QM II fall 2018

Questions for oral exam. This is the final list, with 2 questions (21 and 38) added after the draft list.

You may also answer the questions in Finnish.

Week 1: Introduction and reminder of QM I

1. What are the operators $a, a^\dagger$ for a harmonic oscillator (you don't need to remember formulas, but what do they do?)?
2. How does one solve the energy levels of a harmonic oscillator with the operators $a, a^\dagger$?
3. What are the quantum numbers characterizing the states of the hydrogen atom?
4. What is the relation between a wave function $\psi(x)$ and the state ket $|\psi\rangle$
5. What is a wave function for a particle with a definite momentum $\mathbf{p}$?
6. What is the relation between a position space wave function $\psi(\mathbf{x})$ and a momentum space one $\psi(\mathbf{p})$?
7. For a system with a Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$, how do you calculate corrections to the energies of the eigenstates in leading order perturbation theory?

Weeks 2 and 3: Time-dependent phenomena

Time dependent perturbation theory

8. What are Schrödinger, Heisenberg and interaction pictures of quantum dynamics?
9. Define a time-evolution operator $\hat{U}_S(t, t_0)$ in the Schrödinger picture and find $\hat{U}_S(t, t_0)$ in the case of a time independent Hamiltonian $\hat{H}_S$. Remember, the Schrödinger equation is $i\hbar\partial_t|\psi(t)\rangle = \hat{H}_S|\psi(t)\rangle$.
10. Express the time-dependent matrix elements of an operator $\hat{A}$ in Schrödinger and Heisenberg pictures. What can you conclude?
11. Assuming that the matrix elements of any given operator $\hat{A}$ are the same both in Schrödinger and Heisenberg pictures, find an expression for the time dependence of the operators on the Heisenberg picture.
12. Explain the Heisenberg equation of motion. Signs are not important.
13. Use the Heisenberg equation of motion to find the time dependence of the momentum $\hat{p}_H = ip_0(a^\dagger - a)$ of a harmonic oscillator with Hamiltonian $H = \hbar\omega a^\dagger a$.
14. The states in an interaction picture are defined as $|\psi(t)\rangle_I = e^{iH_0t/\hbar}|\psi(t)\rangle_S$. Find out the time dependence of the operators $\hat{A}_I$.
15. The states in an interaction picture are defined as $|\psi(t)\rangle_I = e^{iH_0t/\hbar}|\psi(t)\rangle_S$. Find out the equation of motion of those states.
16. The time evolution operator $\hat{U}_I(t, t_0)$ in the interaction picture satisfies $\hat{U}_I(t_0, t_0) = \hat{1}$ and

$$i\hbar\frac{\partial}{\partial t}\hat{U}_I(t, t_0) = \hat{V}_I(t)\hat{U}_I(t, t_0) \tag{1}$$

Use these results to pose a series expansion for $\hat{U}_I(t, t_0)$ in terms of $V_I(t)$.

17. The time-evolution operator $\hat{U}_I(t,t_0)$ in the interaction picture can be expressed as a time-ordered exponential,
$$\hat{U}_I(t,t_0) = T[e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt' \hat{V}_I(t')}], \tag{2}$$
where $\hat{V}_I(t) = e^{\frac{i}{\hbar}\hat{H}_0 t} V_S(t) e^{-\frac{i}{\hbar}\hat{H}_0 t}$. The transition probability between (different) energy eigenstates of H_0, $|\varphi_{i/f}\rangle$ is $P_{fi} = |\langle\varphi_f|\hat{U}_I(t,t_0)|\varphi_i\rangle|^2$. Use these two results to find a lowest-order expression for the transition probability in terms of the matrix elements of $\hat{V}$.

18. The transition rate W_{fi} is connected to the transition matrix element via the Fermi Golden rule
$$W_{fi} = \frac{2\pi}{\hbar}\delta(E_f - E_i)|T_{fi}|^2 \tag{3}$$
For what type of perturbing potential $\hat{V}_S$ is this valid? What should we assume about the times of the initial and final states?

19. The transition rate W_{fi} is connected to the transition matrix element via the Fermi Golden rule
$$W_{fi} = \frac{2\pi}{\hbar}\delta(E_f - E_i)|T_{fi}|^2 \tag{4}$$
and T_{fi} satisfies the perturbation series
$$T_{fi} = \sum_{n=0}^{\infty}\langle\varphi_f|\hat{V}_S\left(\frac{1}{E_i - \hat{H}_0 + i\epsilon}\hat{V}_S\right)^n|\varphi_i\rangle. \tag{5}$$
Find out the fourth-order (in $\hat{V}_S$) contribution to the transition rate.

