QM IIA spring 2020

Exercise 5, discussed in the tutorial session Thu Feb 6th, return by Fri Feb 7th at 21h and again self-graded by Mon Feb 10th by 14h.

1. (Double points for this problem, see Tuominen) Consider the 3d "hard sphere" potential

$$V(r) = \begin{cases} \infty, & r < a \\ 0, & r > a \end{cases} \tag{1}$$

   (a) Can you calculate the cross section with the Born approximation?

   (b) Using the ansatz $\psi(\mathbf{r}) = R_\ell(kr) Y_\ell^m(\theta, \varphi)$ in the Schrödinger equation $\hbar = 1$

$$\left(-\nabla^2 + 2mV(r)\right)\psi = k^2\psi \tag{2}$$

   show that the radial equation is a spherical Bessel equation. What is the boundary condition?

   (c) Write down the general solution ψ that satisfies the boundary condition and is independent of the azimuthal angle φ in terms of the spherical Bessel or Hankel functions and Legendre polynomials. Using their asymptotical forms and the relation

$$e^{ikr\cos\theta} \underset{kr\gg 1}{\approx} \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)\left(\frac{e^{ikr}}{r} + (-1)^{\ell+1}\frac{e^{-ikr}}{r}\right)P_\ell(\cos\theta) \tag{3}$$

   choose the remaining constants so that ψ reduces to

$$\psi(\mathbf{r}) = C\left[e^{ikz} + \left(\sum_{\ell=0}^{\infty}\frac{(2\ell+1)}{k} f_\ell P_\ell(\cos\theta)\right)\frac{e^{ikr}}{r}\right] \tag{4}$$

   in the limit $r \to \infty$. What are f_ℓ in terms of k, a?

   (d) Determine the phase shifts δ_ℓ and the total cross section.

   (e) What is the scattering length, including its sign? What is the total cross section in the limit $k \to 0$? Compare to the classical cross section for a hard sphere (the transverse area).

2. (Double points also for this problem, see Tuominen) Now consider a softer attractive sphere potential

$$2mV(r) = \begin{cases} -q^2, & r < a \\ 0, & r > a \end{cases} \tag{5}$$

   (a) Write down the radial equation for the S-wave $\ell = 0$. Find out the number of bound states as a function of qa; what are the values $(qa)_c$ when a new bound state appears. What is the energy of this new bound state exactly at this threshold?

   (b) Calculate the S-wave phase shift δ_0. What is the scattering length? Show that in the limit when the potential is weak, the scattering length is the same as given by the Born approximation, including the sign! What happens to the scattering length when $qa = (qa)_c$?

   [Tuominen 4.9.1]

3. Derive the relation (3) You could derive the full result in terms of $j_\ell(kr)$ and use the asymptotics for $j_\ell(kr)$ (Tuominen gives hints for one way to do this). It is however easier to directly take the large r limit as early as possible. In both cases you need the Legendre polynomial orthogonality relation. In the latter way you can partially integrate the integral

$$\int_{-1}^{1} \mathrm{d}u P_\ell(u) e^{ikru}, \tag{6}$$

   and show that each new partial integration generates terms that are suppressed by additional powers of $1/(kr)$ and use the known values of $P_\ell(\pm 1)$.