FYSA2042 osa B, kevät 2019

Koe pe 10.5.2019. Kesto 4 tuntia. Kaavakokoelma lopussa.
Exam Friday May 10th, 2019. Duration: 4 hours. Questions in English and a collection of formulae at the end of the sheet

1. Vastaa lyhyesti seuraaviin kysymyksiin:

   (a) (1p) Miten kemiallinen potentiaali μ liittyy hiukkasluvun odotusarvoon $\langle N\rangle$ suurkanonisessa joukossa, eli miten μ:n nosto tai lasku vaikuttaa odotusarvoon $\langle N\rangle$?

   (b) (1p) Mitä tarkoitetaan degeneraatiolla, kun lasketaan summa yli tilojen? Esimerkiksi. mikä on degeneraatio laskettaessa fermioneille summa $\sum_i f(\mathbf{k}_i)$, missä suure f riippuu vain aaltoluvusta, muttei spinistä?

   (c) (1p) Mitä tarkoittaa Bose-Einstein kondensaatio?

   (d) (1p) Mitä tarkoitetaan degeneroituneella elektronikaasulla?

   (e) (2p) Mitä statistisessa mekaniikassa tarkoittaa tasapainotilalla ja toisaalta muuttumattomalla eli stationaarisella tilalla?

   (f) (2p) Mitä kiteen ominaislämmön Debyen mallissa oletetaan fononien spektristä $\epsilon(k)$?

   (g) (2p) Boltzmannin (kuljetus)teoriassa ratkaistaan jakaumaa $f(\mathbf{r};\mathbf{p};t)$. Mitä jakauma kuvaa?

2. (9p) Relativististen elektronien yksihiukkasenergia on $\epsilon_{\mathbf{p}} = cp = \hbar ck$, missä $p = |\mathbf{p}|$. Elektronikaasun lämpötila $T = 0$ ja siinä on N hiukkasta tilavuudessa V. Laske Fermi-energia, sisäenergia ja paine N:n ja V:n funktiona.

3. (a) (2p) Mitä tarkoittavat keskimääräinen vapaa matka ja keskimääräinen vapaa aika?

   (b) (7p) Metalleissa elektronitiheys on usein hyvin korkea, silti elektronien vapaa matka on varsin pitkä. Esim. kuparissa on n. $8{,}5 \times 10^{28}$ elektronia kuutiometrissä, joten elektronien välinen etäisyys on luokkaa

   $$r \approx n^{-1/3} \approx 0{,}2\,\mathrm{nm}\ .$$

   Silti huoneenlämmössä kuparin elektronien vapaa matka on $\lambda \approx 40\,\mathrm{nm}$. Miksi elektronien vapaa matka metalleissa huoneenlämmössä on paljon suurempi kuin elektronien välinen keskietäisyys? Ja miten lämpötila vaikuttaa asiaan?

4. (10p) Gibbsin entropia on

   $$S = -k_B \sum_i p_i \ln p_i\ ,$$

   missä summa on yli mikrotilojen ja p_i on tilan i todennäköisyys. Johda tästä suurkanonisen jakauman p_i:n lauseke maksimoimalla entropia reunaehdoilla (i) $\sum_i p_1 = 1$ (p_i on normitettu todennäköisyysjakauma) (ii) $\sum_i p_i E_i = \langle E\rangle := E$ (termodyn. sisäenergia) ja (iii) $\sum_i p_i N_i = \langle N\rangle := N$ (termod. hiukkasluku).
   Vihje: ota reunaehdot Lagrangen kertoimina, maksimoi S ja kirjoita Lagrangen kertoimet lämpötilan ja kemiallisen potentiaalin avulla käyttäen yhtälöitä $\sum_i p_1 = 1$ ja $dE = TdS - PdV + \mu dN$.

5. (a) (6p) Eristetyn laatikon tilavuus V ja sen sisäseinien lämpötila on T. Osoita, että termodynaamisessa tasapainossa laatikossa on keskimäärin

$$N = V \left(\frac{k_B T}{\hbar c} \right)^3 \times a$$

fotonia, missä a on dimensioton luku.
Voit käyttää hyväksi tietoa, että fotonikaasun tilatiheys on $g = V\epsilon^2/(\pi^2\hbar^3 c^3)$

(b) (4p) Osoita edellisen tuloksen avulla, että fotonikaasun ominaislämpö on

$$c_V \propto T^3 \ .$$

1. Give brief answers to following questions:

   (a) (1p) In the grand canonical ensemble, what is the relation of chemical potential μ and the expectation value of particle number $\langle N\rangle$, that is, how does rising or lowering μ affect the expectation value $\langle N\rangle$?

   (b) (1p) What is degeneration in summing over states? For example, what is degeneracy if one calculates the fermion sum $\sum_i f(\mathbf{k}_i)$, where the quantity f depends on the wavenumber, but not on spin?

   (c) (1p) What is Bose-Einstein condensation?

   (d) (1p) What is degenerate electron gas?