20. The transition rate W_{fi} is connected to the transition matrix element via the Fermi Golden rule
$$W_{fi} = \frac{2\pi}{\hbar}\delta(E_f - E_i)|T_{fi}|^2 \tag{6}$$
For $E_i = E_f$ this seems to diverge. How would you get a finite result?

21. Write down an expression for T_{fi} that you need to calculate the ionization rate of atomic hydrogen under a perturbation potential $V(t) = V_0\cos(\omega t))$ with Fermi's golden rule.

22. What is the Fermi golden rule for a harmonic perturbation of the form $\hat{V}_S = \hat{K}e^{-i\omega t} + \hat{K}^\dagger e^{i\omega t}$? (Prefactors are not essential)

23. How do you change the sum over final states for particles in a 3D box of size $V = L^3$ to an integral, i.e. how do you calculate the density of states $\rho(\mathbf{k})$ in
$$\sum_{\mathbf{k}} = \int \mathrm{d}k\rho(k)? \tag{7}$$

24. When you calculate the ionization rate into free electrons for a hydrogen atom under a perturbation potential using Fermi's golden rule, the normalization of plane wave states depends on the size of your box L^3. However, in the physical result this dependence cancels, how?

Interaction with a classical electromagnetic field

25. What are kinetic and canonical momenta?

26. How is the coupling of a charged particle with an electromagnetic field incorporated in the Schrödinger equation ("minimal substitution")?

27. The Pauli Hamiltonian, describing a spin-1/2 particle coupled with an electromagnetic field, reads
$$H_{s=1/2} = \frac{1}{2m}(\hat{p} - q\vec{A}(\hat{x},t))^2\hat{1}_{2\times 2} - \frac{q\hbar}{2m}\vec{\sigma}\cdot\vec{B}(\hat{x},t) + q\varphi(\vec{x},t)\hat{1}_{2\times 2} \tag{8}$$
Show that for a position independent magnetic field $\vec{B}(t) = \nabla\times\vec{A}(t)$, the spin and spatial dependence of the wavefunction are separated.

28. Explain the gauge invariance of a classical electromagnetic field coupled with a quantum-mechanical charged particle. Small typos are not relevant.

29. Show that the electric and magnetic fields are invariant under the gauge transformation $\vec{A} \mapsto \vec{A} + \nabla f$, $\varphi \mapsto \varphi - \partial_t f$.

30. The gauge invariance of the electromagnetic field can be expressed via $\vec{A} \mapsto \vec{A} + \nabla f$, $\varphi \mapsto \varphi - \partial_t f$. What should one do with the wave function?

31. What are Landau levels?

32. How do you derive the energy states for a particle in a constant magnetic field from the Pauli equation (8)?

33. Consider electrons passing a double-slit interferometer that contains a solenoid with a magnetic field $\vec{B} = \nabla \times \vec{A}$ that vanishes within the "classical" path of the electrons. Explain why the interference pattern formed in a film placed behind the double slit is a periodic function of the magnetic flux Φ, and find the corresponding flux period. You can use the fact that the wave function for the electrons satisfies $\psi_{\vec{A}} = e^{\frac{iq}{\hbar}\int_C d\vec{x}'\cdot\vec{A}}\psi_{\vec{A}=0}$. Remember also that $\oint d\vec{x} \cdot \vec{A} = \int d\vec{S}(\nabla \times \vec{A})$.

34. Explain the Aharonov-Bohm effect.

35. Consider an atom in a classical electromagnetic field, $\vec{A} = A_0\vec{\epsilon}\cos(\vec{k}\cdot\vec{x} - \omega t)$. How does one usually approximate this potential for atomic transitions?

36. How does the Pauli Hamiltonian

$$H_{s=1/2} = \frac{1}{2m}(\hat{p} - q\vec{A}(\hat{x},t))^2\hat{1}_{2\times 2} - \frac{q\hbar}{2m}\vec{\sigma}\cdot\vec{B}(\hat{x},t) + q\varphi(\vec{x},t)\hat{1}_{2\times 2} \tag{9}$$

simplify in the radiation gauge?

