

QM IIA spring 2020, week 5

Reading assignment for Tuesday Feb 4th: more features of scattering theory

- Optical theorem
- Partial wave amplitudes, the phase shift
- Eikonal approximation, resonance scattering
- Read from
 - Tuominen: Secs 4.6-4.12
 - Sakurai (old edition): Secs 7.3-7.8 (see also rest of Chapter 7)
 - Sakurai and Napolitano Secs 6.4-6.7 (see also the rest of Chapter 6)
 - Eskola, p. 45-50 and 74-99
 - Bransden Secs 13.3-13.4 and 13.7-13.8.

Preliminary exercises Do these before the class of Tuesday Feb 4th and be prepared to present your solutions in class.

1. Consider a wave scattered from a potential (that is localized close to the origin $\mathbf{r} = \mathbf{0}$):

$$\psi(\mathbf{r}) = e^{ikz} + f(\theta) \frac{e^{ikr}}{r}. \quad (1)$$

Now let us look at this wave on a disk A perpendicular to the z -axis, behind the target at $z \gg 1/k$; $\mathbf{r} = (x, y, z) = (\mathbf{x}_\perp, z) \in A$. Now we want to calculate the integral of $|\psi(\mathbf{r})|^2$ over A . Write an expression for this integral, expanding to leading nontrivial order in \mathbf{x}_\perp^2 in the limit $z \gg x, y$. Now use the result

$$\int_A d^2\mathbf{x}_\perp e^{-\mathbf{x}_\perp^2/(2a^2i)} \approx \int_{\mathbb{R}^2} d^2\mathbf{x}_\perp e^{-\mathbf{x}_\perp^2/(2a^2i)} = 2\pi a^2 i, \quad (2)$$

to show that

$$\int_A d^2\mathbf{x}_\perp |\psi(\mathbf{r})|^2 = A - \frac{4\pi}{k} \text{Im} f(\theta = 0). \quad (3)$$

What does this tell you about what happened to density of particles originally coming in from $z = -\infty$? Interpret as the *optical theorem*. The approximation (2) strictly speaking requires the limit $z \rightarrow \infty$, $A \rightarrow \infty$ such that the opening angle $\theta^2 \sim A/z^2 \ll 1$ stays fixed. Draw a figure to illustrate the coordinates/angles!

2. Write down a decomposition of the scattering amplitude $f(\theta)$ in terms of Legendre polynomials, i.e. partial waves, with coefficients expressed in terms of the *phase shifts*. If you know the potential, describe what you have to do to calculate the phase shifts. Square the amplitude and integrate over θ to get an expression for the total cross section. For this you need to find somewhere the orthogonality property and the value at 1 of the Legendre polynomials. Show that you have recovered the optical theorem. What is the “partial wave unitarity bound”? What does it mean to have an “ S -wave scattering”? What happened to the other angle φ ?
3. (a) Assume that one partial wave ℓ is much larger than the others, and that for scattering energies E close to some value E_0 we have $\cot \delta_\ell \approx 2(E - E_0)/\Gamma$. Calculate how the cross section depends on $E - E_0$ and show that it has a local maximum at $E = E_0$.
 (b) What is the definition of the *scattering length*? What does it tell you?