   (e) (2p) In statistical mechanics, what does equilibrium state and stationary state mean?

   (f) (2p) What assumptions about the phonon spectrum $\epsilon(k)$ does the Debye model of lattice specific heat make?

   (g) (2p) In the Boltzmann (transport) theory one solves the distribution $f(\mathbf{r};\mathbf{p};t)$. What does the distribution describe?

2. (9p) Relativistic electrons have single-particle energies $\epsilon_{\mathbf{p}} = cp = \hbar ck$, where $p = |\mathbf{p}|$. The temperature of elektron gas is $T = 0$, and there are N electrons in volume V. Calculate the Fermi-energy, internal energy and pressure as functions of N and V.

3. (a) (2p) What does one mean with the mean free path and the mean free time?

   (b) (7p) The electron density in metals is often very high, yet the electron mean free path is quite long. For example, in copper there are about 8.5×10^{28} electrons per cubic meter, so the distance between electrons is about

   $$r \approx n^{-1/3} \approx 0.2\,\mathrm{nm} \ .$$

   Still, at room temperature the mean free path of electrons in copper is $\lambda \approx 40\,\mathrm{nm}$. Why is the electron mean free path at room temperature so much longer than the average distance between electrons? What is the role of temperature?

4. Gibbs entropy is

   $$S = -k_B \sum_i p_i \ln p_i \ ,$$

   where the sum is over microstates and p_i is the probability of state i. Derive p_i in the grand canonical ensemble by maximizing entropy with boundary conditions (i) $\sum_i p_1 = 1$ (p_i is normalized probability distribution) (ii) $\sum_i p_i E_i = \langle E\rangle := E$ (thermod. internal energy), and (iii) $\sum_i p_i N_i = \langle N\rangle := N$ (thermod. number of particles).
   Hint: take the boundary conditions as Lagrange multipliers, maximize S, and write the Lagrange multipliers in terms of temperature and chemical potential with the aid of relations $\sum_i p_1 = 1$ and $dE = TdS - PdV + \mu dN$.

5. (a) (6p) An isolated box has volume v and the inner wall temperature is T. Show, that in thermodynamical equilibrium there are on average

   $$N = V\left(\frac{k_B T}{\hbar c}\right)^3 \times a$$

photons in the box, where a is a dimensionless constant. You may use the fact, that the density of states of photon gas is $g = V\epsilon^2/(\pi^2\hbar^3c^3)$.

(b) (4p) Using the previous result, show that the spefific heat of photon gas is

$$c_V \propto T^3 \ .$$

Mahdollisesti hyödyllisiä tietoja / potentially useful information

$k_B = 1.3805 \times 10^{-23}\,\mathrm{J/K}$, $R = k_B N_A = 8.3143\,\mathrm{J/(mol\,K)}$, $N_A = 6.022 \times 10^{23}\,1/\mathrm{mol}$

$k_B \times 11600\ \mathrm{K} \approx 1\ \mathrm{eV}$, $0\,^\circ\mathrm{C} = 273.15\,\mathrm{K}$, $1\ \mathrm{atm} = 1.013 \times 10^5\,\mathrm{Pa}$, $g = 9.82\,\mathrm{m/s^2}$, $c = 2.998 \times 10^8\,\mathrm{m/s}$

$\hbar = 1.054 \times 10^{-34}\,\mathrm{Js}$, $1\,\mathrm{J} = 1.602 \times 10^{-19}\,\mathrm{eV}$, $m_e c^2 = 511\,\mathrm{keV}$ $1u = 1.66 \times 10^{-27}\,\mathrm{kg}$

$dU = \delta Q + \delta W \overset{\mathrm{rev}}{=} TdS - PdV$, $dU = TdS - PdV + \mu dN$

$F = U - TS$, $G = U - TS + PV$, $H = U + PV$, $\Omega = U - TS - \mu N = -PV$

$S = k_B \ln \Omega$, $F = -k_B T \ln Z$, $U = -\frac{\partial}{\partial \beta} \ln Z$, $ln(n!) \approx n \ln n - n$, $\binom{N}{n} \equiv \frac{N!}{n!(N-n)!}$

$C_V = T\left(\frac{\partial S}{\partial T}\right)_{V,N} = \left(\frac{\partial U}{\partial T}\right)_{V,N}$, $C_P = T\left(\frac{\partial S}{\partial T}\right)_{P,N} = \left(\frac{\partial H}{\partial T}\right)_{P,N}$, $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,N}$, $\kappa_S = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{S,N}$

$\left(\frac{\partial x}{\partial y}\right)_z = \left[\left(\frac{\partial y}{\partial x}\right)_z\right]^{-1}$, $\left(\frac{\partial x}{\partial y}\right)_z = \left(\frac{\partial x}{\partial w}\right)_z \left(\frac{\partial w}{\partial y}\right)_z$, $\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$