37. Starting from the Hamiltonian for a spin-1/2 particle in an electromagnetic field

$$H_{s=1/2} = \frac{1}{2m}(\hat{p} - q\vec{A}(\hat{x},t))^2\hat{1}_{2\times 2} - \frac{q\hbar}{2m}\vec{\sigma}\cdot\vec{B}(\hat{x},t) + q\varphi(\vec{x},t)\hat{1}_{2\times 2}, \tag{10}$$

how does one get the dipole approximation for atomic transitions?

38. Write down an expression for T_{fi} that you need in order to calculate the ionization rate of atomic hydrogen in ultraviolet light with Fermi's golden rule.

Adiabatic approximation

39. Explain the essence of the adiabatic approximation (in the non-degenerate case). What is the requirement for the validity of the approximation?

40. Consider changing N parameters $\{R_i\}_{i=1}^N$ of a system slowly. Within adiabatic approximation, the time-dependent state (starting in the nth eigenstate, assuming lack of degeneracies) of the system can be written as

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\int_0^t E_n(t')dt' - \int_0^t dt'\langle\psi_n(t')|\dot{\psi}_n(t')\rangle}|\psi_n(t)\rangle \tag{11}$$

Interpret the different phases and explain their qualitative difference regarding the speed of changing the parameters.

41. Consider changing N parameters $\{R_i\}_{i=1}^N$ of a system slowly. Within adiabatic approximation, the time-dependent state (starting in the nth eigenstate, assuming lack of degeneracies) of the system can be written as

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\int_0^t E_n(t')dt' - \int_0^t dt'\langle\psi_n(t')|\dot{\psi}_n(t')\rangle}|\psi_n(t)\rangle \tag{12}$$

Consider a periodic change of parameters and derive a formula for the Berry phase (independent of the time t).

Weeks 4 and 5: scattering theory

42. What is a scattering cross section? What does it measure?

43. What is the differential scattering cross section $\frac{d\sigma}{d\Omega}$?

44. The current density in QM can be defined in terms of the wave function via

$$\mathbf{j} = \frac{\hbar}{m}\text{Im}[\Psi^* \nabla \Psi].$$

Find out the current density of a plane wave $\Psi(r) = e^{i\mathbf{k}\cdot\mathbf{r}}/\sqrt{2\pi}$.

45. Quantum scattering problem (in 3 dimensions). For a wave function satisfying

$$(\nabla^2 + k^2)\Psi(\mathbf{r}) = \frac{2\mu}{\hbar^2}V(\mathbf{r})\Psi(\mathbf{r}) \tag{13}$$

with the boundary condition

$$\Psi(\mathbf{r}) \xrightarrow{\mathbf{r}\to\infty} \frac{1}{(2\pi)^{3/2}}\left[e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{e^{ikr}}{r}f_k(\theta,\varphi)\right] \tag{14}$$

Interpret the different parts of the boundary condition (14). What is the relation to the (differential) cross section?

46. Consider the (normalized) Schrödinger equation, Eq. (13), and use the solution $\phi_{\mathbf{k}}(r)$ of the corresponding homogeneous equation and the Green's function satisfying

$$(\nabla^2 + k^2)G_{\mathbf{k}}(\mathbf{r},\mathbf{r}') = \delta^{(3)}(\mathbf{r}-\mathbf{r}') \tag{15}$$

to pose Eq. (13) in an integral equation form.

47. Sketch the Lippmann-Schwinger integral equation corresponding to the quantum-mechanical 3D elastic scattering problem. Which part of the equation represents the fact that you are solving a scattering problem?

48. Calculate the Green's function solving

$$(\nabla^2 + k^2)G_{\mathbf{k}}(\mathbf{r},\mathbf{r}') = \delta^{(3)}(\mathbf{r}-\mathbf{r}') \tag{16}$$

needed in the 3D scattering problem. Hint: use $k \mapsto k + i\varepsilon$. You can start from the form $G_{\mathbf{k}}(r) = I_k(r)/(4\pi^2 ri)$ with

$$I_k(r) = \int_{-\infty}^{\infty} \mathrm{d}q \frac{qe^{iqr}}{(k+i\epsilon)^2 - q^2}.$$

Why does the choice $k + i\varepsilon$ correspond to the physical situation of scattering?

