QM IIa fall 2018, week 5

- Reading assignment for Monday Oct 8th: more features of scattering theory
  - Optical theorem
  - Partial wave amplitudes, the phase shift
  - Eikonal approximation, resonance scattering
  - Tuominen: Secs 4.6-4.12 or Sakurai (old edition): Secs 7.3-7.8 (see also rest of Chapter 7) or Sakurai and Napolitano Secs 6.4-6.7 (see also the rest of Chapter 6) or Eskola, p. 45-50 and 74-99 or Bransden Secs 13.3-13.4 and 13.7-13.8.

- Preliminary exercises, do these before the class of Mon Oct 8th and be prepared to present your solutions in class.

  1. Consider a wave scattered from a potential (that is localized close to the origin $\mathbf{r} = \mathbf{0}$):
  $$\psi(\mathbf{r}) = e^{ikz} + f(\theta)\frac{e^{ikr}}{r}. \tag{1}$$
  Now let us look at this wave on a disk A perpendicular to the z-axis, behind the target at $z \gg 1/k$; $\mathbf{r} = (x, y, z) = (\mathbf{x}_\perp, z) \in A$. Now we want to calculate the integral of $|\psi(\mathbf{r})|^2$ over A. Write an expression for this integral, expanding to leading nontrivial order in $\mathbf{x}_\perp^2$ in the limit $z \gg x, y$. Now use the result
  $$\int_A \mathrm{d}^2\mathbf{x}_\perp e^{-\mathbf{x}_\perp^2/(2a^2 i)} \approx \int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{x}_\perp e^{-\mathbf{x}_\perp^2/(2a^2 i)} = 2\pi a^2 i, \tag{2}$$
  to show that
  $$\int_A \mathrm{d}^2\mathbf{x}_\perp |\psi(\mathbf{r})|^2 = A - \frac{4\pi}{k}\mathrm{Im} f(\theta = 0). \tag{3}$$
  What does this tell you about what happened to density of particles originally coming in from $z = -\infty$? Interpret as the optical theorem. The approximation (2) strictly speaking requires the limit $z \to \infty$, $A \to \infty$ such that the opening angle $\theta^2 \sim A/z^2 \ll 1$ stays fixed. Draw a figure to illustrate the coordinates/angles!

  2. Write down a decomposition of the scattering amplitude $f(\theta)$ in terms of Legendre polynomials, i.e. partial waves, with coefficients expressed in terms of the *phase shifts*. If you know the potential, how would you calculate the phase shifts? Square the amplitude and integrate over θ to get an expression for the total cross section. For this you need to find somewhere the orthogonality property and the value at 1 of the Legendre polynomials. Show that you have recovered the optical theorem. What is the "partial wave unitarity bound"? What does it mean to have an "S-wave scattering"? What happened to the other angle φ?

  3. (a) Assume that one partial wave ℓ is much larger than the others, and that for scattering energies E close to some value E_0 we have $\cot\delta_l \approx 2(E - E_0)/\Gamma$. Calculate how the cross section depends on $E - E_0$ and show that it has a local maximum at $E = E_0$.

     (b) What is the *scattering length*?