$S = -k_B \sum_i p_i \ln p_i$, $p_i = \frac{1}{Z} e^{-\beta E_i}$, $Z = \sum_i e^{-\beta E_i}$, $\beta \equiv \frac{1}{k_B T}$

$\Omega(T,V,\mu) = -k_B T \ln \mathcal{Z}$, $p_i = \frac{1}{\mathcal{Z}} e^{-\beta(E_i - \mu N_i)}$, $\mathcal{Z} = \sum_i e^{-\beta(E_i - \mu N_i)}$

$\lambda_T = \left(\frac{2\pi\hbar^2}{m k_B T}\right)^{1/2}$, $Z_1(T,V) = V\lambda_T^{-3}$, $Z_N(T,V) = \frac{1}{N!}\left[Z_1(T,V)\right]^N$

$PV = N k_B T = nRT$, $U = \frac{3}{2} N k_B T$, $\left(\frac{dP}{dT}\right)_{\mathrm{cx}} = \frac{\Delta S}{\Delta V} = \frac{L_{1\to 2}}{T \Delta V_{1\to 2}}$

$\langle n_i \rangle_{\mathrm{FD}} = \frac{1}{e^{\beta(\epsilon_i - \mu)} + 1} \overset{T\to 0}{\to} \theta(\epsilon_F - \epsilon_i)$, $\frac{\partial \langle n_i \rangle_{\mathrm{FD}}}{\partial \epsilon_i} \overset{T\to 0}{\to} -\delta(\epsilon_i - \epsilon_F)$, $\langle n_i \rangle_{\mathrm{BE}} = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}$, $\langle n_i \rangle_{\mathrm{MB}} = e^{-\beta(\epsilon_i - \mu)}$

$\Omega_{\mathrm{FD}} = -k_B T \sum_i \ln[1 + e^{-\beta(\epsilon - \mu)}] \overset{T\to 0}{\to} -\sum_i (\mu - \epsilon_i)\theta(\mu - \epsilon_i)$, $\Omega_{\mathrm{BE}} = k_B T \sum_i \ln[1 - e^{-\beta(\epsilon - \mu)}]$

$\sum_i f(k_i) \approx \frac{gV}{2\pi^2} \int_0^\infty dk k^2 f(k)$, $\epsilon(k) = \frac{\hbar^2 k^2}{2m}$ (NR) , $\epsilon(k) = \hbar c k$ (UR)

$u(\omega,T) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}$, $u(T) = aT^4$, $I(T) = \sigma T^4$, $a = \frac{\pi^2 k_B^4}{15\hbar^3 c^3}$, $\sigma = \frac{ac}{4}$

$\int_0^\infty dx \frac{x^p}{e^x - 1} = p!\zeta(p+1)$, $\int_0^\infty dx \frac{x^p}{e^x + 1} = (1 - 2^{-p})\zeta(p+1)$

$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$, $\zeta(\frac{3}{2}) \approx 2.61$, $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(\frac{5}{2}) \approx 1.34$, $\zeta(3) \approx 1.20$, $\zeta(\frac{7}{2}) \approx 1.13$, $\zeta(4) = \frac{\pi^4}{90}$

$\Gamma(p+1) = p!$, $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\int_0^\infty dx e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}$, $\int_0^\infty dx x^{2n} e^{-ax^2} = (-1)^n \left[\frac{d^n}{da^n} \frac{1}{2}\sqrt{\frac{\pi}{a}}\right]_{a=1}$

$\int_0^\infty dx x e^{-ax^2} = \frac{1}{2a}$, $\int_0^\infty dx x^{2n+1} e^{-ax^2} = (-1)^n \left[\frac{d^n}{da^n} \frac{1}{2a}\right]_{a=1}$

$\int d\epsilon \frac{f(\epsilon)}{e^{\beta(\epsilon - \mu)} + 1} \approx \int_0^\mu d\epsilon f(\epsilon) + \frac{\pi^2}{6}(k_B T)^2 \left.\frac{df(\epsilon)}{d\epsilon}\right|_\mu + \mathcal{O}(T^4)$

$\sum_{n=0}^\infty a x^n = \frac{a}{1-x}, |x| < 1,$, $\sum_{n=1}^\infty \frac{x^n}{n} = -\ln(1-x),\ |x| < 1$, $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$, $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$

$\frac{\partial f}{\partial t} + \dot{\mathbf{r}} \cdot \boldsymbol{\nabla} f + \dot{\mathbf{p}} \cdot \boldsymbol{\nabla}_{\mathbf{p}} f = -\frac{1}{\tau}(f - f_0)$, $\dot{\mathbf{r}} = \mathbf{v}$, $\dot{\mathbf{p}} = \mathbf{F}$

If $\frac{\partial f}{\partial t} = 0$ and $f = f_0 + f'$ and $f' \ll f_0$, then$f' \approx -\tau(\dot{\mathbf{r}} \cdot \boldsymbol{\nabla} f_0 + \dot{\mathbf{p}} \cdot \boldsymbol{\nabla}_{\mathbf{p}} f_0)$