49. The Lippmann-Schwinger integral equation for the scattering problem is of the form

$$\Psi_{\mathbf{k}}(\mathbf{r}) = \phi_{\mathbf{k}}(\mathbf{r}) + \int \mathrm{d}^3\mathbf{r}' G_{\mathbf{k}}(\mathbf{r},\mathbf{r}')U(\mathbf{r}')\Psi_{\mathbf{k}}(\mathbf{r}') \tag{17}$$

with $G_{\mathbf{k}}(\mathbf{r},\mathbf{r}') = G_{\mathbf{k}}(\mathbf{r}-\mathbf{r}') = -\frac{1}{4\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$. Using this result and the boundary condition (14), express the scattering amplitude in terms of the total wave function.

50. Sketch the Born series solution to the the Lippmann-Schwinger equation

$$\Psi_{\mathbf{k}}(\mathbf{r}) = \phi_{\mathbf{k}}(\mathbf{r}) + \int \mathrm{d}^3\mathbf{r}' G_{\mathbf{k}}(\mathbf{r},\mathbf{r}')U(\mathbf{r}')\Psi_{\mathbf{k}}(\mathbf{r}'). \tag{18}$$

51. Using the Lippmann-Schwinger equation

$$\Psi_{\mathbf{k}}(\mathbf{r}) = \phi_{\mathbf{k}}(\mathbf{r}) + \int \mathrm{d}^3\mathbf{r}' G_{\mathbf{k}}(\mathbf{r}, \mathbf{r}') U(\mathbf{r}') \Psi_{\mathbf{k}}(\mathbf{r}') \tag{19}$$

extract the Born series for the scattering amplitude. Give also a graphical interpretation of the different terms in this series.

52. Explain Born series and Born approximation.

53. In the first Born approximation the scattering amplitude is given by

$$f_B(\theta, \varphi) = -\frac{1}{4\pi} \int \mathrm{d}^3\mathbf{r} e^{i(\mathbf{k}_i - \mathbf{k}_f)\cdot\mathbf{r}} U(\mathbf{r}). \tag{20}$$

Consider a spherically symmetric potential, $U(\mathbf{r}) = U(|\mathbf{r}|)$. How does $f_B(\theta, \varphi)$ depend on φ? What is the symmetry of $f_B(\theta, \varphi)$ on inversion of θ to $-\theta$?

54. What is the *optical theorem*?

55. What is the scattering length?

56. Calculate the scattering length in the Born approximation

$$f_B(\theta, \varphi) = -\frac{1}{4\pi} \int \mathrm{d}^3\mathbf{r} e^{i(\mathbf{k}_i - \mathbf{k}_f)\cdot\mathbf{r}} U(\mathbf{r}). \tag{21}$$

57. From the decomposition

$$f(\theta) = \frac{1}{k} \sum_{\ell=0} (2\ell + 1) e^{i\delta_\ell} \sin \delta_\ell P_\ell(\cos\theta) \tag{22}$$

calculate the total cross section.

58. Based on

$$\sigma = \frac{4\pi}{k} \sum_{\ell=0} (2\ell + 1) \sin^2 \delta_\ell \tag{23}$$

explain the partial wave unitarity bound .

59. What is a sufficient condition that the scattering amplitude can be expressed as

$$f(\theta) = \frac{1}{k} \sum_{\ell=0} (2\ell + 1) f_\ell P_\ell(\cos\theta). \tag{24}$$

60. If you know the potential that one scatters off, describe how you calculate δ_ℓ in

$$f(\theta) = \frac{1}{k} \sum_{\ell=0} (2\ell + 1) e^{i\delta_\ell} \sin \delta_\ell P_\ell(\cos\theta) \tag{25}$$

61. For

$$f(\theta) = \frac{1}{k} \sum_{\ell=0} (2\ell + 1) e^{i\delta_\ell} \sin \delta_\ell P_\ell(\cos\theta) \tag{26}$$

when is there a *resonance*?

62. Using

$$f(\theta) = \frac{1}{k} \sum_{\ell=0} (2\ell + 1) e^{i\delta_\ell} \sin \delta_\ell P_\ell(\cos\theta) \tag{27}$$

assume that one partial wave ℓ is much larger than the others, and that for scattering energies E close to some value E_0 we have $\cot \delta_l \approx c(E - E_0)$. Calculate how the cross section depends on $E - E_0$.

Weeks 6 and 7: rotations

The CG table at the end is at your free disposal.

