FYSH560 spring 2011

Exercise 2, return by Tue Feb 1st at 14.15, discussed Wed Feb 2nd at 8.15 in FL140

1. Derive the Green's function for the Helmholtz equation that was skipped in the lectures. Start from the definition $(\omega^2 + \nabla^2)G(\mathbf{x}, \mathbf{y}) = -\delta^3(\mathbf{x} - \mathbf{y})$. Defining the Fourier transform as
$$G(\mathbf{x}, \mathbf{y}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} G(\mathbf{p})$$
calculate first $G(\mathbf{p})$ and then $G(\mathbf{x}, \mathbf{y})$. You have to regularize the $\mathbf{p}$ integral; doing this with the substitution $\omega \to \omega + i\varepsilon$ will give the result we want here. What do you get if you replace $\omega \to \omega - i\varepsilon$ (we are assuming $\omega > 0$)?

2. (a) Show (this is easy) that if
$$\frac{\mathrm{d}\sigma_{\mathrm{el.}}}{\mathrm{d}^2\mathbf{q}_T} = \left| \frac{i}{2\pi} \int \mathrm{d}^2\mathbf{b}_T e^{-i\mathbf{q}_T\cdot\mathbf{b}_T} \Gamma(\mathbf{b}_T) \right|^2$$
then
$$\sigma_{\mathrm{el}} = \int \mathrm{d}^2\mathbf{b}_T |\Gamma(\mathbf{b}_T)|^2$$
(b) The total cross section is
$$\sigma_{\mathrm{tot}} = 2 \int \mathrm{d}^2\mathbf{b}_T \mathrm{Re}[\Gamma(\mathbf{b}_T)],$$
and the partial wave unitarity bound is $|\Gamma(\mathbf{b}_T)|^2 \leq 2\mathrm{Re}[\Gamma(\mathbf{b}_T)]$, which leads to $\sigma_{\mathrm{el}} \leq \sigma_{\mathrm{tot}}$ (a pretty natural requirement). Where in the complex plane can $\Gamma(\mathbf{b}_T)$ be to satisfy this?

3. Let's rederive the cross section in the eikonal approximation via the Lippman-Schwinger equation. We want to solve the Schroedinger equation
$$\left[\nabla^2 - U(\vec{r}) + k^2\right] \psi(\vec{r}) = 0 \tag{1}$$
with the initial condition $\psi(\vec{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$ for $z \to -\infty$.

   (a) Show that the solution Lippman-Schwinger integral equation
$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{1}{4\pi} \int \mathrm{d}^3\mathbf{r}' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} U(\mathbf{r}')\psi(\mathbf{r}') \tag{2}$$
satisfies Eq. (1).

   (b) We are interested in the asymptotic solution $\psi(\mathbf{r})|_{r\to\infty}$ that should look like
$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + f(\mathbf{k}, \mathbf{k}')\frac{e^{ikr}}{r},$$
where $f(\mathbf{k}, \mathbf{k}')$ is the scattering amplitude. Assuming that $rU(\mathbf{r}) \to 0$ for $r \to \infty$ Eq. (2) immediately gives
$$f(\mathbf{k}, \mathbf{k}') = -\frac{1}{4\pi} \int \mathrm{d}^3\mathbf{r}' e^{i\mathbf{k}'\cdot\mathbf{r}} U(\mathbf{r}')\psi(\mathbf{r}').$$
In the eikonal approximation the wavefunction is
$$\psi(\mathbf{r}) \approx \exp\left\{ i\mathbf{k}\cdot\mathbf{r} - \frac{i}{2k} \int_{-\infty}^{z} \mathrm{d}z' U(x, y, z') \right\}.$$
Calculate the scattering amplitude $f(\mathbf{k}, \mathbf{k}')$ neglecting the z-component of the momentum transfer $\mathbf{q} = \mathbf{k}' - \mathbf{k}$.

4. Take the Feynman rules for the graviton propagator (26) and the graviton-spin-0 particle vertex (37) from `http://homepages.nyu.edu/~jll419/gravity.pdf` .
Calculate the elastic scattering cross section $\mathrm{d}\sigma/\mathrm{d}t$ between two unidentical massless spin-0 particles via the exchange of a graviton. What is the dimension of the coupling κ so that the cross section has the right dimensions? What is the high energy behavior?