1p) Mitä tarkoittaa Bose-Einstein kondensaatio?
   (d) (1p) Mitä on Brownin liike?
   (e) (2p) Miksi ideaalisten bosonien kemiallinen potentiaali on aina negatiivinen?
   (f) (2p) Kuvaile valkoisen kääpiötähden tasapainoa.
   (g) (2p) Hahmottele muutamia ideaalisen bosonikaasun (P, V)-tason isotermejä ja (P, T)-tason kuvaaja.

2. (9p) Relativististen elektronien yksihiukkasenergia on $\epsilon_{\mathbf{p}} = cp = \hbar ck$, missä $p = |\mathbf{p}|$. Elektronikaasun lämpötila $T = 0$ ja siinä on N hiukkasta tilavuudessa V. Laske Fermi-energia, sisäenergia ja paine N:n ja V:n funktiona.

3. Aineen kanoninen partitiofunktio on

$$Z_N = \frac{1}{N!}\left(\frac{V}{\lambda_T^3}\right)^N ,$$

   missä λ_T on terminen (DeBroglie'n) aallonpituus.

   (a) (3p) Osoita, että suurkanoninen partitiofunktio on (z on fugasiteetti)

$$\mathcal{Z} = \exp\left(zV/\lambda_T^3\right) .$$

   Vihje: $\mathcal{Z} = \sum_{N=0}^{\infty} z^N Z_N$; tämän johtamisesta saa bonuspisteitä.

   (b) (6p) Millaista ainetta partitiofunktio kuvaa?
   Vihje: Laske paine $P = P(N, V, T)$; mikä on aineen tilayhtälö?

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4. Tarkastellaan N:ää vuorovaikuttamatonta bosonia L^2 kokoisessa neliössä (2D kaasu) termodynaamisella rajalla. Energiaspektri on $\epsilon_k = \hbar^2 k^2/(2m)$.

   (a) (6p) Osoita, että niiden tilojen lukumäärä, joiden energia on alle ϵ, on

   $$N(\epsilon) = \sum_n \theta(\epsilon - \epsilon_n) = \frac{gL^2}{4\pi}\left(\frac{2m\epsilon}{\hbar^2}\right) \ ,$$

   missä θ on steppifunktio ja summa on yli mikrotilojen.

   (b) (1p) Osoita, että tilatiheys on

   $$g(\epsilon) = \sum_n \delta(\epsilon - \epsilon_n) = \frac{dN(\epsilon)}{d\epsilon} = \frac{gL^2}{4\pi}\left(\frac{2m}{\hbar^2}\right) \ .$$

   (c) (3p) Osoita tilatiheyden avulla, ettei 2-ulotteisessa ideaalisessa bosonikaasussa ole Bose-Einstein kondensaatiota.
   Vihje: Kondensaatiolämpötilassa T_c on kemiallinen potentiaali $\mu = 0^-$, mutta silloin $N = \langle N \rangle$ on integraali, joka ei suppene.

5. (a) (6p) Eristetyn laatikon tilavuus V ja sen sisäseinien lämpötila on T. Osoita, että termodynaamisessa tasapainossa laatikossa on keskimäärin

   $$N = V\left(\frac{k_B T}{\hbar c}\right)^3 \times a$$

   fotonia, missä a on dimensioton luku.
   Voit käyttää hyväksi tietoa, että fotonikaasun tilatiheys on $g = V\epsilon^2/(\pi^2\hbar^3c^3)$

   (b) (4p) Osoita, että fotonikaasun lämpökapasiteetti on

   $$C_V \propto T^3 \ .$$

1. Give brief answers to following questions:

   (a) (1p) What does one mean with the mean free path and the mean free time?

   (b) (1p) When is the classical Maxwell-Boltzmann statistics accurate enough to describe identical particles?

   (c) (1p) What is Bose-Einstein condensation?

   (d) (1p) What is brownian motion?

   (e) (2p) Why is the the chemical potential of ideal bosons always negative?

   (f) (2p) Describe the balance of white dwarf stars.

   (g) (2p) Scketch a few isotherms in (P, V)-plane and the (P, T)-plane diagram of ideal bosons.

2. (9p) Relativistic electrons have single-particle energies $\epsilon_{\mathbf{p}} = cp = \hbar ck$, where $p = |\mathbf{p}|$. The temperature of elektron gas is $T = 0$, and there are N electrons in volume V. Calculate the Fermi-energy, internal energy and pressure as functions of N and V.

3. The canonical partition function of matter is

$$Z_N = \frac{1}{N!}\left(\frac{V}{\lambda_T^3}\right)^N ,$$

   where λ_T is the thermal (DeBroglie) wave length.