63. What is the algebra of SU(2) generators?
64. What is the Wigner-Eckart theorem?
65. What is a spherical tensor?
66. What is the spherical basis for a 3-component vector V^i?
67. Write down an expression for the transformation matrix for a two-component vector describing a spin-$\frac{1}{2}$-state under a rotation of angle θ around the axis $\mathbf{n}$.
68. What are the Euler angles?
69. In the CG table you will find $d^{1/2}_{1/2,-1/2}(\theta) = -\sin\frac{\theta}{2}$. What is it?
70. What is $(J_x + iJ_y)|j\, m = j\rangle$?
71. Calculate the coefficient C_{jm} in $(J_x + iJ_y)|j\, m\rangle = C_{jm}|j\, m + 1\rangle$
72. Express $|j_1 = \frac{1}{2},\, j_2 = \frac{1}{2},\, J = 1,\, m = 0\rangle$ in terms of $|j_1 = \frac{1}{2},\, m_1\rangle \otimes |j_2 = \frac{1}{2},\, m_2\rangle$.
73. Express $|j_1 = \frac{1}{2},\, j_2 = \frac{1}{2},\, J = 0,\, m = 0\rangle$ in terms of $|j_1 = \frac{1}{2},\, m_1\rangle \otimes |j_2 = \frac{1}{2},\, m_2\rangle$.
74. Express $|j_1 = 1,\, j_2 = \frac{1}{2},\, J = \frac{1}{2},\, m = \frac{1}{2}\rangle$ in terms of $|j_1 = 1,\, m_1\rangle \otimes |j_2 = \frac{1}{2},\, m_2\rangle$.
75. Show that a matrix that rotates 3-dimensional vectors $\mathbf{r}$ is an SO(3) matrix.
76. What are a, b, c in $11 \otimes 3 = a \oplus b \oplus c$? (Numbers are $2j + 1$).
77. What is $\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = \langle j_1 j_2 j_1 j_2 | j_1 j_2 (j_1 + j_2)(j_1 + j_2)\rangle$?
78. Calculate $\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = \langle 1\, 1\, m_1 m_2 | 1\, 1\, 1\, 1\rangle$.
79. What are the selection rules of lowest-order transitions caused by a radiation field on a charged particle in a rotationally invariant potential?
80. Show that $\mathbf{r}$ is an irreducible spherical tensor.
81. When the operator xz is written as a sum of irreducible spherical tensors $T^{(k)}_q$, what values of k and q appear?
82. The Wigner-Eckart theorem states that

$$\langle \alpha_1 j_1 m_1 | T^{(k)}_q | \alpha_2 j_2 m_2\rangle = \frac{1}{\sqrt{2j_1 + 1}} \langle j_2 k m_2 q | j_2 k j_1 m_1\rangle (\alpha_1 j_1 | T^{(k)} | \alpha_2 j_2) \tag{28}$$

    What do the symbols in this equation mean?

83. The Wigner-Eckart theorem states that

$$\langle \alpha_1 j_1 m_1 | T^{(k)}_q | \alpha_2 j_2 m_2\rangle = \frac{1}{\sqrt{2j_1 + 1}} \langle j_2 k m_2 q | j_2 k j_1 m_1\rangle (\alpha_1 j_1 | T^{(k)} | \alpha_2 j_2). \tag{29}$$

    How can you use it to express $\langle \ell' m' | x | \ell m\rangle$ in terms of $\langle \ell' m' | z | \ell m\rangle$?

84. The Wigner-Eckart theorem states that

$$\langle \alpha_1 j_1 m_1 | T^{(k)}_q | \alpha_2 j_2 m_2\rangle = \frac{1}{\sqrt{2j_1 + 1}} \langle j_2 k m_2 q | j_2 k j_1 m_1\rangle (\alpha_1 j_1 | T^{(k)} | \alpha_2 j_2). \tag{30}$$

    Why does it lead to the selection rule $m - m' = 0, \pm 1$ for the dipole transition matrix element $\langle \ell' m' | \mathbf{r} | \ell m\rangle$?

85. The Wigner-Eckart theorem states that

$$\langle \alpha_1 j_1 m_1 | T_q^{(k)} | \alpha_2 j_2 m_2 \rangle = \frac{1}{\sqrt{2j_1+1}} \langle j_2 k m_2 q | j_2 k j_1 m_1 \rangle (\alpha_1 j_1 | T^{(k)} | \alpha_2 j_2). \tag{31}$$

Why does it lead to the selection rule $\ell - \ell' = 0, \pm 1$ for the dipole transition matrix element $\langle \ell' m' | \mathbf{r} | \ell m \rangle$?

