QM II fall 2018

Exercise 2, discussed in the tutorial session Thu Jan 16th, return by Fri Jan 17th at 21h.

1. Let us consider a two-level system with energy eigenstates $\{|\phi_1\rangle, |\phi_2\rangle\}$, which is perturbed by a potential oscillating with angular frequency ω

$$V(t) = \begin{pmatrix} 0 & V_{12} \\ V_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \gamma e^{i\omega t} \\ \gamma e^{-i\omega t} & 0 \end{pmatrix},$$

where $\gamma \in \mathbb{R}$. Let the system be initially in state $|\Psi(t=0)\rangle = |\phi_1\rangle = (1 \quad 0)^T$. Solve the time evolution of the system exactly and show that the probability of finding the system at later times t in the state $|\phi_2\rangle$ is given by the **Rabi formula**

$$P_2 = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega - \delta\omega)^2/4} \sin^2\left(\sqrt{\gamma^2/\hbar^2 + \frac{(\omega - \delta\omega)^2}{4}} t\right),$$

where $\delta\omega \equiv (E_2 - E_1)/\hbar$.

2. Compute the corresponding transition probabilities using first order time-dependent perturbation theory and compare the results by expanding the exact result in powers of γ, when $\gamma^2 \ll \hbar^2 (\omega - \delta\omega)^2/4$. What happens at the resonance frequency $\omega = \delta\omega$?

3. Consider a harmonic oscillator whose frequency depends on time:

$$\omega(t) = \omega_0 + \delta \cos(\alpha t) \tag{1}$$

Assuming that the oscillator is in the ground state at $t_0 = 0$, use perturbation theory (i.e. assuming that $\delta \ll \omega_0$) to calculate the probability that it will be in the state $|n\rangle$ at the time $t > 0$.

4. Rederive Fermi's golden rule for a perturbation with a harmonic time dependence by considering, instead of a time-independent perturbation turned on at $t = 0$, a perturbation with a harmonic time dependence that is turned on slowly $\hat{V}(t) = \hat{V} e^{-i\omega t + \varepsilon t} + \hat{V}^\dagger e^{i\omega t + \varepsilon t}$, calculating $\mathrm{d}P_{i\to f}/\mathrm{d}t$ and then taking the limit $\varepsilon \to 0$. See the literature.

5. Consider a hydrogen atom in the ground state

$$\psi_{n=0,\ell=0}(\mathbf{r}) = \frac{1}{a_0^{3/2}\sqrt{\pi}} e^{-r/a_0} \tag{2}$$

with the electron subject to a perturbation potential

$$V(\mathbf{r}, t) = V_0 \cos(kz - \omega t), \quad \omega > 0 \tag{3}$$

Use Fermi's golden rule for a harmonic perturbation to calculate the ionization rate to free plane wave electron states in a box of size $V = L^3$

$$\psi_f(\mathbf{r}) = L^{-3/2} e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} \tag{4}$$

differentially in $\cos\theta$, where θ is the angle between $\mathbf{p}$ and the z-axis, i.e. between $\mathbf{p}$ and $\mathbf{k}$. The size of the box L should cancel in the final result, and L has to be much larger than what quantity? You need to recall the density of states for the plane waves from e.g. your statistical mechanics course. Check that your result has the correct dimensions. You do not need to integrate over θ.