

Reading assignment for Tuesday Feb 11th: rotations

- Groups and representations
- Rotation matrices
- Angular momentum algebra (reminder from QMI)
- Read from:
 - Tuominen: Secs 5.1-5.4
 - Sakurai (old edition or Sakurai & Napolitano): Secs 3.1-3.3, 3.5-3.6
 - Bransden Secs 6.1-6.9
 - Eskola, p. 241-285
 - Niskanen Secs. 2.1-2.5

Preliminary exercises do these before the class of Tue Feb 11th and be prepared to present your solutions in class.

- (a) What is a group? What is a representation? Why do the matrices that rotate 3-dimensional vectors form a group? What is the name of this group?
- (b) What are the matrices for infinitesimally small rotations of a 3-dimensional vector around the x, y and z axes? From these identify the generators of the group. What are the commutation relations of the generators?
- Recall the spectrum of the angular momentum operator, taking $\hbar = 1$.
 - Why do the operators \mathbf{J}^2 and J_z have simultaneous eigenstates: $|\lambda_{\mathbf{J}^2}, \lambda_{J_z}\rangle$ with $\mathbf{J}^2|\lambda_{\mathbf{J}^2}, \lambda_{J_z}\rangle = \lambda_{\mathbf{J}^2}|\lambda_{\mathbf{J}^2}, \lambda_{J_z}\rangle$ and $J_z|\lambda_{\mathbf{J}^2}, \lambda_{J_z}\rangle = \lambda_{J_z}|\lambda_{\mathbf{J}^2}, \lambda_{J_z}\rangle$?
 - What are the possible eigenvalues $\lambda_{\mathbf{J}^2}$ and λ_{J_z} , how does one find them from the commutation relations?
- What is the matrix that rotates a $j = 1/2$ spinor by an angle ω around the axis $\mathbf{n}, \mathbf{n}^2 = 1$? What happens with $\omega = 2\pi$? What is the matrix that rotates the same spinor by first an angle γ around the z -axis, then an angle β around the y -axis and finally an angle α around the z -axis? Convince yourself that all rotations can be expressed in both ways. What is $d_{m,m'}^j(\beta)$? Read off from your result $d_{1/2,1/2}^{1/2}$ and $d_{1/2,-1/2}^{1/2}$.

$$Y_{\ell}^{-m} = (-1)^m Y_{\ell}^{m*} \quad \begin{bmatrix} -1 & 0 & 1/2 & -1/2 \\ -1 & -1 & 1 & 1 \end{bmatrix} d_{m,0}^{\ell} = \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell}^m e^{-im\phi} \quad \begin{bmatrix} -2 & 0 & 1/3 & -2/3 \\ -2 & -1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} \langle j_1 j_2 m_1 m_2 | j_1 j_2 J M \rangle \\ = (-1)^{J-j_1-j_2} \langle j_2 j_1 m_2 m_1 | j_2 j_1 J M \rangle \end{bmatrix}$$

$$d_{m',m}^j = (-1)^{m-m'} d_{-m,-m'}^j = d_{-m,-m'}^j \quad \begin{bmatrix} 3/2 \times 3/2 \\ +3/2 & +3/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1/2 & 1/2 \end{bmatrix} \quad d_{0,0}^1 = \cos \theta \quad d_{1/2,1/2}^1 = \cos \frac{\theta}{2} \quad d_{1,1}^1 = \frac{1+\cos \theta}{2}$$

$$2 \times 3/2 \quad \begin{bmatrix} 7/2 & 5/2 \\ +7/2 & +5/2 \end{bmatrix} \begin{bmatrix} 7/2 & 5/2 \\ 3/2 & 3/2 \end{bmatrix} \begin{bmatrix} 7/2 & 5/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1/2 & 1/2 \end{bmatrix} \quad d_{1/2,-1/2}^1 = -\sin \frac{\theta}{2} \quad d_{1,0}^1 = -\frac{\sin \theta}{\sqrt{2}}$$

$$2 \times 2 \quad \begin{bmatrix} 4 & 3 \\ +4 & +3 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1/2 & 1/2 \end{bmatrix} \quad d_{1,-1}^1 = \frac{1-\cos \theta}{2}$$

$$d_{3/2,3/2}^{3/2} = \frac{1+\cos \theta}{2} \cos \frac{\theta}{2} \quad d_{3/2,1/2}^{3/2} = -\sqrt{3} \frac{1+\cos \theta}{2} \sin \frac{\theta}{2} \quad d_{3/2,-1/2}^{3/2} = \sqrt{3} \frac{1-\cos \theta}{2} \cos \frac{\theta}{2} \quad d_{3/2,-3/2}^{3/2} = \frac{1-\cos \theta}{2} \sin \frac{\theta}{2}$$

$$d_{1/2,1/2}^{3/2} = \frac{3 \cos \theta - 1}{2} \cos \frac{\theta}{2} \quad d_{1/2,-1/2}^{3/2} = -\frac{3 \cos \theta + 1}{2} \sin \frac{\theta}{2}$$

$$d_{2,2}^2 = \left(\frac{1+\cos \theta}{2} \right)^2 \quad d_{2,1}^2 = -\frac{1+\cos \theta}{2} \sin \theta \quad d_{2,0}^2 = \frac{\sqrt{6}}{4} \sin^2 \theta \quad d_{2,-1}^2 = -\frac{1-\cos \theta}{2} \sin \theta \quad d_{2,-2}^2 = \left(\frac{1-\cos \theta}{2} \right)^2$$

$$d_{1,1}^2 = \frac{1+\cos \theta}{2} (2 \cos \theta - 1) \quad d_{1,0}^2 = -\sqrt{\frac{3}{2}} \sin \theta \cos \theta \quad d_{1,-1}^2 = \frac{1-\cos \theta}{2} (2 \cos \theta + 1) \quad d_{0,0}^2 = \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

Figure 44.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).