86. How can you obtain the selection rule $\ell \neq \ell'$ for the dipole transition matrix element $\langle \ell' m' | \mathbf{r} | \ell m \rangle$?

87. Show that the Gaunt formula

$$\int \mathrm{d}\Omega (Y_\ell^m)^* Y_{\ell_1}^{m_1} Y_{\ell_2}^{m_2} = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}} \langle \ell_1 \ell_2 m_1 m_2 | \ell_1 \ell_2 \ell m \rangle \langle \ell_1 \ell_2 0 0 | \ell_1 \ell_2 \ell 0 \rangle \tag{32}$$

leads to the selection rule $|\ell - \ell'| \leq 1$ for dipole transitions.

44. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

m_1	m_2	J / M	J / M	...
m_1	m_2	Coefficients		
.	.			

$1/2 \times 1/2$

m_1	m_2	$J=1$, $M=+1$
+1/2	+1/2	1

m_1	m_2	$J=1$, $M=0$	$J=0$, $M=0$
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	$J=1$, $M=-1$
−1/2	−1/2	1

$1 \times 1/2$

m_1	m_2	$J=3/2$, $M=+3/2$
+1	+1/2	1

m_1	m_2	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	$J=3/2$, $M=-3/2$
−1	−1/2	1

2×1

m_1	m_2	$J=3$, $M=+3$
+2	+1	1

m_1	m_2	$J=3$, $M=+2$	$J=2$, $M=+2$
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	$J=3$, $M=-2$	$J=2$, $M=-2$
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	$J=3$, $M=-3$
−2	−1	1

1×1

m_1	m_2	$J=2$, $M=+2$
+1	+1	1

m_1	m_2	$J=2$, $M=+1$	$J=1$, $M=+1$
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	$J=2$, $M=-1$	$J=1$, $M=-1$
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	$J=2$, $M=-2$
−1	−1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$2 \times 1/2$

m_1	m_2	$J=5/2$, $M=+5/2$
+2	+1/2	1

m_1	m_2	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	$J=5/2$, $M=-5/2$
−2	−1/2	1

$3/2 \times 1/2$

m_1	m_2	$J=2$, $M=+2$
+3/2	+1/2	1

m_1	m_2	$J=2$, $M=+1$	$J=1$, $M=+1$
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	$J=2$, $M=0$	$J=1$, $M=0$
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	$J=2$, $M=-1$	$J=1$, $M=-1$
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	$J=2$, $M=-2$
−3/2	−1/2	1

$3/2 \times 1$

m_1	m_2	$J=5/2$, $M=+5/2$
+3/2	+1	1

m_1	m_2	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	$J=5/2$, $M=-5/2$
−3/2	−1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

$3/2 \times 3/2$

m_1	m_2	$J=3$, $M=+3$
+3/2	+3/2	1

m_1	m_2	$J=3$, $M=+2$	$J=2$, $M=+2$
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	$J=3$, $M=-2$	$J=2$, $M=-2$
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	$J=3$, $M=-3$
−3/2	−3/2	1

$2 \times 3/2$

m_1	m_2	$J=7/2$, $M=+7/2$
+2	+3/2	1

m_1	m_2	$J=7/2$, $M=+5/2$	$J=5/2$, $M=+5/2$
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	$J=7/2$, $M=+3/2$	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	$J=7/2$, $M=+1/2$	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	$J=7/2$, $M=-1/2$	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	$J=7/2$, $M=-3/2$	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	$J=7/2$, $M=-5/2$	$J=5/2$, $M=-5/2$
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	$J=7/2$, $M=-7/2$
−2	−3/2	1

2×2

m_1	m_2	$J=4$, $M=+4$
+2	+2	1

m_1	m_2	$J=4$, $M=+3$	$J=3$, $M=+3$
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	$J=4$, $M=+2$	$J=3$, $M=+2$	$J=2$, $M=+2$
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	$J=4$, $M=+1$	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	$J=4$, $M=0$	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	$J=4$, $M=-1$	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	$J=4$, $M=-2$	$J=3$, $M=-2$	$J=2$, $M=-2$
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	$J=4$, $M=-3$	$J=3$, $M=-3$
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	$J=4$, $M=-4$
−2	−2	1

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 44.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).