FYSH555 spring 2014

Exercise 2, return by Mon Jan 27th at 10.00., discussed Mon Jan 27th, at 16.15 in FYS5

1. Let's rederive the cross section in the eikonal approximation via the Lippman-Schwinger equation. We want to solve the Schroedinger equation

$$\left[\nabla^2 - U(\vec{r}) + k^2\right]\psi(\vec{r}) = 0 \tag{1}$$

with the initial condition $\psi(\vec{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$ for $z \to -\infty$.

   (a) Show that the solution Lippman-Schwinger integral equation

$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{1}{4\pi}\int \mathrm{d}^3\mathbf{r}' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} U(\mathbf{r}')\psi(\mathbf{r}') \tag{2}$$

   satisfies Eq. (1).

   (b) We are interested in the asymptotic solution $\psi(\mathbf{r})|_{r\to\infty}$ that should look like

$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + f(\mathbf{k},\mathbf{k}')\frac{e^{ikr}}{r},$$

   where $f(\mathbf{k},\mathbf{k}')$ is the scattering amplitude. Assuming that $rU(\mathbf{r}) \to 0$ for $r \to \infty$ Eq. (2) immediately gives

$$f(\mathbf{k},\mathbf{k}') = -\frac{1}{4\pi}\int \mathrm{d}^3\mathbf{r}' e^{-i\mathbf{k}'\cdot\mathbf{r}'} U(\mathbf{r}')\psi(\mathbf{r}'). \tag{3}$$

   In the eikonal approximation the wavefunction is

$$\psi(\mathbf{r}) \approx \exp\left\{ i\mathbf{k}\cdot\mathbf{r} - \frac{i}{2k}\int_{-\infty}^{z} \mathrm{d}z' U(x,y,z') \right\}. \tag{4}$$

   Calculate the scattering amplitude $f(\mathbf{k},\mathbf{k}')$ by inserting Eq. (4) into Eq. (3) neglecting the z-component of the momentum transfer $\mathbf{q} = \mathbf{k}' - \mathbf{k}$.

2. Take the Feynman rules for the graviton propagator (39) and the graviton-spin-0 particle vertex (first line of (40)) from gr-qc/9512024, taking $m = 0$
Calculate the elastic scattering cross section $\mathrm{d}\sigma/\mathrm{d}t$ between two unidentical massless spin-0 particles via the exchange of a graviton (one Feynman diagram). What is the dimension of the coupling κ so that the cross section has the right dimensions? What is the high energy behavior?

3. Let us try to get a handle on dispersion relations. Assume that we have an amplitude with imaginary part $\mathrm{Im}A(s,t) = s$ for $s > s_+$. Calculate the real part from the twice subtracted dispersion relation

$$\mathrm{Re}A(s,t) = \frac{s^2}{\pi}\left[P\int_{s_+}^{\infty} \frac{\mathrm{d}s'}{s'}\frac{1}{s'-s} \right],$$

where P denotes the principal value (i.e. integrate up to $s-\varepsilon$ and up from $s+\varepsilon$ and take the limit $\epsilon \to 0$). Take the limit $s \to \infty$ and combine the real and imaginary parts into an analytic function of s and t. Do you get the sign of the imaginary part consistently? You need to know that $\ln(-s) = \ln s - i\pi$; how is this related to our convention that the physical $A(s,t)$ is $\lim_{\varepsilon\to 0^+} A(s+i\varepsilon,t)$? The logarithm is chosen to have a branch cut on the negative real axis.

4. Now we shall learn about string theory which, as you might know, was a failed attempt to describe the strong interaction before the discovery of QCD. Consider a simple model of a meson as a system of two massless quarks connected by a string. The string has a linear energy density (in its own rest frame) of κ (units: GeV/fm). The length of the string is $2R$. The system is rotating rigidly so that the quark at the end move at the speed of light $c = 1$ on a circular trajectory. Remember that if the energy of a piece (length $\mathrm{d}l$) of the string is $\kappa \mathrm{d}l$ in its rest frame (rest mass of the string), it is $\kappa \mathrm{d}l/\sqrt{1-v^2}$ in a frame where the piece is moving at velocity v. Calculate the *energy* of the spinning string (in the frame where the center of the meson is at rest). This is the mass of the meson m. Now calculate the *angular momentum* with respect to the center of the meson (remembering that the angular momentum of a point particle is pr, and for p you need the relativistic momentum of a particle at velocity v). This is the spin of the meson, which must be an integer (mesons are bosons). Verify that these mesons lie on a Regge trajectory; i.e. $J = \alpha' m^2$.

5. A famous ansatz for the scattering amplitude of these objects is the *Veneziano amplitude*:

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))},$$

where the trajectory is a linear function $\alpha(x) = \alpha_0 + \alpha' x$ with $\alpha' > 0$. It is derived in in any string theory textbook as the scattering amplitude of bosonic strings.

   (a) Where are the poles (singularities) of this amplitude in the physical s-channel region $s > 0$, $t < 0$? The amplitude has been constructed as the simplest one with singularities in these locations. Why must they be there?

   (b) Using Stirling's approximation for the Γ function ($\Gamma(n+1) = n!$) find out the high energy limit of the Veneziano amplitude. The Stirling formula does not apply on the negative real axis but we are assuming that s has a small imaginary part so it is OK here.

   (c) The hard scattering limit is $s \to \infty$, $t \to -\infty$, t/s fixed. For point-like particles scattering amplitudes should decrease like a power in s or t in this limit. How does the Veneziano amplitude behave in this limit? Does it describe point-like particles?