QM IIA spring 2020

Exercise 4, discussed in the tutorial session Thursday Jan 30th return by Fri Jan 31st at 21h and again self-graded by Mon Feb 3rd by 14h.

1. Derive the Green function for the Helmholtz equation that was needed with the Lippmann-Schwinger equation. To do this solve the differential equation

$$\left[\nabla^2_{\mathbf{x}} + k^2\right] G(\mathbf{x}, \mathbf{x}') = \delta^{(3)}(\mathbf{x} - \mathbf{x}') \tag{1}$$

by writing $G(\mathbf{x}, \mathbf{x}')$ as a Fourier-transform, solving the equation and then Fourier-transforming back using contour integration. To have a solution for the equation you need to regularize a division by zero, show how doing this with $k^2 \to (k + i\varepsilon)^2$ and $k^2 \to (k - i\varepsilon)^2$ [you can take $k > 0$] lead to different results. Use the limit $k = 0$ and the differential version of the Gauss law $\nabla \cdot \mathbf{E} = \rho(\mathbf{x})/\varepsilon_0$ to get an integral expression for the Coulomb potential of an electric charge distribution $\rho(\mathbf{x})$.

2. Compute the scattering amplitude $f(\theta, \varphi)$, the differential cross-section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$, and the total cross-section σ in the Born approximation for the following potentials:

   (a) Yukawa potential:

$$V_{\text{Yukawa}}(r) = V_0 \frac{e^{-\kappa r}}{r} \tag{2}$$

   (b) Gaussian potential:

$$V_{\text{Gauss}}(r) = V_0 e^{-r^2/\lambda^2} \tag{3}$$

3. Rederive the Born approximation for the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$ from Fermi's golden rule for the transition rate:

$$w_{i\to f} = \frac{2\pi}{\hbar} \left|\langle f|\hat{V}|i\rangle\right|^2 \delta(E_f - E_i). \tag{4}$$

The calculation is not complicated. Hints to get started: you need to

- Consider the initial and final states $|i\rangle$ and $|f\rangle$ as box-normalized plane waves,
- Relate the transition rate to the cross section using the particle flux for the plane wave.
- Integrate over the energy, i.e. the absolute value of the outgoing momentum, since the cross section is differential only in the angle.

4. Consider an elastic electron-atom scattering. In a simple model, the charge distribution of an atom with atomic number Z can be described by

$$\rho(r) = e\left(Z\delta^{(3)}(\mathbf{r}) - \rho_e(r)\right) \tag{5}$$

where $\rho_e(r)$ is a sperically symmetric electron density normalized so that there are Z electrons in total. Show that the differential cross-section in the Born approximation takes a form ($\hbar = 1$)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{e^2[Z - F(q)]}{16\pi\varepsilon_0 E \sin^2(\theta/2)}\right)^2, \quad q = \sqrt{8m_e E}\sin\theta/2, \tag{6}$$

where the form factor $F(q)$ is the Fourier transform of the electron density, m_e is the electron mass, and E is the collision energy. Hint: use problem 1 to get the potential from the charge density, although it is better to stay in momentum space. Why is $F(0) = Z$?

5. Continuation of the previous problem: assume that the electron density is

$$\rho_e(r) = \frac{Ze^{-2r/a}}{\pi a^3}. \tag{7}$$

Compute the form factor $F(q)$. What is the energy dependence of the differential cross section in the small and large scattering angle limits? Interpret: what does the and electron scatter off in different regimes; relate to the famous Rutherford experiment.