

FYSA241 osa A

Koe pe 16.6.2017. Kesto 4 tuntia. Kaavakokoelma lopussa.

Exam Friday, June 16th, 2017. Duration: 4 hours. Questions in English and a collection of formulae at the end of the sheet

1. Kerro lyhyesti:

- (a) (5p) Yksiatomisen ideaalikaasun sisäenergia on $E = \frac{3}{2}Nk_B T$. Miksi tämä ei riitä ideaalikaasun tilayhtälön $PV = Nk_B T$ johtamiseen? Miksi E ei riipu tilavuudesta?
- (b) (2p) Miksi lämpövoimakoneen hyötysuhteella on teoreettinen yläraja eikä se ole 1, eli kaikkea kuumasta lämpövarastosta otettua lämpöä ei voi muuttaa työksi?
- (c) (3p) Luonnossa ei tapahdu mitään reversiibeliä prosessia, se on yhtä epätodellinen kuin kitkaton pinta tai massaton lanka. Silti systeemin entropian muutos lasketaan reversiibelinä prosessina alkutilasta lopputilaan, vaikka todellinen prosessi onkin irreversiibeli. Miksi tulos on oikea?

2. (9p) Osoita, että sisäenergian E ja Helmholtzin vapaan energian F välillä on relaatio

$$E = -T^2 \left(\frac{\partial(F/T)}{\partial T} \right)_{V,N} .$$

3. Landaun mallissa spinsysteemin Helmholtzin vapaa energia on magnetoituman \mathcal{M} funktiona

$$F = F_0(T) + a_2(T - T_c)\mathcal{M}^2 + a_4\mathcal{M}^4 - B\mathcal{M} .$$

- (a) (1p) Osoita, että korkeissa lämpötiloissa ($T \gg T_c$, pieni \mathcal{M}) on termodynaamisessa tasapainossa voimassa Curien laki $\mathcal{M} = cB/T$ (vakio c).
- (b) (2p) Osoita, että termodynaamisessa tasapainossa spontaani ($B = 0$) magnetoituma saa arvot

$$\begin{cases} \mathcal{M} = 0 & , \text{ jos } T > T_c \\ \mathcal{M} = \pm \sqrt{\frac{a_2}{2a_4}(T_c - T)} & , \text{ jos } T < T_c . \end{cases}$$

- (c) (4p) Jos $B = 0$, osoita, että faasitransitiossa $(\partial F/\partial T)_V$ on jatkuva, mutta $(\partial^2 F/\partial T^2)_V$ epäjatkuva. Onko faasitransitio 1. kertaluvun transiitio vai jatkuva transiitio; miksi?
- (d) (3p) Miten ominaislämpö c_V käyttäytyy faasitransitiossa? Edelleen $B = 0$.

4. (10p) Tilan i energia on E_i ja sen todennäköisyys saadaan Boltzmannin jakaumasta

$$p_i = \frac{1}{Z} e^{-\beta E_i} \quad , \quad Z = \sum_i e^{-\beta E_i} . \quad (1)$$

- (a) Johda entropia todennäköisyyden p_i avulla lausuttuna.
- (b) Jos systeemi koostuu kahdesta osasta A ja B, jotka eivät vuorovaikuta keskenään, osoita että partitiofunktio on osien partitiofunktioiden tulo, eli $Z = Z_A Z_B$.
- (c) Jos systeemin energia koostuu rotaatiosta, vibraatiosta ja translaatiosta, niin miksi partitiofunktio on jälleen tulo $Z = Z_{\text{rotaatio}} Z_{\text{vibraatio}} Z_{\text{translaatio}}$?

JATKUU SEURAAVALLA SIVULLA!

5. (10p) Värähtelyn energiatasot ovat $\epsilon_n = \hbar\omega(n + \frac{1}{2})$, missä $n = 0, 1, \dots$ ja ω on värähtelytaajuus. Johda värähtelyyn liittyvän ominaislämmön lauseke

$$c_{\text{vib}} = k_B e^{-\beta\hbar\omega} \left(\frac{\beta\hbar\omega}{1 - e^{-\beta\hbar\omega}} \right)^2 .$$



1. Explain briefly:

- (a) (5p) The internal energy of a monatomic ideal gas is $E = \frac{3}{2}Nk_B T$. Why isn't this enough for derivation of the equation of state of ideal gas, $PV = Nk_B T$? Why doesn't E depend on volume V ?
- (b) (2p) Why is there a theoretical upper limit to the efficiency of a heat engine, and it's not 1, meaning that not all heat taken from a hot reservoir cannot be converted to work?
- (c) (3p) There are no reversible processes in nature, it is as unreal as a frictionless surface or a massless string. Still, the change in the entropy of a system is calculated for a reversible process from the initial state to the final state, even though in reality the process is irreversible. Why is the result correct?

