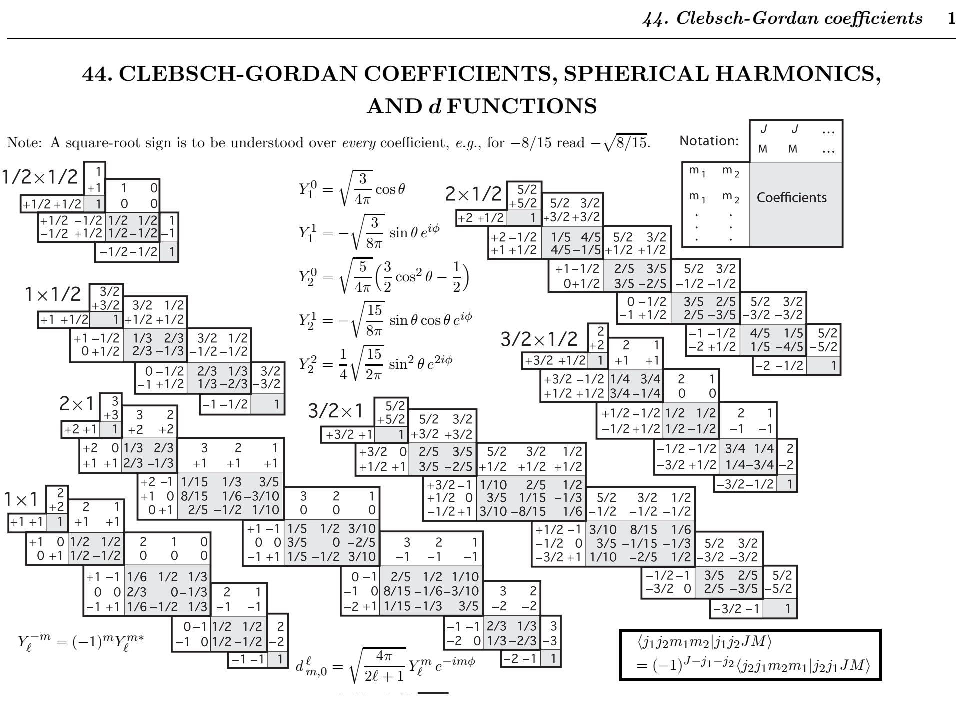
QM IIA spring 2020, week 7

Reading assignment for Tuesday Feb 18th: more on rotations

- Addition of angular momenta
- Spherical tensors and Wigner-Eckart theorem (WET)
- Read from
  - Tuominen: Secs 5.5-5.7
  - Sakurai (old edition): Secs 3.7, 3.10
  - Sakurai and Napolitano Secs 3.8, 3.11
  - Bransden Secs. 6.9 and 6.10 (does not discuss spherical tensor operators and WET)
  - Niskanen Secs. 2.6-2.8
  - Eskola, p. 286-325
- Note preliminary list of exam questions in moodle

Preliminary exercises Do these before the class of Tue Feb 18th and be prepared to present your solutions in class.

1. Let's get familiar with Clebsch-Gordans by explicitly coupling a triplet and a doublet: $3 \otimes 2 = 4 \oplus 2$. This could be e.g. a hydrogen atom in the P-wave $\ell = 1$ state, with electron spin $s = 1/2$. [Using the notation $2j+1$ for the representation, i.e. $3 \otimes 2$ means $(j_1 = 1) \otimes (j_2 = 1/2)$]. What do the "2,3,4", "$\otimes$" and "$\oplus$" mean? How many states are there? Construct explicitly the coupled basis states in as linear combinations of the uncoupled ones using the CG table below (pay attention to the "note" about the square root in the table). What values j_1, j_2 momenta are coupled when $5 \otimes 3 = 7 \oplus 5 \oplus 3$? Locate these in the table below. What are "$3j$, $6j$ and $9j$ symbols"?
2. Calculate the Clebsch-Gordan coefficients for $2 \otimes 2 = 3 \oplus 1$ (denoted $1/2 \times 1/2$ in the table) using the ladder operators. Write the coefficients as a unitary Clebsch-Gordan matrix. What is the dimensionality of this matrix?
3. Write the components of the position vector $\mathbf{r}$ as superpositions of spherical tensors $\sum_{kq} c_q^k T_q^{(k)}$. What is the rank k of the spherical tensor $\mathbf{r}$? Do the same for a few components of the tensor $x^i x^j$ (e.g. z^2, x^2, or xy) and convince yourself that it is a superposition of rank $k = 0$ and $k = 2$ tensors. Why does the Wigner-Eckart theorem lead to the *selection rule* $|\ell - \ell'| = 1$, $m - m' = \pm 1, 0$ in electromagnetic dipole transitions $\langle n'\ell'm'|\mathbf{r}|n\ell m\rangle$? (Note that $|\ell - \ell'| = \pm 1, 0$ is straightforward, but separately one needs to show $|\ell - \ell'| \neq 0$ which is more complicated).



44. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation: columns headed J, M; rows labelled m_1, m_2; entries are the coefficients.

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1 m_2	J, M		
	1, +1		
+1/2 +1/2	1		
	1, 0	0, 0	
+1/2 −1/2	1/2	1/2	
−1/2 +1/2	1/2	−1/2	
	1, −1		
−1/2 −1/2	1		

1×1/2

m_1 m_2	J, M	
	3/2, +3/2	
+1 +1/2	1	
	3/2, +1/2	1/2, +1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3
	3/2, −1/2	1/2, −1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3
	3/2, −3/2	
−1 −1/2	1	

2×1/2

m_1 m_2	J, M	
	5/2, +5/2	
+2 +1/2	1	
	5/2, +3/2	3/2, +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5
	5/2, +1/2	3/2, +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5
	5/2, −1/2	3/2, −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5
	5/2, −3/2	3/2, −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5
	5/2, −5/2	
−2 −1/2	1	

3/2×1/2

m_1 m_2	J, M	
	2, +2	
+3/2 +1/2	1	
	2, +1	1, +1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4
	2, 0	1, 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2
	2, −1	1, −1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4
	2, −2	
−3/2 −1/2	1	

2×1

m_1 m_2	J, M		
	3, +3		
+2 +1	1		
	3, +2	2, +2	
+2 0	1/3	2/3	
+1 +1	2/3	−1/3	
	3, +1	2, +1	1, +1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10
	3, 0	2, 0	1, 0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10
	3, −1	2, −1	1, −1
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5
	3, −2	2, −2	
−1 −1	2/3	1/3	
−2 0	1/3	−2/3	
	3, −3		
−2 −1	1		

1×1

m_1 m_2	J, M		
	2, +2		
+1 +1	1		
	2, +1	1, +1	
+1 0	1/2	1/2	
0 +1	1/2	−1/2	
	2, 0	1, 0	0, 0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3
	2, −1	1, −1	
0 −1	1/2	1/2	
−1 0	1/2	−1/2	
	2, −2		
−1 −1	1		

3/2×1

m_1 m_2	J, M		
	5/2, +5/2		
+3/2 +1	1		
	5/2, +3/2	3/2, +3/2	
+3/2 0	2/5	3/5	
+1/2 +1	3/5	−2/5	
	5/2, +1/2	3/2, +1/2	1/2, +1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6
	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2
	5/2, −3/2	3/2, −3/2	
−1/2 −1	3/5	2/5	
−3/2 0	2/5	−3/5	
	5/2, −5/2		
−3/2 −1	1		