

QM II fall 2018

Exercise 4, discussed in the tutorial session Wed Oct 3rd, return by Fri Oct 5th at 15h.

- Derive the Green function for the Helmholtz equation that was needed with the Lippmann-Schwinger equation. To do this solve the differential equation

$$[\nabla_{\mathbf{x}}^2 + k^2] G(\mathbf{x}, \mathbf{x}') = \delta^{(3)}(\mathbf{x} - \mathbf{x}') \quad (1)$$

by writing $G(\mathbf{x}, \mathbf{x}')$ as a Fourier-transform, solving the equation and then Fourier-transforming back using contour integration. To have a solution for the equation you need to regularize a division by zero, show how doing this with $k^2 \rightarrow (k + i\varepsilon)^2$ and $k^2 \rightarrow (k - i\varepsilon)^2$ [you can take $k > 0$] lead to different results. Use the limit $k = 0$ and the differential version of the Gauss law $\nabla \cdot \mathbf{E} = \rho(\mathbf{x})/\varepsilon_0$ to get an integral expression for the Coulomb potential of an electric charge distribution $\rho(\mathbf{x})$.

- Compute the scattering amplitude $f(\theta, \varphi)$, the differential cross-section $\frac{d\sigma}{d\Omega}$, and the total cross-section σ in the Born approximation for the following potentials:

- Yukawa potential:

$$V_{\text{Yukawa}}(r) = V_0 \frac{e^{-\kappa r}}{r} \quad (2)$$

- Gaussian potential:

$$V_{\text{Gauss}}(r) = V_0 e^{-r^2/\lambda^2} \quad (3)$$

- Rederive the Born approximation for the differential cross section $d\sigma/d\Omega$ from Fermi's golden rule for the transition rate:

$$w_{i \rightarrow f} = \frac{2\pi}{\hbar} \left| \langle f | \hat{V} | i \rangle \right|^2 \delta(E_f - E_i). \quad (4)$$

The calculation is not complicated. Hints to get started: you need to

- Consider the initial and final states $|i\rangle$ and $|f\rangle$ as box-normalized plane waves,
 - Relate the transition rate to the cross section using the particle flux for the plane wave.
 - Integrate over the energy, i.e. the absolute value of the outgoing momentum, since the cross section is differential only in the angle.
- Slightly modifying the derivation based on Fermi's Golden rule in problem 3 you can easily generalize the Born approximation to a harmonic time dependent potential. Calculate, with this generalized Born approximation, the differential cross section for scattering off the potential

$$V(r, t) = 2 \cos(\omega t) V(\mathbf{r}) \quad (5)$$

- Consider an elastic electron-atom scattering. In a simple model, the charge distribution of an atom with atomic number Z can be described by

$$\rho(r) = e \left(Z \delta^{(3)}(\mathbf{r}) - \rho_e(r) \right) \quad (6)$$

where $\rho_e(r)$ is a spherically symmetric electron density normalized so that there are Z electrons in total.

- Show that the differential cross-section in the Born approximation takes a form ($\hbar = 1$)

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2 [Z - F(q)]}{16\pi\varepsilon_0 E \sin^2(\theta/2)} \right)^2, \quad q = \sqrt{8m_e E} \sin \theta/2, \quad (7)$$

where the form factor $F(q)$ is the Fourier transform of the electron density, m_e is the electron mass, and E is the collision energy. Hint: see problem 1 for the potential, although it is better to stay in momentum space.

- Show that $F(0) = Z$

- Assume that the electron density is

$$\rho_e(r) = \frac{Ze^{-2r/a}}{\pi a^3}. \quad (8)$$

Compute the form factor $F(q)$. What is the energy dependence of the differential cross section in the small and large scattering angle limits?