2. (9p) Show, that the internal energy E and the Helmholtz free energy F have the relation

$$E = -T^2 \left(\frac{\partial(F/T)}{\partial T} \right)_{V,N} .$$

3. In the Landau model the Helmholtz free energy of a spin system is a function of magnetization \mathcal{M}

$$F = F_0(T) + a_2(T - T_c)\mathcal{M}^2 + a_4\mathcal{M}^4 - B\mathcal{M} .$$

- (a) (1p) Show that at high temperatures ($T \gg T_c$, small \mathcal{M}) in thermodynamical equilibrium the Curie law $\mathcal{M} = cB/T$ (constant c) is valid.
- (b) (2p) Show that in thermodynamical equilibrium spontaneous ($B = 0$) magnetization has the values

$$\begin{cases} \mathcal{M} = 0 & , \text{ if } T > T_c \\ \mathcal{M} = \pm \sqrt{\frac{a_2}{2a_4}(T_c - T)} & , \text{ if } T < T_c. \end{cases}$$

- (c) (4p) At $B = 0$, show that the phase transition has continuous $(\partial F/\partial T)_V$, but discontinuous $(\partial^2 F/\partial T^2)_V$. Is the transition 1st order or continuous; why?
- (d) (3p) How does the specific heat c_V behave at the phase transition? We still have $B = 0$.

4. (10p) The energy of state i is E_i , and its probability is given by the Boltzmann distribution

$$p_i = \frac{1}{Z} e^{-\beta E_i} \quad , \quad Z = \sum_i e^{-\beta E_i} . \quad (2)$$

- (a) Write entropy in terms of p_i .
- (b) If the system is made of two parts A and B, which don't interact, show that the partition function is the product of the partition functions of the parts, $Z = Z_A Z_B$.
- (c) If the system energy comprises of rotation, vibration and translation, why is the partition function again a product, $Z = Z_{\text{rotation}} Z_{\text{vibration}} Z_{\text{translation}}$?

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5. (10p) The vibration energy levels are $\epsilon_n = \hbar\omega(n + \frac{1}{2})$, where $n = 0, 1, \dots$ and ω is the vibration frequency. Derive the heat capacity of vibrations,

$$c_{\text{vib}} = k_B e^{-\beta\hbar\omega} \left(\frac{\beta\hbar\omega}{1 - e^{-\beta\hbar\omega}} \right)^2 .$$

Mahdollisesti hyödyllisiä tietoja / potentially useful information

$$k_B = 1.3805 \times 10^{-23} \text{ J/K} \quad R = k_B N_A = 8.3143 \text{ J/molK} \quad N_A = 6.022 \times 10^{23} / \text{mol}$$

$$k_B \cdot 300 \text{ K} \approx \frac{1}{40} \text{ eV} \quad 0^\circ\text{C} = 273.15 \text{ K} \quad 1 \text{ atm} = 101.3 \text{ kPa} \quad g = 9.82 \text{ m/s}^2$$

$$dE = \delta Q + \delta W \quad dE = TdS - PdV + \mu dN$$

$$F = E - TS \quad G = E - TS + PV \quad H = E + PV$$

$$S = k_B \ln \Omega \quad \ln n! \approx n \ln n - n \quad \binom{N}{n} \equiv \frac{N!}{n!(N-n)!}$$

$$F = -k_B T \ln Z \quad \langle E \rangle = -\frac{\partial}{\partial \beta} \ln Z \quad (\text{thermodyn. } E = \text{stat. phys. } \langle E \rangle)$$

$$C_V \equiv T \left(\frac{dS}{dT} \right)_{V,N} = \left(\frac{dE}{dT} \right)_{V,N} \quad C_P \equiv T \left(\frac{dS}{dT} \right)_{P,N}$$

$$\kappa_T \equiv -\frac{1}{V} \left(\frac{dV}{dP} \right)_{T,N} \quad \kappa_S \equiv -\frac{1}{V} \left(\frac{dV}{dP} \right)_{S,N}$$

$$\left(\frac{\partial x}{\partial y} \right)_z = \left[\left(\frac{\partial y}{\partial x} \right)_z \right]^{-1} \quad \left(\frac{\partial x}{\partial y} \right)_z = \left(\frac{\partial x}{\partial w} \right)_z \left(\frac{\partial w}{\partial y} \right)_z \quad \left(\frac{\partial x}{\partial y} \right)_z \left(\frac{\partial y}{\partial z} \right)_x \left(\frac{\partial z}{\partial x} \right)_y = -1$$

$$S = -k_B \sum_{\nu} p_{\nu} \ln p_{\nu} \quad p_{\nu} = \frac{1}{Z} e^{-\beta E_{\nu}} \quad Z = \sum_{\nu} e^{-\beta E_{\nu}} \quad \beta \equiv 1/(k_B T)$$

$$PV = Nk_B T = nRT \quad E = \frac{3}{2} Nk_B T \quad \left(\frac{dP}{dT} \right)_{\text{cx}} = \frac{\Delta S}{\Delta V} = \frac{L_{1 \rightarrow 2}(T)}{T \Delta V}$$

$$\sinh x \equiv \frac{1}{2} (e^x - e^{-x}) \quad \cosh x \equiv \frac{1}{2} (e^x + e^{-x}) \quad \tanh x \equiv \frac{\sinh x}{\cosh x}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, |x| < 1 \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (a+b)^N = \sum_{n=0}^N \binom{N}{n} a^n b^{N-n}$$