   (a) (3p) Show, that the grand canonical partition function is (z is fugacity)

$$\mathcal{Z} = \exp\left(zV/\lambda_T^3\right) .$$

   Hint: $\mathcal{Z} = \sum_{N=0}^{\infty} z^N Z_N$; derive this formula for bonus points.

   (b) (6p) What kind of matter does the partition function describe?
   Hint: Calculate pressure $P = P(N, V, T)$; what is the equation of state of this matter?

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4. Lets examine N noninteracting bosons in a 2D square L^2 (2D gas) in the thermodynamical limit . The energy spectrum is $\epsilon_k = \hbar^2 k^2/(2m)$.

   (a) (6p) Show, that the number of states below energy ϵ is

   $$N(\epsilon) = \sum_n \theta(\epsilon - \epsilon_n) = \frac{gL^2}{(4\pi} \left(\frac{2m\epsilon}{\hbar^2}\right) \ ,$$

   where θ is the step function and the sum is over microstates.

   (b) (1p) Show, that the density of states is

   $$g(\epsilon) = \frac{dN(\epsilon)}{d\epsilon} = \frac{gL^2}{4\pi} \left(\frac{2m}{\hbar^2}\right) \ .$$

   (c) (3p) Based on the density of states, show that there is no Bose-Einstein condensation in 2-dimensional ideal boson gas.
   Hint: At the condensation temperature T_c the chemical potential is $\mu = 0^-$, but then $N = \langle N \rangle$ is a divergent integral.

5. (a) (6p) An isolated box has volume v and the inner wall temperature is T. Show, that in thermodynamical equilibrium there are on average

   $$N = V \left(\frac{k_B T}{\hbar c}\right)^3 \times a$$

   photons in the box, where a is a dimensionless constant. You may use the fact, that the density of states of photon gas is $g = V\epsilon^2/(\pi^2\hbar^3c^3)$.

   (b) (4p) Show that the heat capasity of photon gas is

   $$C_V \propto T^3 \ .$$

Mahdollisesti hyödyllisiä tietoja / potentially useful information

$k_B = 1.3805 \times 10^{-23}\,\mathrm{J/K}$, $R = k_B N_A = 8.3143\,\mathrm{J/(mol\,K)}$, $N_A = 6.022 \times 10^{23}\,\mathrm{1/mol}$

$k_B \times 11600$ K ≈ 1 eV , $0\,^\circ\mathrm{C} = 273.15\,\mathrm{K}$, 1 atm $= 1.013 \times 10^5\,\mathrm{Pa}$, $g = 9.82\,\mathrm{m/s^2}$, $c = 2.998 \times 10^8\,\mathrm{m/s}$

$\hbar = 1.054 \times 10^{-34}$ Js , $1\,\mathrm{J} = 1.602 \times 10^{-19}$ eV , $m_e c^2 = 511\,\mathrm{keV}$ $1u = 1.66 \times 10^{-27}$ kg

$dE = \delta Q + \delta W \overset{\mathrm{rev.}}{=} TdS - PdV$, $dE = TdS - PdV + \mu dN$

$F = E - TS$, $G = E - TS + PV$, $H = E + PV$, $\Omega = E - TS - \mu N = -PV$

$S = k_B \ln \Omega$, $F = -k_B T \ln Z$, $E = -\frac{\partial}{\partial \beta} \ln Z$, $ln(n!) \approx n \ln n - n$, $\binom{N}{n} \equiv \frac{N!}{n!(N-n)!}$

$C_V = T\left(\frac{\partial S}{\partial T}\right)_{V,N} = \left(\frac{\partial E}{\partial T}\right)_{V,N}$, $C_P = T\left(\frac{\partial S}{\partial T}\right)_{P,N} = \left(\frac{\partial H}{\partial T}\right)_{P,N}$, $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,N}$, $\kappa_S = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{S,N}$

$\left(\frac{\partial x}{\partial y}\right)_z = \left[\left(\frac{\partial y}{\partial x}\right)_z\right]^{-1}$, $\left(\frac{\partial x}{\partial y}\right)_z = \left(\frac{\partial x}{\partial w}\right)_z \left(\frac{\partial w}{\partial y}\right)_z$, $\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$

$S = -k_B \sum_i p_i \ln p_i$, $p_i = \frac{1}{Z} e^{-\beta E_i}$, $Z = \sum_i e^{-\beta E_i}$, $\beta \equiv \frac{1}{k_B T}$

$\Omega_G(T,V,\mu) = -k_B T \ln \mathcal{Z}$, $p_i = \frac{1}{\mathcal{Z}} e^{-\beta(E_i - \mu N_i)}$, $\mathcal{Z} = \sum_i e^{-\beta(E_i - \mu N_i)}$

$\lambda_T = \left(\frac{2\pi\hbar^2}{m k_B T}\right)^{1/2}$, $Z_1(T,V) = V \lambda_T^{-3}$, $Z_N(T,V) = \frac{1}{N!}\left[Z_1(T,V)\right]^N$

$PV = N k_B T = nRT$, $E = \frac{3}{2} N k_B T$, $\left(\frac{dP}{dT}\right)_{\mathrm{cx}} = \frac{\Delta S}{\Delta V} = \frac{L_{1\to 2}}{T \Delta V_{1\to 2}}$

$\langle n_i \rangle_{\mathrm{FD}} = \frac{1}{e^{\beta(\epsilon_i - \mu)} + 1} \overset{T\to 0}{\to} \theta(\epsilon_F - \epsilon_i)$, $\frac{\partial \langle n_i \rangle_{\mathrm{FD}}}{\partial \epsilon_i} \overset{T\to 0}{\to} -\delta(\epsilon_i - \epsilon_F)$, $\langle n_i \rangle_{\mathrm{BE}} = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}$, $\langle n_i \rangle_{\mathrm{MB}} = e^{-\beta(\epsilon_i - \mu)}$

$\Omega_{\mathrm{FD}} = -k_B T \sum_i \ln[1 + e^{-\beta(\epsilon - \mu)}] \overset{T\to 0}{\to} -\sum_i (\mu - \epsilon_i)\theta(\mu - \epsilon_i)$, $\Omega_{\mathrm{BE}} = k_B T \sum_i \ln[1 - e^{-\beta(\epsilon - \mu)}]$

$\sum_i f(k_i) \approx \frac{gV}{2\pi^2} \int_0^\infty dk k^2 f(k)$, $\epsilon(k) = \frac{\hbar^2 k^2}{2m}$ (NR) , $\epsilon(k) = \hbar c k$ (UR)

$u(\omega, T) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}$, $u(T) = aT^4$, $I(T) = \sigma T^4$, $a = \frac{\pi^2 k_B^4}{15 \hbar^3 c^3}$, $\sigma = \frac{ac}{4}$

$\int_0^\infty dx \frac{x^p}{e^x - 1} = p! \zeta(p+1)$, $\int_0^\infty dx \frac{x^p}{e^x + 1} = (1 - 2^{-p}) \zeta(p+1)$

$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$, $\zeta(\frac{3}{2}) \approx 2.61$, $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(\frac{5}{2}) \approx 1.34$, $\zeta(3) \approx 1.20$, $\zeta(\frac{7}{2}) \approx 1.13$, $\zeta(4) = \frac{\pi^4}{90}$

$\Gamma(p+1) = p!$, $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\int_0^\infty dx e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}$, $\int_0^\infty dx x^{2n} e^{-ax^2} = (-1)^n \left[\frac{d^n}{da^n} \frac{1}{2}\sqrt{\frac{\pi}{a}}\right]_{a=1}$

$\int_0^\infty dx x e^{-ax^2} = \frac{1}{2a}$, $\int_0^\infty dx x^{2n+1} e^{-ax^2} = (-1)^n \left[\frac{d^n}{da^n} \frac{1}{2a}\right]_{a=1}$

$\int d\epsilon \frac{f(\epsilon)}{e^{\beta(\epsilon - \mu)} + 1} \approx \int_0^\mu d\epsilon f(\epsilon) + \frac{\pi^2}{6}(k_B T)^2 \left.\frac{df(\epsilon)}{d\epsilon}\right|_\mu + \mathcal{O}(T^4)$

$\sum_{n=0}^\infty a x^n = \frac{a}{1-x}, |x| < 1,$, $\sum_{n=1}^\infty \frac{x^n}{n} = -\ln(1-x),\ |x| < 1$, $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$, $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$

$\frac{\partial f}{\partial t} + \dot{\mathbf{r}} \cdot \boldsymbol{\nabla} f + \dot{\mathbf{p}} \cdot \boldsymbol{\nabla}_{\mathbf{p}} f = -\frac{1}{\tau}(f - f_0)$, $\dot{\mathbf{r}} = \mathbf{v}$, $\dot{\mathbf{p}} = \mathbf{F}$

Jos $\frac{\partial f}{\partial t} = 0$ ja $f = f_0 + f'$ ja $f' \ll f_0$, niin $f' \approx -\tau(\dot{\mathbf{r}} \cdot \boldsymbol{\nabla} f_0 + \dot{\mathbf{p}} \cdot \boldsymbol{\nabla}_{\mathbf{p}} f_0)$